

Categorical Commutator Theory

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The theory of commutators [1] can be considered as an extension of the classical commutator theory for groups to more general varieties of algebras. A description of commutator of equivalence relations in Mal'tsev varieties was developed by Smith [9], and then extended to a categorical context by Pedicchio [8], while a first categorical notion of commutator of subobjects was given by Huq in [4].

In this mini-course we will follow the approaches of Higgins [3] and Gumm-Ursini [2] based on the notion of *commutator word*, introduced by Higgins in the context of a variety of Ω -groups (groups equipped with additional algebraic operations of signature Ω). In [6] an internal interpretation of the commutator words is given by means of the so called *formal commutator*, which let us provide an intrinsic notion of Higgins commutator in the context of ideal determined categories [5]. In the same context, following a similar approach given in [7], we can also introduce another notion of commutator, due to Ursini in the case of varieties, and we will see how the latter is related to Higgins, Huq and Smith-Pedicchio commutators.

Furthermore, we will illustrate some useful applications of commutator theory in categorical algebra.

References

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