Rings and their spectrum

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Noncommutative geometry

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Thus we have functors $F\colon \{\text{spaces}\} \to \{\text{commutative rings}\}$ and $G\colon \{\text{commutative rings}\} \to \{\text{spaces}\}$.

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Conversely, if $\mathfrak{a}$ is an ideal in $C(X)$, let

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be the common vanishing locus of the functions in $\mathfrak{a}$. Then we get a mapping $V\colon \mathcal{L}(C(X)) \to \mathcal{P}(X)$.

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More is true: not only do they reverse the orders, they also send the greatest element X of $\mathcal{P}(X)$ to the least element 0 of $\mathcal{L}(C(X))$ and the least element $\emptyset$ of $\mathcal{P}(X)$ to the greatest element $C(X)$ of $\mathcal{L}(C(X))$,

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Moreover, $A \subseteq V(I(A))$, $\mathfrak{a} \subseteq I(V(\mathfrak{a}))$, $V(I(V(\mathfrak{a}))) = V(\mathfrak{a})$ and $I(V(I(A))) = I(A)$. (Antitone Galois connection=...

Antitone Galois connection

If $(A, \leq)$ and $(B, \leq)$ are two partially ordered sets, an *antitone Galois connection* is a pair of order-reversing mappings $\alpha\colon A \to B$ and $\beta\colon B \to A$ such that $b \leq \alpha(a)$ if and only if $a \leq \beta(b)$ for every $a \in A$, $b \in B$.

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If $x_0 \in X$ is a point, then $I(x_0)$ is a maximal ideal, because it is the kernel of the surjective ring morphism $C(X) \to \mathbb{R}$, $f \mapsto f(x_0)$.

Proposition

If X is a compact Hausdorff space, every maximal ideal is the ideal of a point.

Thus if X is a compact Hausdorff space, we can recover X from $C(X)$.

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If X is not compact, this is false: the ideal $\mathfrak{a}$ of functions with compact support is proper, hence is contained in a maximal ideal, but $V(\mathfrak{a}) = \emptyset$. In this general case, the maximal ideals of $C(X)$ are in bijection with the points of the Stone-Čech compactification of X.

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Throughout, all rings are associative, with identity $1 \neq 0$, and all ring morphisms send 1 to 1 (We don't want fields with characteristic one, with one element...)

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There is a mapping $V\colon \mathcal{L}(R) \to \mathcal{P}(X)$, defined, for every ideal $\mathfrak{a}$ of R, by

$$V(\mathfrak{a}) := \{\, P \in X \mid P \supseteq \mathfrak{a} \,\}.$$

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R any commutative ring, $X := \mathrm{Spec}(R)$,
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We find again that V is a lattice antihomomorphism (it reverses the inclusion), it sents to the greatest element X of $\mathcal{P}(X)$ the least element 0 of $\mathcal{L}(R)$ and to the least element $\emptyset$ of $\mathcal{P}(X)$ the greatest element R of $\mathcal{L}(R)$,

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As a corollary: the image of the mapping $V\colon \mathcal{L}(R) \to \mathcal{P}(X)$ is a family of subsets of X that satisfies the axioms for closed sets in a topological space. The resulting topology on X is called the *Zariski topology* and the topological space X is called the *prime spectrum* of R.

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A basis of compact open sets

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and an open subset of X is compact if and only if it is a finite union of sets X_f.

Irreducible topological spaces

A topological space X is said to be *irreducible* if $X \neq \emptyset$ and $A \cap B \neq \emptyset$ for every pair A, B of non-empty open sets in X, or, equivalently, every non-empty open set is dense in X.

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In $\mathrm{Spec}(R)$ the irreducible closed subsets are the subsets $V(\mathfrak{p})$ with $\mathfrak{p}$ a prime ideal of R, that is, the closures $V(\mathfrak{p})$ of the points $\mathfrak{p}$ of $\mathrm{Spec}(R)$.

Hochster

M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969), 43–60:

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Let X be a topological space and let $K^{\circ}(X)$ be the set of all its compact open subsets. Then X is homeomorphic to the spectrum of a commutative ring if and only if it is a *spectral space*, i.e., it satisfies all of the following conditions:

(1) X is compact and T_0.

(2) $K^{\circ}(X)$ is a basis of open subsets for the topology of X.

(3) $K^{\circ}(X)$ is closed under finite intersections.

(4) Every irreducible closed subset of X is the closure of a point of X.

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(3) $K^\circ(X)$ is closed under finite intersections.
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(Taking the empty intersection in (3) yields that X is compact).

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Given any ring morphism $\varphi\colon R \to S$, associate to φ the mapping $\varphi^*\colon\ \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, $\mathfrak{q} \in \mathrm{Spec}(S) \mapsto \varphi^{-1}(\mathfrak{q})$.

Dozens of generalizations to noncommutative rings

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Dozens of generalizations to noncommutative rings.

Here is one:
Guoyin Zhang , Wenting Tong and Fanggui Wang, Spectrum of a Noncommutative Ring, Comm. Algebra 34 (2006), 2795–2810.

Here we have that $\mathrm{Spec}_r(R)$ is the set of all prime right ideals of R. For every right ideal I of R, set $U_r(I) := \{ P \in \mathrm{Spec}_r(R) \mid P \not\supseteq I \}$. The topology of $\mathrm{Spec}_r(R)$ with basis of open subsets the sets $U_r(I)$ is called the *weak Zariski topology* of $\mathrm{Spec}_r(R)$.

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Recall that a right ideal I of R is *prime* if, for every $a, b \in R$, $aRb \subseteq I$ implies $a \in I$ or $b \in I$.

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Recall that a right ideal I of R is *prime* if, for every $a, b \in R$, $aRb \subseteq I$ implies $a \in I$ or $b \in I$. Unluckily, the inverse image of a prime ideal via a ring morphism is not necessarily a prime ideal. Cf. $\mathbb{Q} \times \mathbb{Q} \hookrightarrow \mathbb{M}_2(\mathbb{Q})$.

Another possibility: Cohn's spectrum

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Cohn's spectrum $\mathbf{X}(R)$ of the ring R is the topological space $\mathbf{X}(R)$ of all epic R-fields, up to isomorphism. A ring morphism $f\colon R \to D$ is an *epic R-field* (P. M. Cohn) if D is a division ring and there is no division ring different from D between $f(R)$ and D. There are rings R for which there is no epic R-field $R \to D$. For instance, if R is a ring that is not IBN, there is no ring morphism $R \to D$, for any division ring D.

A wonderful paper

M. Reyes, Obstructing extensions of the functor Spec to noncommutative rings, Israel J. Math. 192 (2012), 667–698.

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Theorem

Let F be a contravariant functor from the category of rings to the category of sets whose restriction to the full subcategory of commutative rings is naturally isomorphic to the functor Spec. Then F assigns the empty set to the rings of matrices $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$.

More precisely:
First of all one must fix the desired properties of the "noncommutative spectrum" in question. Consider the prime spectrum Spec. From the viewpoint of $\operatorname{Spec}(R)$ as an underlying point-set, two facts of key importance are (1) the spectrum of every nonzero commutative ring is non-empty, and (2) the prime spectrum construction can be regarded as a contravariant functor from the category of commutative rings to the category of sets,

$$\operatorname{Spec}\colon \mathsf{CommRing} \to \mathsf{Set}.$$

Theorem (Reyes)

Let F be a rule assigning to each ring R a set $F(R)$, such that for every commutative ring C one has $F(C) \cong \mathrm{Spec}(C)$.

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Then one of the following two properties does not hold:

Property A: For every nonzero ring R, the set $F(R)$ is non-empty.

Property B: The assignment $R \mapsto F(R)$ is the object part of a functor F whose restriction to the category of commutative rings is isomorphic to Spec.

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The set of prime right ideals satisfies A but not B.

The set of completely prime ideals satisfies B but not A. (An ideal I of R is *completely prime* if, for every $r, s \in R$, $rs \in I$ implies $r \in I$ or $s \in I$.)

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Property B: The invariant F can be made into a set-valued functor extending Spec, in the sense that the assignment $R \mapsto F(R)$ is the object part of a functor F whose restriction to the category of commutative rings is isomorphic to Spec.

O. Goldman, Rings and modules of quotients, J, Algebra 13 (1969), 10–47.

Goldman's prime torsion theories (suitable torsion theories associated to suitable Gabriel filters on R).

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A different approach: another "space" attached to a ring

– and L. Heidari Zadeh, On a partially ordered set associated to ring morphisms, J. Algebra 535 (2019), 456-479.

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Let R be a ring. The group of all invertible elements of R will be denoted by $U(R)$.

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Let R be a ring. The group of all invertible elements of R will be denoted by $U(R)$. We associate to each ring morphism $\varphi\colon R \to S$ into any other ring S the pair $(\mathfrak{a}, M)$, where $\mathfrak{a} := \ker(\varphi)$ is the kernel of φ and $M := \varphi^{-1}(U(S))$ is the inverse image of the group of units $U(S)$ of S.

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Let R be a ring. The group of all invertible elements of R will be denoted by $U(R)$. We associate to each ring morphism $\varphi\colon R \to S$ into any other ring S the pair $(\mathfrak{a}, M)$, where $\mathfrak{a} := \ker(\varphi)$ is the kernel of φ and $M := \varphi^{-1}(U(S))$ is the inverse image of the group of units $U(S)$ of S. Thus M is a submonoid of the multiplicative monoid R that contains $U(R)$.

The set $\mathrm{Hom}(R)$

We associate to any ring R with identity a partially ordered set $\mathrm{Hom}(R)$, whose elements are all pairs $(\mathfrak{a}, M)$, where $\mathfrak{a} = \ker \varphi$ and $M = \varphi^{-1}(U(S))$ for some ring morphism φ of R into an arbitrary ring S.

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Proposition

Let Ring *be the category of rings with identity and* ParOrd *the category of partially ordered sets. Then* $\mathrm{Hom}(-)\colon \mathrm{Ring} \to \mathrm{ParOrd}$ *is a contravariant functor.*

The contravariant functor $\mathrm{Hom}(-)\colon \mathsf{Ring} \to \mathsf{ParOrd}$

It assigns to each ring morphism $f\colon R \to R'$ the increasing mapping

$$\mathrm{Hom}(f)\colon \mathrm{Hom}(R') \to \mathrm{Hom}(R)$$

$$(\mathfrak{a}', M') \in \mathrm{Hom}(R') \mapsto (f^{-1}(\mathfrak{a}'), f^{-1}(M')).$$

Notice that if $\varphi'\colon R' \to S$ is a ring morphism, $\mathfrak{a}' := \ker(\varphi')$ and $M' := \varphi'^{-1}(U(S))$, then $\varphi' f\colon R \to S$ is a ring morphism,

$$f^{-1}(\mathfrak{a}') = \ker(\varphi' f)$$

and

$$f^{-1}(M') = (\varphi' f)^{-1}(U(S)).$$

The contravariant functor $\mathrm{Hom}(-)\colon \mathsf{Ring} \to \mathsf{ParOrd}$

This functor has a good behaviour. For instance:

Theorem

$$\mathrm{Hom}(\varinjlim R_i) \cong \varprojlim \mathrm{Hom}(R_i).$$

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Let $\mathfrak{a}$ be an ideal of a ring R such that $(\mathfrak{a}, R \setminus \mathfrak{a}) \in \mathrm{Hom}(R)$. Then $\mathfrak{a}$ is a completely prime ideal of R, the universal mapping inverting all non-zero elements of the ring $R/\mathfrak{a}$ is injective, and $(\mathfrak{a}, R \setminus \mathfrak{a}) \in \mathrm{Hom}(R)$ is a maximal element of $\mathrm{Hom}(R)$.

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Corollary

For every commutative ring R, the set $\mathsf{Max}(R)$ of all maximal elements of $\mathrm{Hom}(R)$ is in one-to-one correspondence with $\mathrm{Spec}(R)$.

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Corollary

For every commutative ring R, the set $\mathsf{Max}(R)$ of all maximal elements of $\mathrm{Hom}(R)$ is in one-to-one correspondence with $\mathrm{Spec}(R)$.

Thus the set $\mathsf{Max}(R)$ of all maximal elements of $\mathrm{Hom}(R)$ could be used as a good replacement for the spectrum of a non-commutative ring R.

The set $\mathsf{Max}(R)$ of all maximal elements of $\mathrm{Hom}(R)$

The set of all maximal elements of $\mathrm{Hom}(R)$ is never empty:

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For every ring R, the partially ordered set $\mathrm{Hom}(R)$ has maximal elements.

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Ops!: a contravariant functor and never empty! A contradiction to Reyes' result? No: Our contravariant functor $\mathrm{Hom}(-)\colon \mathsf{Ring} \to \mathsf{Set}$ does not extend $\mathrm{Spec}\colon \mathsf{CommRing} \to \mathsf{Set}$.

Theorem

There is no contravariant functor from the category of rings to the category of sets that assigns to each ring R the set of all maximal elements of $\mathrm{Hom}(R)$.

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It is something very similar to what we get in the commutative case: we have that $\mathsf{Max}\colon \mathsf{CommRing} \to \mathsf{Set}$ is always non-empty, but it is not a contravariant functor.

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(For an arbitrary topological space X, $K^\circ(X)$ is not closed under finite intersections, in general. When X is a spectral topological space, $K^\circ(X)$ is a bounded distributive lattice.)

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As far as (1) is concerned, notice that finite topological space = finite preordered set, and finite T_0 topological space = finite partially ordered set.

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for every $x, y \in L, x \wedge y \in I \Leftrightarrow x \in I$ or $y \in I$).

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Moreover, $L \cong K^{\circ}(X)$.

The lattice $\overline{\mathrm{Hom}(R)}$

We now enlarge the partially ordered set $\mathrm{Hom}(R)$ adjoining to it a further element, a new greatest element 1, setting
$\overline{\mathrm{Hom}(R)} := \mathrm{Hom}(R) \,\dot\cup\, \{1\}$.

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The contravariant functor $\mathrm{Hom}(-)\colon \mathsf{Ring} \to \mathsf{ParOrd}$ extends to a contravariant functor $\overline{\mathrm{Hom}(-)}\colon \mathsf{Ring} \to \mathsf{ParOrd}$ simply extending, for every ring morphism $\varphi\colon R \to S$, the mapping $\mathrm{Hom}(\varphi)\colon \mathrm{Hom}(S) \to \mathrm{Hom}(R)$ to the mapping $\overline{\mathrm{Hom}(\varphi)}\colon \overline{\mathrm{Hom}(S)} \to \overline{\mathrm{Hom}(R)}$, where $\overline{\mathrm{Hom}(\varphi)}(1) = 1$.

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(It is not a distributive lattice in general. If $R := \mathbb{F}_2 \propto V$ is the trivial extension of the field $\mathbb{F}_2$ with two elements with the two-dimensional vector space V over $\mathbb{F}_2$, then $\mathrm{Hom}(R)$ is the non-distributive lattice M_3, so that $\overline{\mathrm{Hom}}(R)$ cannot be distributive.)

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Proposition

Let R_1 and R_2 be rings. Then $\overline{\mathrm{Hom}}(R_1 \times R_2) \cong \overline{\mathrm{Hom}}(R_1) \times \overline{\mathrm{Hom}}(R_2)$.

Now, in the commutative case, for every prime $\mathfrak{p}$ of the commutative ring R, we have not only the point $\mathfrak{p}$ in the topological space $\mathrm{Spec}(R)$, but also the *localization* $R_{\mathfrak{p}}$ and the canonical ring morphism $R \to R_{\mathfrak{p}}$.

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Given any point $(\mathfrak{a}, M) \in \mathrm{Hom}(R)$, we can "localize" R at $(\mathfrak{a}, M)$: first, we factor out $\mathfrak{a}$, constructing the ring $R/\mathfrak{a}$; then we formally invert the elements of $R/\mathfrak{a}$ that are in the monoid $M/\mathfrak{a}$.

R-rings

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If R is a ring and X is a set, we will denote by $R\{X\}$ the *free R-ring over X*. Here the elements of X are non-commuting indeterminates. The ring $R\{X\}$ is the R-ring containing X that is the solution of the following universal problem: for every R-ring S and every mapping $\psi \colon X \to S$, there is a unique R-ring homomorphism $R\{X\} \to S$ whose restriction to X is equal to ψ.

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R-rings

As a right R-module, $R\{X\}$ is the free right R-module having as a free set of generators the set whose elements are 1 and all the formal expressions of the type $r_1 x_{i_1} r_2 x_{i_2} \dots r_n x_{i_n}$, where $n \geq 1$, $r_1, r_2, \dots, r_n \in R$ and $x_{i_1}, x_{i_2}, \dots, x_{i_n} \in X$.

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and then extended by $\mathbb{Z}$-bilinearity.

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and then extended by $\mathbb{Z}$-bilinearity. The ring $R\{X\}$ is a graded ring over the monoid $\mathbb{N}_0$ of all non-negative integers, and its component of degree 0 is isomorphic to R, so that the structure of R-ring on $R\{X\}$ is given by an isomorphism between R and the component of degree 0 of $R\{X\}$.

The "localization" at a point of $\mathrm{Hom}(R)$

Suppose that $(\mathfrak{a}, M) \in \mathrm{Hom}(R)$, that is, $\mathfrak{a} = \ker \varphi$ and $M = \varphi^{-1}(U(S))$ for some ring ring S and some ring morphism $\varphi \colon R \to S$.

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Let I be the two-sided ideal of $(R/\mathfrak{a})\{X\}$ generated by the subset $\{\, x_n n - 1 \mid n \in N \,\} \cup \{\, n x_n - 1 \mid n \in N \,\}$ and $R_{(\mathfrak{a},m)} := (R/\mathfrak{a})\{X\}/I$.

The "localization" at a point of $\mathrm{Hom}(R)$

Theorem

Let R be a ring and $(\mathfrak{a}, M)$ be an element of $\mathrm{Hom}(R)$. Then there is a canonical morphism $\psi\colon R \to R_{(\mathfrak{a},m)}$, $R_{(\mathfrak{a},m)}$ is a non-zero ring, $\ker(\psi) = \mathfrak{a}$ and $\psi^{-1}(U(R_{(\mathfrak{a},m)})) = M$. Moreover, for any ring morphism $f\colon R \to S$ such that $\ker(f) \supseteq \mathfrak{a}$ and $f^{-1}(U(S)) \supseteq M$, there is a unique ring morphism $g\colon R_{(\mathfrak{a},m)} \to S$ such that $g\psi = f$.

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Thus, we have a “localization” of R at any point $(\mathfrak{a}, M)$ of $\mathrm{Hom}(R)$. Hence we can construct the analogue of a ringed space.

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A ringed partially ordered set with zero is a pair (L, F), where L is a partially ordered set with a least element 0_L and $F\colon L \to \mathrm{Ring}$ is a covariant functor. Here the partially ordered set L is given a category structure in the usual way and Ring denotes the category of associative rings with identity.

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