Price dynamics on a stock market with asymmetric information

Bernard De Meyer - PSE Université Paris 1
Introduction

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- *Informed agents have the power to influence the future prices.*
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• Informed agents have the power to influence the future prices.

• Game theoretical models.

• Informational asymmetries are driving the prices.
  → The price process should be locally a CMMV
    (Continuous Martingale of Maximal Variation)
General idea

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  → *Uninformed agents analyze informed agents’ moves.*
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• *Avoiding too fast revelation*
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  → *Martingale optimization problem*
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4. The ingredients of the proof
5. How do CMMV fit the real data?
Definition of CMMV

- A martingale $\Pi_t$ is a CMMV if $\Pi_t = f(B_t, t)$ where $B = \text{M.B.}$ and $f(x, t)$ is increasing in $x$. 
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Notations:
- $\Delta^2 := \{\}$ of probabilities $\mu$ on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ such that $\int \! x^2 d\mu(x) < \infty$. 
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  - $\Pi_\mu^t := E[f_\mu(B_1) | (B_s)_{s \leq t}]$ is a CMMV s. th. $[\Pi_1^\mu] = \mu$. 

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  • If $\mu \in \Delta^2$ and $Z \sim \mathcal{N}(0, 1)$, there exists a unique increasing function $f_\mu$ such that $[f_\mu(Z)] = \mu$.
  • $\Pi^\mu_t := E[f_\mu(B_1) \mid (B_s)_{s \leq t}]$ is a CMMV s. th. $[\Pi^\mu_1] = \mu$.
  • $\mu = \mathcal{N} \rightarrow \text{Bachelier}; \mu = \text{Log}\mathcal{N} \rightarrow \text{Black and Scholes}$. 
The Market as a game:

- P1 = risk neutral informed investor.
- P2 = remaining part of the market.
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$P_1$ and $P_2$ are trading a risky asset $R$ against a numéraire $N$.

- Information asymmetry:

  $P_1$ receives initially a message $m \in M$ with law $\nu$.
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  At a future date $D$, $m$ will be publicly revealed.
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  At a future date $D$, $m$ will be publicly revealed.
  At date $D$, the value of $R$ on the market will be $L = L(m)$. The value of $N$ will be 1. The function $L(\cdot)$ is known by both players.
The Market as a game:

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- Information asymmetry:
  
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  At a future date $D$, $m$ will be publicly revealed.
  
  At date $D$, the value of $R$ on the market will be $L = L(m)$. The value of $N$ will be 1. The function $L(.)$ is known by both players.

- The message $m$ can be identified with $L(m)$.
  
  $\mu = $law of $L(m)$.
The game $\Gamma_n(\mu)$:

- stage 0:
  
  Nature chooses $L \sim \mu$
  
  $P_1$ is informed of $L$ not $P_2$.
  
  $P_1$ and $P_2$ know $\mu$. 

The game $\Gamma_n(\mu)$:

- stage 0:
- $n$ transaction periods before $D$. 

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The game $\Gamma_n(\mu)$:

- stage 0:
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- They use a general trading mechanism $\langle I, J, T \rangle$:
  $I, J =$P1’s and P2’s action spaces.
  $T : I \times J \rightarrow \mathbb{R}^2$.
  If choices=$(i, j)$, $T(i, j) = (A_{ij}, B_{ij})$ where $A_{ij}$ and $B_{ij}$ are the numbers of $R$ and $N$ shares that P2 gives to P1.
The game $\Gamma_n(\mu)$:

- stage 0:
- $n$ transaction periods before $D$.
- They use a general trading mechanism $\langle I, J, T \rangle$:
- At stage $q$: P1 and P2 chose simultaneously $(i_q, j_q)$.

$(i_q, j_q)$ is then publicly announced.

$y_q = (y_q^R, y_q^N), z_q = (z_q^R, z_q^N)$: P1’s and P2’s portfolios after $q$

$y_q = y_{q-1} + T(i_q, j_q)$ and $z_q = z_{q-1} - T(i_q, j_q)$
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- $n$ transaction periods before $D$.
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- $P1$ aims to maximize the liquidation value of his final portfolio.
  $y_0 = (0, 0)$. 
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The game $\Omega_n(\mu)$:

- At stage $q$: P1 and P2 chose simultaneously $(i_q, j_q)$. $(i_q, j_q)$ is then publicly announced.

  $y_q = (y_q^R, y_q^N), z_q = (z_q^R, z_q^N)$: P1’s and P2’s portfolios after $q$

  $y_q = y_{q-1} + T(i_q, j_q)$ and $z_q = z_{q-1} - T(i_q, j_q)$

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  A cautious P1 will play his max-min strategy
The game $\Gamma_n(\mu)$:

- At stage $q$: P1 and P2 chose simultaneously $(i_q, j_q)$. $(i_q, j_q)$ is then publicly announced.

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A cautious P1 will play his max-min strategy

=Equilibrium strategy in the 0-sum game where a risk neutral P2 aims to maximize the liquidation value of his final portfolio.

\[
z_0 = (0, 0)
\]
A strategy \( \sigma \) for P1 is \( \sigma = (\sigma_1, \ldots, \sigma_n) \) where
\[
\sigma_q : \mathbb{R} \times (I \times J)^{q-1} \rightarrow \Delta(I).
\]
A strategy $\sigma$ for P1 is $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_q : \mathbb{R} \times (I \times J)^{q-1} \rightarrow \Delta(I)$.

A strategy $\tau$ for P2 is $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_q : (I \times J)^{q-1} \rightarrow \Delta(J)$.
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$\tau_q : (I \times J)^{q-1} \rightarrow \Delta(J)$.

$(\mu, \sigma, \tau) \rightarrow$ probability $\pi(\mu, \sigma, \tau)$ on $\mathbb{R} \times (I \times J)^n$. 
Strategies in $\Gamma_n(\mu)$

- A strategy $\sigma$ for P1 is $\sigma = (\sigma_1, \ldots, \sigma_n)$ where
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- $(\mu, \sigma, \tau) \rightarrow$ probability $\pi_{(\mu,\sigma,\tau)}$ on $\mathbb{R} \times (I \times J)^n$

- P1’s payoff: $g_n(\mu, \sigma, \tau) := E_{\pi_{(\mu,\sigma,\tau)}}[y_n^R L + y_n^N]$. 
Strategies in $\Gamma_n(\mu)$

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- Zero sum game.
A strategy $\sigma$ for P1 is $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_q : \mathbb{R} \times (I \times J)^{q-1} \rightarrow \Delta(I)$.

A strategy $\tau$ for P2 is $\tau = (\tau_1, \ldots, \tau_n)$ where $\tau_q : (I \times J)^{q-1} \rightarrow \Delta(J)$.

$(\mu, \sigma, \tau) \rightarrow$ probability $\pi_{(\mu,\sigma,\tau)}$ on $\mathbb{R} \times (I \times J)^n$

P1’s payoff: $g_n(\mu, \sigma, \tau) \equiv E_{\pi_{(\mu,\sigma,\tau)}}[y_n^R L + y_n^N]$.

Zero sum game.

An equilibrium in $\Gamma_n(\mu)$ is a pair $(\sigma^*, \tau^*)$ s. th. $\forall \sigma, \tau : g_n(\mu, \sigma, \tau^*) \leq g_n(\mu, \sigma^*, \tau^*) \leq g_n(\mu, \sigma^*, \tau)$.
Value of $\Gamma_n(\mu)$.

If $\sup_\sigma \inf_\tau g_n(\mu, \sigma, \tau) = \inf_\tau \sup_\sigma g_n(\mu, \sigma, \tau) =: V_n(\mu)$, $V_n(\mu)$ is called the value of $\Gamma_n(\mu)$. 
Value of $\Gamma_n(\mu)$.

- If $\sup_{\sigma} \inf_{\tau} g_n(\mu, \sigma, \tau) = \inf_{\tau} \sup_{\sigma} g_n(\mu, \sigma, \tau) =: V_n(\mu)$, $V_n(\mu)$ is called the value of $\Gamma_n(\mu)$.
- If $\Gamma_n(\mu)$ has an equilibrium $(\sigma^*, \tau^*)$, the game has a value $V_n(\mu) = g_n(\mu, \sigma^*, \tau^*)$. 
Natural exchange mechanism
A trading mechanism \( \langle I, J, T \rangle \) is natural if

- Numéraire scale invariance
- Invariance with respect to the riskless part of the risky asset.
- Existence of the value
- Positive value of information.
- Continuity of the value
A trading mechanism $\langle I, J, T \rangle$ is natural if

- Numéraire scale invariance
  - If trading $R$ against $\$ \text{ or against the cent}$, 
    $\rightarrow$ same transactions in value.

- Invariance with respect to the riskless part of the risky asset.

- Existence of the value

- Positive value of information.

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A trading mechanism $⟨I, J, T⟩$ is natural if

- Numéraire scale invariance
  \[ \Rightarrow \forall \alpha > 0, \forall X : V_1([\alpha \cdot X]) = \alpha \cdot V_1([X]) \]

- Invariance with respect to the riskless part of the risky asset.

- Existence of the value

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- Continuity of the value
A trading mechanism \( \langle I, J, T \rangle \) is natural if

- Numéraire scale invariance
- Invariance with respect to the riskless part of the risky asset.
  - If trading \( R \) or \( R + $100 \) against $ \( \rightarrow \) same transactions in value.
- Existence of the value
- Positive value of information.
- Continuity of the value
A trading mechanism $\langle I, J, T \rangle$ is natural if

- Numéraire scale invariance
- Invariance with respect to the riskless part of the risky asset.
  - $\Rightarrow \forall \text{ constant } \beta, \forall X : V_1([X + \beta]) = V_1([X])$
- Existence of the value
- Positive value of information.
- Continuity of the value
A trading mechanism $\langle I, J, T \rangle$ is natural if

- Numéraire scale invariance
- Invariance with respect to the riskless part of the risky asset.
- Existence of the value
  - $\forall \mu \in \Delta^2, \Gamma_n(\mu)$ has an equilibrium.
- Positive value of information.
- Continuity of the value
A trading mechanism $\langle I, J, T \rangle$ is natural if

- Numéraire scale invariance
- Invariance with respect to the riskless part of the risky asset.
- Existence of the value
- Positive value of information.
  - $\exists \mu \in \Delta^2 : V_1(\mu) > 0$
- Continuity of the value
A trading mechanism \( \langle I, J, T \rangle \) is natural if

- Numéraire scale invariance
- Invariance with respect to the riskless part of the risky asset.
- Existence of the value
- Positive value of information.
- Continuity of the value

\[ \exists p \in [1, 2[, \exists A \text{ s. th. } \forall \text{ v.a. } X, Y : \quad |V_1([X]) - V_1([Y])| \leq A \|X - Y\|_{L^p} \]
• How can we define the price of $R$ at stage $q$?
The price at stage \( q \)

- How can we define the price of \( R \) at stage \( q \)?
- Possibility 1: \( \frac{-B_{i_q j_q}}{A_{i_q j_q}} \)
The price at stage $q$

• How can we define the price of $R$ at stage $q$?
• Possibility 1: $\frac{-B_{iqjq}}{A_{iqjq}}$
  $\rightarrow$ problem if $A_{iqjq} = 0$. 
The price at stage $q$

- How can we define the price of $R_t$ at stage $q$?
- Possibility 1: $\frac{-B_{iqjq}}{A_{iqjq}}$
  \[ \rightarrow \text{problem if } A_{iqjq} = 0. \]
- Possibility 2: price $= L^n_q := E[L|i_s, j_s; s \leq q]$. 
  It is the price at which P2 would agree to trade with another uninformed player.
Theorem 1:

- If the exchange mechanism is natural
- If, \( \forall n, (\sigma^n, \tau^n) \) is an equilibrium in \( \Gamma_n(\mu) \)
- If \( L^n_q := E_{\pi(\mu, \sigma^n, \tau^n)}[L | i_s, j_s; s \leq q] \) and \( \Pi^n_t := L^n_{[nt]} \)

Then \( \Pi^n \) converges in finite-dimensional law to the CMMV \( \Pi^\mu \).
Theorem 1:

- If the exchange mechanism is natural
- if, $\forall n$, $(\sigma^n, \tau^n)$ is an equilibrium in $\Gamma_n(\mu)$
- if $L^n_q := E_{\pi(\mu, \sigma^n, \tau^n)}[L| i_s, j_s; s \leq q]$ and $\Pi^n_t := L^n_{[nt]}$

Then $\Pi^n$ converges in finite-dimensional law to the CMMV $\Pi^\mu$.

The asymptotic price process is thus independent of the trading mechanism!
P1’s martingale optimization problem

\[ M_n(\mu) := \{ \} \text{ of martingales } X \text{ of length } n + 1 \text{ s. th. } [X_{n+1}] = \mu. \]
P1’s martingale optimization problem

- \( \mathcal{M}_n(\mu) := \{\} \) of martingales \( X \) of length \( n + 1 \) s. th. \( [X_{n+1}] = \mu \).

Example: \( L^n \in \mathcal{M}(\mu) \) where \( L^n \) is defined as

\[
L^n := E[L|i_s, j_s; s \leq q] \quad \text{and} \quad L^n_{n+1} := L.
\]
P1’s martingale optimization problem

• $\mathcal{M}_n(\mu) := \{\} \text{ of martingales } X \text{ of length } n+1 \text{ s. th. } [X_{n+1}] = \mu.$

  Example: $L^n \in \mathcal{M}(\mu)$ where $L^n$ is defined as $L^n_q := E[L|i_s, j_s; s \leq q]$ and $L^n_{n+1} := L.$

• P1 controls $L^n_q.$
  → He can choose any martingale $L^n \in \mathcal{M}_n(\mu).$
P1’s martingale optimization problem

- Let $X$ be in $\mathcal{M}_n(\mu)$. 
P1’s martingale optimization problem

• Let $X$ be in $\mathcal{M}_n(\mu)$.
  • stage 0: Nature uses $X$ to select $L := X_{n+1}$
P1’s martingale optimization problem

- Let $X$ be in $\mathcal{M}_n(\mu)$.
  - stage 0: Nature uses $X$ to select $L := X_{n+1}$
  - $P1$ observes $X$
P1’s martingale optimization problem

• Let $X$ be in $\mathcal{M}_n(\mu)$.
  • stage 0: Nature uses $X$ to select $L := X_{n+1}$
  • P1 observes $X$
  • At stage $q$, P1 plays $i_q(X_s, s \leq q)$.  

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P1’s martingale optimization problem

• Let $X$ be in $\mathcal{M}_n(\mu)$.
  • stage 0: Nature uses $X$ to select $L := X_{n+1}$
  • P1 observes $X$
  • At stage $q$, P1 plays $i_q(X_s, s \leq q)$.
  • More information to P2: after stage $q$, P2 is informed of $X_q$.  


Let $X$ be in $\mathcal{M}_n(\mu)$.

- stage 0: Nature uses $X$ to select $L := X_{n+1}$
- $P1$ observes $X$
- At stage $q$, $P1$ plays $i_q(X_s, s \leq q)$.
- More information to $P2$: after stage $q$, $P2$ is informed of $X_q$.
- $i_q(.)$ is then chosen to maximize stage $q$ payoff.

No revelation problem anymore
P1’s martingale optimization problem

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The game at stage $q$ is then $\Gamma_1([X_q|X_s, s < q])$

where $[X_q|X_s, s < q] = \text{law of } X_q \text{ conditional to } X_s, s < q$. 
P1’s martingale optimization problem

- Let $X$ be in $\mathcal{M}_n(\mu)$.
  - P1 observes $X$
  - At stage $q$, P1 plays $i_q(X_s, s \leq q)$.
  - More information to P2: after stage $q$, P2 is informed of $X_q$.
  - $i_q(.)$ is then chosen to maximize stage $q$ payoff.
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  - The game at stage $q$ is then $\Gamma_1([X_q|X_s, s < q])$
    where $[X_q|X_s, s < q] = \text{law of } X_q \text{ conditional to } X_s, s < q$.
  - If $i_q(.)$ is optimal in $\Gamma_1([X_q|X_s, s < q])$, P1 gets at least $V_n(X) := \sum_{q=1}^{n} E[V_1([X_q|X_s, s < q])]$
P1’s martingale optimization problem

• Let $X$ be in $\mathcal{M}_n(\mu)$.

• More information to P2: after stage $q$, P2 is informed of $X_q$.

• $i_q(.)$ is then chosen to maximize stage $q$ payoff.

  No revelation problem anymore

• The game at stage $q$ is then $\Gamma_1([X_q|X_s, s < q])$

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• If $i_q(.)$ is optimal in $\Gamma_1([X_q|X_s, s < q])$, P1 gets at least

  $\mathcal{V}_n(X) := \sum_{q=1}^{n} E[V_1([X_q|X_s, s < q])]$

• $\overline{\mathcal{V}}_n(\mu) := \sup\{\mathcal{V}_n(X) : X \in \mathcal{M}_n(\mu)\}$

  $\Rightarrow V_n(\mu) \geq \overline{\mathcal{V}}_n(\mu)$
**P1’s martingale optimization problem**

- Proposition: $V_n(\mu) = \overline{V}_n(\mu)$.  
  If $(\sigma^n, \tau^n)$ is an equilibrium in $\Gamma_n(\mu)$ if $L^n_q := E_{\pi(\mu, \sigma^n, \tau^n)}[L|i_s, j_s; s \leq q]$ and $L^n_{n+1} := L$ then $L^n$ is optimal in the problem $\overline{V}_n(\mu)$.

- $\mathcal{V}_n(X) := \sum_{q=1}^{n} E[V_1([X_q|X_s, s < q])]$

- $\overline{V}_n(\mu) := \sup\{\mathcal{V}_n(X) : X \in \mathcal{M}_n(\mu)\}$. 
P1’s martingale optimization problem

**Proposition:** \( V_n(\mu) = \overline{V}_n(\mu) \).

If \((\sigma^n, \tau^n)\) is an equilibrium in \( \Gamma_n(\mu) \)

\[
L^n_q := E_{\pi(\mu, \sigma^n, \tau^n)}[L|i_s, j_s; s \leq q]
\]

and \( L^n_{n+1} := L \)

then \( L^n \) is optimal in the problem \( \overline{V}_n(\mu) \).

**\( \mathcal{V}_n(X) := \sum_{q=1}^{n} E[V_1([X_q|X_s, s < q])] \)**

**\( \overline{V}_n(\mu) := \sup\{\mathcal{V}_n(X) : X \in \mathcal{M}_n(\mu)\} \).**

**Invariance with respect to the riskless part of \( R \)

\( \Rightarrow \) \( \forall \) constant \( \beta \) : \( V_1([X + \beta]) = V_1([X]) \),
**P1’s martingale optimization problem**

- **Proposition:** \( V_n(\mu) = \overline{V}_n(\mu) \).
  
  If \((\sigma^n, \tau^n)\) is an equilibrium in \( \Gamma_n(\mu) \)
  
  if \( L^n_q := E_{\pi(\mu, \sigma^n, \tau^n)} [L|i_s, j_s; s \leq q] \) and \( L^n_{n+1} := L \)
  
  then \( L^n \) is optimal in the problem \( \overline{V}_n(\mu) \).

- \( \mathcal{V}_n(X) := \sum_{q=1}^{n} E[V_1([X_q|X_s, s < q])] \)

- \( \overline{V}_n(\mu) := \sup\{ \mathcal{V}_n(X) : X \in \mathcal{M}_n(\mu) \} \).

- **Invariance with respect to the riskless part of \( R \)**
  
  \( \Rightarrow \) \( \forall \) constant \( \beta : V_1([X + \beta]) = V_1([X]) \),

  \( \Rightarrow \)

  \( V_1[X_q|X_s, s < q] = V_1[X_q - X_{q-1}|X_s, s < q] \)
Let $M : \Delta^2 \rightarrow \mathbb{R}$ and $X \in \mathcal{M}_n(\mu)$,

The $M$-variation of $X$ is defined as:

$$\mathcal{V}^M_n(X) := \sum_{q=1}^n E[M([X_q - X_{q-1}|X_s, s < q])]$$
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$$\mathcal{V}_n(X) = \mathcal{V}^{V_1}_n(X)$$
Let $M : \Delta^2 \to \mathbb{R}$ and $X \in \mathcal{M}_n(\mu)$, 

The $M$-variation of $X$ is defined as:

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- $V_n(X) = V_{n1}^V(X)$
- $V_n^M(\mu) := \sup\{V_n^M(X) : X \in \mathcal{M}_n(\mu)\}$
**$M$-variation**

- Let $M : \Delta^2 \to \mathbb{R}$ and $X \in \mathcal{M}_n(\mu)$,

  The $M$-variation of $X$ is defined as:

  $$V_n^M(X) := \sum_{q=1}^n E[M(X_q - X_{q-1}|X_s, s < q)]$$

- $V_n(X) = V_n^{V_1}(X)$

- $\overline{V}_n^M(\mu) := \sup\{V_n^M(X) : X \in \mathcal{M}_n(\mu)\}$

- **Mass Transportation Problem**:

  $$\gamma(\mu) := \sup\{E[ZL] : Z \sim \mathcal{N}(0, 1); L \sim \mu\}$$
Let $M : \Delta^2 \to \mathbb{R}$ and $X \in \mathcal{M}_n(\mu)$, 

The $M$-variation of $X$ is defined as:

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- **Mass Transporation Problem:**

$$\gamma(\mu) := \sup\{E[ZL] : Z \sim \mathcal{N}(0, 1); L \sim \mu\}$$

$\rightarrow L = increasing\ function\ in\ Z$.
Let $M : \Delta^2 \to \mathbb{R}$ and $X \in \mathcal{M}_n(\mu)$, the $M$-variation of $X$ is defined as:

\[ V_n^M(X) := \sum_{q=1}^{n} E[M([X_q - X_{q-1}|X_s, s < q])] \]

- $V_n(X) = V_{n1}^v(X)$

- $\overline{V}_n^M(\mu) := \sup\{V_n^M(X) : X \in \mathcal{M}_n(\mu)\}$

**Mass Transporation Problem:**

$\gamma(\mu) := \sup\{E[ZL] : Z \sim \mathcal{N}(0, 1); L \sim \mu\}$

$\Rightarrow L = increasing\ function\ in\ Z$.

$\Rightarrow L = f_\mu(Z)\ and\ \gamma(\mu) = E[Zf_\mu(Z)]$
Theorem 2:

If $\forall \alpha > 0, \forall Y : M([\alpha Y]) = \alpha M([Y])$, if $\exists p \in [1, 2[, A \in \mathbb{R} : \forall X, Y : |(M([X]) - M([Y])| \leq A \|X - Y\|_{L^p}$

Then $\frac{\sum_{n}^{M(\mu)}}{\sqrt{n}} \xrightarrow{n \to \infty} \rho \gamma(\mu)$.

where $\rho := \sup\{M([X]) : \|X\|_{L^2} \leq 1\}$
Theorem 2:

- If \( \forall \alpha > 0, \forall Y : M([\alpha Y]) = \alpha M([Y]) \), if \( \exists \rho \in [1, 2[, A \in \mathbb{R} : \forall X, Y : |(M([X]) - M([Y])| \leq A\|X - Y\|_{L^p} \)

Then \( \frac{\nabla^M_n (\mu)}{\sqrt{n}} \xrightarrow{n \to \infty} \rho \gamma(\mu) \).

where \( \rho := \sup \{ M([X]) : \|X\|_{L^2} \leq 1 \} \)

- If \( \rho > 0 \) and if, \( \forall n, L^n \in \mathcal{M}_n(\mu) \) satisfies

\( \nabla^M_n (L^n) = \nabla^M_n (\mu) \)

Then \( \Pi^n_t := L^n_{\lfloor nt \rfloor} \) converges in finite-dimensional laws to the CMMV \( \Pi^\mu_t \)
• The limit $\Pi^\mu$ does not depend on $M$
Remarks

• The limit $\Pi^\mu$ does not depend on $M$

• This result explains the terminology CMMV
Remarks

• The limit $\Pi^\mu$ does not depend on $M$
• This result explains the terminology CMMV
• The ingredients of the proof are
  • duality
  • CLT
  • martingale embedding techniques.
How robust is this class of dynamics?

• It is independent of $T$
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How robust is this class of dynamics?

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  ∃ an equilibrium in the risk averse game, where, up to a change of probability, P1’s strategy is optimal in a risk neutral game.
How robust is this class of dynamics?

- It is independent of $T$
- If one adds derivatives?
- If $P_2$ is risk averse? $\rightarrow$ Non zero-sum
  $\exists$ an equilibrium in the risk averse game, where, up to a change of probability, $P_1$’s strategy is optimal in a risk neutral game.
  $\rightarrow$ CMMV under a change of probability
How do CMMV fit the real data?

• Conjecture: Under the risk neutral probability, the actualized price process is locally a CMMV
How do CMMV fit the real data?

• Conjecture: *Under the risk neutral probability, the actualized price process is locally a CMMV*

• *This can be tested on real data.*
How do CMMV fit the real data?

• Conjecture: Under the risk neutral probability, the actualized price process is locally a CMMV.

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  Black and Scholes model is a CMMV.
How do CMMV fit the real data?

- Conjecture: *Under the risk neutral probability, the actualized price process is locally a CMMV*

- This can be tested on real data.
  
  *Black and Scholes model is a CMMV.*

- *Is CMMV a good model when there is a volatility smile?*
CMMV with a volatility smile

- $S_t = \text{Forward price of an underlying asset at time } t.$
CMMV with a volatility smile

- \( S_t = \text{Forward price of an underlying asset at time } t. \)
- \( C_{K,t} = \text{forward price of a european Call with strike } K, \text{ maturity 1.} \)
CMMV with a volatility smile

• $S_t = \text{Forward price of an underlying asset at time } t.$

• $C_{K,t} = \text{forward price of a european Call with strike } K, \text{ maturity 1.} \quad C_{K,0} = E[(S_1 - K)^+]$
CMMV with a volatility smile

- $S_t = \text{Forward price of an underlying asset at time } t.$
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- From date 0 observations, we can recover:
  - the law $\mu$ of $S_1$
CMMV with a volatility smile

- $S_t = \text{Forward price of an underlying asset at time } t.$
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  $C_{K,0} = E[(S_1 - K)^+]$

- From date 0 observations, we can recover:
  - the law $\mu$ of $S_1$
  - $f_\mu$
CMMV with a volatility smile

- $S_t =$ Forward price of an underlying asset at time $t$.
- $C_{K,t} =$ forward price of a european Call with strike $K$, maturity 1.
  
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- From date 0 observations, we can recover:
  - the law $\mu$ of $S_1$
  - $f_\mu$
  - $f(x, t)$ s.th. $S_t = f(B_t, t)$. 

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CMMV with a volatility smile

- $S_t =$ *Forward price of an underlying asset at time* $t$.
- $C_{K,t} =$ *forward price of a european Call with strike* $K$, *maturity* 1.
  
  $C_{K,0} = E[(S_1 - K)^+]$

- *From date 0 observations, we can recover:*
  - *the law* $\mu$ *of* $S_1$
  - $f_{\mu}$
  - $f(x, t)$ *s.th.* $S_t = f(B_t, t)$.

- *At time* $t$:
  - *we observe* $S_t = f(B_t, t)$. 
CMMV with a volatility smile

- $C_{K,t} =$ forward price of a european Call with strike $K$, maturity 1.
  
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  - we get $B_t$
CMMV with a volatility smile

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- At time $t$:
  - we observe $S_t = f(B_t, t)$.
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  - $C_{K,t} = E[(S_1 - K)^+ | B_{s,s<t}]$
CMMV with a volatility smile

• From date 0 observations, we can recover:
  • the law $\mu$ of $S_1$
  • $f_\mu$
  • $f(x, t)$ s.th. $S_t = f(B_t, t)$.

• At time $t$:
  • we observe $S_t = f(B_t, t)$.
  • we get $B_t$
  • $C_{K,t} = E[(S_1 - K)^+|B_{s,s<t}] = E[(f_\mu(B_1) - K)^+|B_t]$
From date 0 observations, we can recover:

- the law \( \mu \) of \( S_1 \)
- \( f_\mu \)
- \( f(x, t) \) s.th. \( S_t = f(B_t, t) \).

At time \( t \):

- we observe \( S_t = f(B_t, t) \).
- we get \( B_t \)
- \( C_{K,t} = E[(S_1 - K)^+ | B_{s,s<t}] = E[(f_\mu(B_1) - K)^+ | B_t] \)
- If the CMMV model is right, we should have \( C_{K,t} = g_{t,S_t}(K) \).
European Call on CAC40
Date 0: 35 days to maturity
European Call on CAC40
Date 20: 15 days to maturity

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European Call on CAC40

Date 20: 15 days to maturity
Proof of theorem 2

\[ \limsup \frac{\gamma_n^M(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu). \]
Proof of theorem 2

\[ \limsup \frac{\mathbb{V}_n^M(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu). \]

- \[ B^q_r := \{ X : \|X\|_{L^q} \leq r \} \]
- \[ B^* := B^2_\rho \cap B^{p'}_{2A} \text{ where } \frac{1}{p} + \frac{1}{p'} = 1 \]
- \[ B([X]) := \sup \{ E[XY] : Y \in B^* \} \]
Proof of theorem 2

• \( \lim \sup \frac{\sqrt[n]{v_n^M(\mu)}}{\sqrt{n}} \leq \rho \gamma(\mu). \)

• \( B^q_r := \{ X : \|X\|_{L^q} \leq r \} \)

• \( B^* := B^2_\rho \cap B^{p'}_{2A} \) where \( 1/p + 1/p' = 1 \)

• \( B([X]) := \sup \{ E[XY] : Y \in B^* \} \)

• **Duality lemma:** For all \( X : M([X]) \leq B([X]) \)
Proof of theorem 2

• \( \lim \sup \frac{\overline{\nu}_n^M(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu) \).

• \( B^q_r := \{ X : \|X\|_{L^q} \leq r \} \)

• \( B^* := B^2_\rho \cap B^{p'}_{2A} \) where \( \frac{1}{p} + \frac{1}{p'} = 1 \)

• \( B([X]) := \sup \{ E[XY] : Y \in B^* \} \)

• **Duality lemma:** For all \( X : M([X]) \leq B([X]) \)

• If \( L \in \mathcal{M}_n(\mu) : \nu_n^M(L) \leq \nu_n^B(L) \).
Proof of theorem 2

- \( \limsup \frac{\sqrt[n]{V_n^M(\mu)}}{\sqrt{n}} \leq \rho\gamma(\mu) \).

- \( B_r^q := \{ X : \|X\|_{L^q} \leq r \} \)

- \( B^* := B^2_\rho \cap B^{p'}_{2A} \) where \( 1/p + 1/p' = 1 \)

- \( B([X]) := \sup\{ E[XY] : Y \in B^* \} \)

- **Duality lemma:** For all \( X : M([X]) \leq B([X]) \)

- If \( L \in M_n(\mu) : V_n^M(L) \leq V_n^B(L) \).

- \( \overline{V}_n^M(\mu) \leq \overline{V}_n^B(\mu) \)
Proof of the duality lemma

• Both $M$ and $B$ are 1-homogeneous

$B := \{X : B([X]) \leq 1\} \subset M := \{X : M([X]) \leq 1\}$
Both $M$ and $B$ are 1-homogeneous

$B := \{ X : B([X]) \leq 1 \} \subset M := \{ X : M([X]) \leq 1 \}$?

$B([X + X']) \leq B([X]) + B([X'])$
Proof of the duality lemma

\[ \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \} \]

\[ B([X + X']) \leq B([X]) + B([X']) \]

So \( \mathcal{B} \) is convex.
Proof of the duality lemma

• \( \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \}? \)

• \( \mathcal{B} \) is convex.

• \( \mathcal{B}^* := B^2_{\rho} \cap B^p_A \subset B^2_{\rho} \)

\[
B([X]) := \sup \{ E[XY] : Y \in \mathcal{B}^* \} \\
\leq \sup \{ E[XY] : Y \in B^2_{\rho} \} = \rho \|X\|_{L^2}
\]
Proof of the duality lemma

1. $\mathcal{B} := \{X : B([X]) \leq 1\} \subset \mathcal{M} := \{X : M([X]) \leq 1\}$?
2. $\mathcal{B}$ is convex.
3. $B^* := B_\rho^2 \cap B_A^p \subset B_A^p$

$$B([X]) \leq \rho \|X\|_{L^2} \text{ and } B([X]) \leq A \|X\|_{L^p}$$
Proof of the duality lemma

- \( \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \} \)?
- \( \mathcal{B} \) is convex.
- \( B([X]) \leq \rho \| X \|_{L^2} \) and \( B([X]) \leq A \| X \|_{L^p} \)
- \( \big( \frac{B^2}{\rho} \cup \frac{B^p}{A} \big) \subset \mathcal{B} \)
Proof of the duality lemma

- $\mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \}$?

- $\mathcal{B}$ is convex.

- $B([X]) \leq \rho \| X \|_{L^2} \quad \text{and} \quad B([X]) \leq A \| X \|_{L^p}$

- $\mathcal{C} := \text{vex}(B_{\frac{1}{\rho}}^2 \cup B_{\frac{1}{A}}^p) \subset \mathcal{B}$
Proof of the duality lemma

\[ \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \} \]

\[ B([X]) \leq \rho \|X\|_{L^2} \quad \text{and} \quad B([X]) \leq A \|X\|_{L^p} \]

\[ \mathcal{C} := \text{vex}(\mathcal{B}^2_{\frac{2}{\rho}} \cup \mathcal{B}^p_{\frac{1}{A}}) \subset \mathcal{B} \]

If \( X \not\in \mathcal{C} : \exists Y : E[XY] > \alpha := \sup\{ E[YZ] : Z \in \mathcal{C} \} \).
Proof of the duality lemma

- \( \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \} \)?

- \( B([X]) \leq \rho \| X \|_{L^2} \) and \( B([X]) \leq A \| X \|_{L^p} \)

- \( \mathcal{C} := \text{vex}(B_{\frac{1}{\rho}}^{2} \cup B_{\frac{1}{A}}^{p}) \subset \mathcal{B} \)

- If \( X \notin \mathcal{C} : \exists Y : E[XY] > \alpha := \sup\{ E[YZ] : Z \in \mathcal{C} \} \).

So \( \alpha \geq \frac{1}{\rho} \| Y \|_{L^2} \)
Proof of the duality lemma

- $\mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \}$?
- $B([X]) \leq \rho \|X\|_{L^2}$ and $B([X]) \leq A \|X\|_{L^p}$
- $\mathcal{C} := \text{vex}(B_{\frac{1}{\rho}}^2 \cup B_{\frac{1}{A}}^p) \subset \mathcal{B}$
- If $X \notin \mathcal{C}$ : $\exists Y : E[XY] > \alpha := \sup \{ E[YZ] : Z \in \mathcal{C} \}$.
  So $\alpha \geq \frac{1}{\rho} \|Y\|_{L^2}$ and $\alpha \geq \frac{1}{A} \|Y\|_{L^{p'}}$
Proof of the duality lemma

- \( \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \} \)

- \( B([X]) \leq \rho \|X\|_{L^2} \) and \( B([X]) \leq A \|X\|_{L^p} \)

- \( \mathcal{C} := \text{vex}(B_{\frac{1}{\rho}}^2 \cup B_{\frac{1}{A}}^p) \subset \mathcal{B} \)

- If \( X \notin \mathcal{C} : \exists Y : E[XY] > \alpha := \sup \{ E[YZ] : Z \in \mathcal{C} \} \).
  
  So \( \alpha \geq \frac{1}{\rho} \|Y\|_{L^2} \) and \( \alpha \geq \frac{1}{A} \|Y\|_{L^p'} \)

Thus \( \frac{Y}{\alpha} \in \mathcal{B}^* := B_{\rho}^2 \cap B_{A}^{p'} \)
Proof of the duality lemma

- \( \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \} \)?
- \( B([X]) \leq \rho \|X\|_{L^2} \) and \( B([X]) \leq A \|X\|_{L^p} \)
- \( \mathcal{C} := \text{vex}(B^2_{1/\rho} \cup B^p_{1/A}) \subset \mathcal{B} \)
- If \( X \not\in \mathcal{C} : \exists Y : E[XY] > \alpha := \sup \{ E[YZ] : Z \in \mathcal{C} \} \).
  So \( \alpha \geq \frac{1}{\rho} \|Y\|_{L^2} \) and \( \alpha \geq \frac{1}{A} \|Y\|_{L^p'} \)
  Thus \( \frac{Y}{\alpha} \in B^* := B^2_{\rho} \cap B^p_{A} \)
  \( B([X]) \geq E\left[X \frac{Y}{\alpha}\right] > 1 \). Therefore: \( X \not\in \mathcal{B} \):
Proof of the duality lemma

\[ \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \}? \]

\[ B([X]) \leq \rho \| X \|_{L^2} \quad \text{and} \quad B([X]) \leq A \| X \|_{L^p} \]

\[ \mathcal{C} := \text{vex}( B_2^2 \cup B_1^p ) \subset \mathcal{B} \]

If \( X \not\in \mathcal{C} : \exists Y : E[XY] > \alpha := \sup \{ E[YZ] : Z \in \mathcal{C} \}. \]

So \( \alpha \geq \frac{1}{\rho} \| Y \|_{L^2} \) and \( \alpha \geq \frac{1}{A} \| Y \|_{L^p} \)

Thus \( \frac{Y}{\alpha} \in \mathcal{B}^* := B_\rho^2 \cap B_A^p \)

\[ B([X]) \geq E[X \frac{Y}{\alpha}] > 1. \quad \text{Therefore:} \quad X \not\in \mathcal{B}: \quad \mathcal{C} = \mathcal{B}. \]
Proof of the duality lemma

• \( \mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \} \)?

• \( B([X]) \leq \rho \|X\|_{L^2} \) and \( B([X]) \leq A \|X\|_{L^p} \)

• \( \mathcal{C} := \text{vex}(B^2_{\frac{1}{\rho}} \cup B^p_{\frac{1}{A}}) = \mathcal{B} \)

• \( \rho := \sup\{ M([X]) : X \in B^2_1 \}. \) So \( M([X]) \leq \rho \|X\|_{L^2} \).
Proof of the duality lemma

• \( B := \{ X : B([X]) \leq 1 \} \subset M := \{ X : M([X]) \leq 1 \} \)?

• \( B([X]) \leq \rho \| X \|_{L^2} \) and \( B([X]) \leq A \| X \|_{L^p} \)

• \( C := \text{vex}(B^2_1 \cup B^p_1) = B \)

• \( \rho := \sup \{ M([X]) : X \in B^2_1 \} \). So \( M([X]) \leq \rho \| X \|_{L^2} \).

• If \( X \in B \): \( \exists X' \in B^2_1, \exists X'' \in B^p_1, \exists \lambda', \lambda'' \geq 0 : X = \lambda' X' + \lambda'' X'' \) and \( 1 = \lambda' + \lambda'' \)
Proof of the duality lemma

\( B := \{ X : B([X]) \leq 1 \} \subset M := \{ X : M([X]) \leq 1 \}? \)

\( B([X]) \leq \rho \| X \|_{L^2} \) and \( B([X]) \leq A \| X \|_{L^p} \)

\( C := \text{vex}(B^2_\frac{1}{\rho} \cup B^p_\frac{1}{A}) = B \)

\( \rho := \sup\{ M([X]) : X \in B^2_1 \}. \) So \( M([X]) \leq \rho \| X \|_{L^2}. \)

If \( X \in B: \exists X' \in B^2_\frac{1}{\rho}, \exists X'' \in B^p_\frac{1}{A}, \exists \lambda', \lambda'' \geq 0 : X = \lambda' X' + \lambda'' X'' \text{ and } 1 = \lambda' + \lambda'' \)

\( M([X]) \leq M([\lambda' X']) + A \| \lambda'' X'' \|_{L^p} \)
Proof of the duality lemma

• $\mathcal{B} := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \}$?

• $B([X]) \leq \rho \|X\|_{L^2}$ and $B([X]) \leq A \|X\|_{L^p}$

• $\mathcal{C} := vex\left( B_1^{2 \rho} \cup B_1^{pA} \right) = \mathcal{B}$

• $\rho := \sup\{ M([X]) : X \in B_1^{2} \}$. So $M([X]) \leq \rho \|X\|_{L^2}$.

• If $X \in \mathcal{B}$: $\exists X' \in B_1^{2 \rho}, \exists X'' \in B_1^{pA}, \exists \lambda', \lambda'' \geq 0$ :

  $X = \lambda'X' + \lambda''X''$ and $1 = \lambda' + \lambda''$

• $M([X]) \leq M([\lambda'X']) + A\|\lambda''X''\|_{L^p}$

  $\leq \rho \lambda'\|X'\|_{L^2} + A\lambda''\|X''\|_{L^p}$
Proof of the duality lemma

Let $B := \{ X : B([X]) \leq 1 \} \subset \mathcal{M} := \{ X : M([X]) \leq 1 \}$.

- $B([X]) \leq \rho \|X\|_{L^2}$ and $B([X]) \leq A \|X\|_{L^p}$

Let $C := \text{vex}(B_{\frac{1}{\rho}}^2 \cup B_{\frac{1}{A}}^p) = B$

$\rho := \sup \{ M([X]) : X \in B_{\frac{1}{\rho}}^2 \}$. So $M([X]) \leq \rho \|X\|_{L^2}$.

If $X \in B$: $\exists X' \in B_{\frac{1}{\rho}}^2, \exists X'' \in B_{\frac{1}{A}}^p, \exists \lambda', \lambda'' \geq 0$:

$X = \lambda' X' + \lambda'' X''$ and $1 = \lambda' + \lambda''$

- $M([X]) \leq M([\lambda' X']) + A \|\lambda'' X''\|_{L^p}$

$\leq \rho \lambda' \|X'\|_{L^2} + A \lambda'' \|X''\|_{L^p'}$

$\leq \lambda' + \lambda'' = 1$
Proof of theorem 2

\[ \limsup \frac{\nabla_n^B(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu). \]
Proof of theorem 2

- \( \limsup \frac{\sqrt{B_n}(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu) \).

- \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

where \( Y_q \) satisfies:

- \( Y_q: \sigma(L_s, s \leq q) \)-measurable.

- \( E[Y_q^2|L_s, s < q] \leq \rho^2 \)

- \( E[|Y_q|^{p'}|L_s, s < q] \leq (2A)^{p'} \)
Proof of theorem 2

• \( \limsup \frac{V_n^B(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu). \)

• \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

where \( Y_q \) satisfies:

• \( Y_q : \sigma(L_s, s \leq q) \)-measurable.

• \( E[Y_q^2|L_s, s < q] \leq \rho^2 \)

• \( E[|Y_q|^{p'}|L_s, s < q] \leq (2A)^{p'} \)

• If \( Y_q' := Y_q - E[Y_q|L_s, s < q] \), then

\[
E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y_q'] = E[L_qY_q']
\]
Proof of theorem 2

- \( \limsup \frac{V_n^{B}(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu) \).
- \( E\left[B\left([L_q - L_{q-1}\mid L_s, s \leq q]\right)\right] = \sup_{Y_q} E\left[(L_q - L_{q-1})Y_q\right] \)
  where \( Y_q \) satisfies:
    - \( Y_q: \sigma(L_s, s \leq q)\)-measurable.
    - \( E[Y_q^2 \mid L_s, s < q] \leq \rho^2 \)
    - \( E[|Y_q|^{p'} \mid L_s, s < q] \leq (2A)^{p'} \)

- If \( Y_q' := Y_q - E[Y_q \mid L_s, s < q] \), then
  \( E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y_q'] = E[L_qY_q'] \)
  - \( Y_q': \sigma(L_s, s \leq q)\)-measurable and \( E[Y_q' \mid L_s, s < q] = 0 \).
Proof of theorem 2

1. \( \lim \sup \frac{\sup_{n} V_{n}(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu) \).

2. 
\[
E[B([L_{q} - L_{q-1}|L_{s}, s \leq q])] = \sup_{Y_{q}} E[(L_{q} - L_{q-1})Y_{q}]
\]
where \( Y_{q} \) satisfies:

   a. \( E[Y_{q}^{2}|L_{s}, s < q] \leq \rho^{2} \)

   b. \( E[|Y_{q}|^{p'}|L_{s}, s < q] \leq (2A)^{p'} \)

3. If \( Y'_{q} := Y_{q} - E[Y_{q}|L_{s}, s < q] \), then
\[
E[(L_{q} - L_{q-1})Y_{q}] = E[(L_{q} - L_{q-1})Y'_{q}] = E[L_{q}Y'_{q}]
\]

   a. \( Y'_{q}: \sigma(L_{s}, s \leq q)\)-measurable and \( E[Y'_{q}|L_{s}, s < q] = 0 \).

   b. \( E[Y'^{2}_{q}|L_{s}, s < q] \leq \rho^{2} \)

Proof of theorem 2

• \( \limsup \sup_n \frac{\text{Var}^B_n(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu) \).

• \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

where \( Y_q \) satisfies:

• \( E[|Y_q|^{p'}|L_s, s < q] \leq (2A)^{p'} \)

• If \( Y'_q := Y_q - E[Y_q|L_s, s < q] \), then
  \( E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y'_q] = E[L_qY'_q] \)

• \( Y'_q: \sigma(L_s, s \leq q)\)-measurable and \( E[Y'_q|L_s, s < q] = 0 \).

• \( E[Y'^2_q|L_s, s < q] \leq \rho^2 \)

• \( E[|Y'_q|^{p'}|L_s, s < q] \leq (4A)^{p'} \)
Proof of theorem 2

• \( \lim \sup \frac{\sqrt[n]{V_n^B(\mu)}}{\sqrt{n}} \leq \rho \gamma(\mu) \).

• \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

• If \( Y'_q := Y_q - E[Y_q|L_s, s < q] \), then
  \( E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y'_q] = E[L_qY'_q] \)

  • \( Y'_q: \sigma(L_s, s \leq q)\)-measurable and \( E[Y'_q|L_s, s < q] = 0 \).

  • \( E[Y'^2_q|L_s, s < q] \leq \rho^2 \)

  • \( E[|Y'_q|^{p'}|L_s, s < q] \leq (4A)^{p'} \)

• If \( Z_q \sim \mathcal{N}(0, 1) \perp \perp L_s, s \leq n + 1 \), if \( \mathcal{H}_q := \sigma(L_s, Z_s, s \leq q) \), if \( Y''_q := Y'_q + \sqrt{\rho^2 - E[Y'^2_q|L_s, s < q]}Z_q \)
Proof of theorem 2

• \( \limsup \frac{\sqrt{V_n^B(\mu)}}{\sqrt{n}} \leq \rho \gamma(\mu). \)

• \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

• If \( Y'_q := Y_q - E[Y_q|L_s, s < q] \), then
  \[ E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y'_q] = E[L_qY'_q] \]

  • \( Y'_q: \sigma(L_s, s \leq q)\)-measurable and \( E[Y'_q|L_s, s < q] = 0 \).

  • \( E[Y'_q^2|L_s, s < q] \leq \rho^2 \)

  • \( E[|Y'_q|^{p'}|L_s, s < q] \leq (4A)^{p'} \)

• If \( Z_q \sim \mathcal{N}(0, 1) \perp \perp L_s, s \leq n + 1 \), if \( \mathcal{H}_q := \sigma(L_s, Z_s, s \leq q) \),
  if \( Y''_q := Y'_q + \sqrt{\rho^2 - E[Y_q^2|L_s, s < q]}Z_q \) then

  • \( Y''_q \) is \( \mathcal{H}_q \)-measurable and \( E[Y''_q|\mathcal{H}_{q-1}] = 0 \).
Proof of theorem 2

• \( \lim \sup \frac{\sqrt{V_n(B_n(\mu))}}{\sqrt{n}} \leq \rho \gamma(\mu) \).

• \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

• If \( Y'_q := Y_q - E[Y_q|L_s, s < q] \), then
  \( E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y'_q] = E[L_qY'_q] \)
    • \( E[Y'_q^2|L_s, s < q] \leq \rho^2 \)
    • \( E[|Y'_q|^p'|L_s, s < q] \leq (4A)^{p'} \)

• If \( Z_q \sim N(0, 1) \perp L_s, s \leq n + 1 \), if \( \mathcal{H}_q := \sigma(L_s, Z_s, s \leq q) \),
  if \( Y''_q := Y'_q + \sqrt{\rho^2 - E[Y'_q^2|L_s, s < q]}Z_q \) then
    • \( Y''_q \) is \( \mathcal{H}_q \)-measurable and \( E[Y''_q|\mathcal{H}_{q-1}] = 0 \).
    • \( E[Y''_q^2|\mathcal{H}_{q-1}] = \rho^2 \)
Proof of theorem 2

- \( \lim \sup \frac{\sqrt{\nu_n^B(\mu)}}{\sqrt{n}} \leq \rho \gamma(\mu). \)

- \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

- If \( Y_q' := Y_q - E[Y_q|L_s, s < q] \), then
  \( E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y_q'] = E[L_qY_q'] \)
  \( E[|Y_q'|^{p'}|L_s, s < q] \leq (4A)^{p'} \)

- If \( Z_q \sim \mathcal{N}(0, 1) \perp L_s, s \leq n + 1 \), if \( \mathcal{H}_q := \sigma(L_s, Z_s, s \leq q) \), if \( Y_q'' := Y_q' + \sqrt{\rho^2 - E[Y_q'^2|L_s, s < q]}Z_q \) then
  \( Y_q'' \) is \( \mathcal{H}_q \)-measurable and \( E[Y_q''|\mathcal{H}_{q-1}] = 0. \)
  \( E[Y_q''^2|\mathcal{H}_{q-1}] = \rho^2 \)
  \( E[|Y_q''|^{p'}|\mathcal{H}_{q-1}] \leq (4A + C)^{p'} =: K^{p'} \)
Proof of theorem 2

- \( \lim \sup \frac{V_n^B(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu). \)
- \( E\left[B\left([L_q - L_{q-1}|L_s, s \leq q]\right)\right] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)
- If \( Y''_q := Y_q - E[Y_q|L_s, s < q] \), then
  \[ E[(L_q - L_{q-1})Y_q] = E[(L_q - L_{q-1})Y'_q] = E[L_qY'_q] \]
- If \( Z_q \sim \mathcal{N}(0,1) \perp L_s, s \leq n + 1 \), if \( \mathcal{H}_q := \sigma(L_s, Z_s, s \leq q) \), if \( Y''_q := Y'_q + \sqrt{\rho^2 - E[Y'^2_q|L_s, s < q]} Z_q \) then
  - \( Y''_q \) is \( \mathcal{H}_q \)-measurable and \( E[Y''_q|\mathcal{H}_{q-1}] = 0. \)
  - \( E[Y''^2_q|\mathcal{H}_{q-1}] = \rho^2 \)
  - \( E[|Y''_q|^p|\mathcal{H}_{q-1}] \leq (4A + C)^p' =: K^p' \)
  \[ E[(L_q - L_{q-1})Y_q] = E[L_qY'_q] = E[L_qY''_q] = E[L_{n+1}Y''_q] \]
Proof of theorem 2

- \( \limsup \frac{V_n^B(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu). \)

- \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

- If \( Z_q \sim \mathcal{N}(0, 1) \perp L_s, s \leq n + 1, \) if \( \mathcal{H}_q := \sigma(L_s, Z_s, s \leq q), \) if \( Y''_q := Y'_q + \sqrt{\rho^2 - E[Y_q'^2|L_s, s < q]}Z_q \)
  
  - \( Y''_q \) is \( \mathcal{H}_q \)-measurable and \( E[Y''_q|\mathcal{H}_{q-1}] = 0. \)
  
  - \( E[Y''_q^2|\mathcal{H}_{q-1}] = \rho^2 \)
  
  - \( E[|Y''_q|^{p'}|\mathcal{H}_{q-1}] \leq (4A + C)^{p'} =: K^{p'} \)

\[
E[(L_q - L_{q-1})Y_q] = E[L_q Y'_q] = E[L_q Y''_q] = E[L_{n+1}Y''_q]
\]

- \( \frac{V_n^B(L)}{\sqrt{n}} \leq \sup_{Y''} E[L_{n+1} \frac{\sum_{q=1}^n Y''_q}{\sqrt{n}}] \)
Proof of theorem 2

- \( \limsup \frac{V_n^B(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu). \)

- \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

- If \( Z_q \sim \mathcal{N}(0, 1) \perp L_s, s \leq n + 1, \) \( H_q := \sigma(L_s, Z_s, s \leq q), \)

  \[
  Y''_q := Y'_q + \sqrt{\rho^2 - E[Y'q^2|L_s, s < q]} Z_q
  \]

  - \( Y''_q \) is \( H_q \)-measurable and \( E[Y''_q|H_{q-1}] = 0. \)
  - \( E[Y''_q^2|H_{q-1}] = \rho^2 \)
  - \( E[|Y''_q|^{p'}|H_{q-1}] \leq (4A + C)^{p'} =: K^{p'} \)

  \[
  E[(L_q - L_{q-1})Y_q] = E[L_q Y'_q] = E[L_q Y''_q] = E[L_{n+1} Y''_q]
  \]

- \( \frac{V_n^B(L)}{\sqrt{n}} \leq \sup_{Y''} E[L_{n+1} \frac{\sum_{q=1}^n Y''_q}{\sqrt{n}}] \overset{CLT}{\approx} E[L_{n+1} Z \rho] \)

  where \( Z \sim \mathcal{N}(0, 1) \)
Proof of theorem 2

• \( \lim \sup \frac{V_n^B(\mu)}{\sqrt{n}} \leq \rho \gamma(\mu) \).

• \( E[B([L_q - L_{q-1}|L_s, s \leq q])] = \sup_{Y_q} E[(L_q - L_{q-1})Y_q] \)

• If \( Z_q \sim N(0, 1) \perp L_s, s \leq q \), if \( \mathcal{H}_q := \sigma(L_s, Z_s, s \leq q) \), if \( Y''_q := Y'_q + \sqrt{\rho^2 - E[Y'^2_q|L_s, s < q]} Z_q \) then

  • \( Y''_q \) is \( \mathcal{H}_q \)-measurable and \( E[Y''_q|\mathcal{H}_{q-1}] = 0 \).

  • \( E[Y''^2_q|\mathcal{H}_{q-1}] = \rho^2 \)

  • \( E[\|Y''\|^p|\mathcal{H}_{q-1}] \leq (4A + C)^p' =: K^{p'} \)

\[ E[(L_q - L_{q-1})Y_q] = E[L_q Y'_q] = E[L_q Y''_q] = E[L_{n+1} Y''_q] \]

• \( \frac{V_n^B(L)}{\sqrt{n}} \leq \sup_{Y''} E[L_{n+1} \frac{\sum_{q=1}^{n} Y''_q}{\sqrt{n}}]^\text{CLT} E[L_{n+1} Z \rho] \leq \rho \gamma(\mu) \)

where \( Z \sim N(0, 1) \) and \( L_{n+1} \sim \mu \).
Skorohod embedding and CLT

Let $B$ be a BM and $\mathcal{F}$ its natural filtration. If $S_q := \sum_{k=1}^{q} \frac{Y_k''}{\rho \sqrt{n}}$

$\exists$ an increasing sequence $\tau_q$ of $\mathcal{F}$-stopping times.

$\exists$ an $\mathcal{F}_\infty$-measurable r.v. $\tilde{L}_{n+1}$

s. th. $[(S_1, \ldots, S_n, L_{n+1})] = [(B_{\tau_1}, \ldots, B_{\tau_n}, \tilde{L}_{n+1})]$. 

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Skorohod embedding and CLT

• Let $B$ be a BM and $\mathcal{F}$ its natural filtration.
  
  If $S_q := \sum_{k=1}^{q} Y_k \frac{\rho}{\sqrt{n}}$
  
  $\exists$ an increasing sequence $\tau_q$ of $\mathcal{F}$-stopping times.
  
  $\exists$ an $\mathcal{F}_\infty$-measurable r.v. $\tilde{L}_{n+1}$
  
  s. th. $[(S_1, \ldots, S_n, L_{n+1})] = [(B_{\tau_1}, \ldots, B_{\tau_n}, \tilde{L}_{n+1})]$.

• $E[\tau_n] = E[B_{\tau_n}^2] = E[S_n^2] = 1$
Skorohod embedding and CLT

- Let $B$ be a BM and $\mathcal{F}$ its natural filtration.
  
  If $S_q := \sum_{k=1}^{q} \frac{Y_k''}{\rho \sqrt{n}}$
  
  $\exists$ an increasing sequence $\tau_q$ of $\mathcal{F}$-stopping times.
  
  $\exists$ an $\mathcal{F}_\infty$-measurable r.v. $\tilde{L}_{n+1}$
  
  s. th. $[(S_1, \ldots, S_n, L_{n+1})] = [(B_{\tau_1}, \ldots, B_{\tau_n}, \tilde{L}_{n+1})]$. 

- $E[\tau_n] = E[B_{\tau_n}^2] = E[S_n^2] = 1$

- $\theta_q := \tau_q - q/n$ is an $\mathcal{F}_{\tau_q}$-martingale:
  
  $E[\tau_q - \tau_{q-1}|\mathcal{F}_{\tau_{q-1}}] = E[(B_{\tau_q} - B_{\tau_{q-1}})^2|\mathcal{F}_{\tau_{q-1}}] = E[(Y_q''/\rho \sqrt{n})^2|S_t, t < q] = 1/n$
Skorohod embedding and CLT

- Let $B$ be a BM and $\mathcal{F}$ its natural filtration.
  
  If $S_q := \frac{\sum_{k=1}^{q} Y_k''}{\rho \sqrt{n}}$
  
  $\exists$ an increasing sequence $\tau_q$ of $\mathcal{F}$-stopping times.

  $\exists$ an $\mathcal{F}_\infty$-measurable r.v. $\tilde{L}_{n+1}$

  s. th. $[(S_1, \ldots, S_n, L_{n+1})] = [(B_{\tau_1}, \ldots, B_{\tau_n}, \tilde{L}_{n+1})]$.

- $E[\tau_n] = E[B_{\tau_n}^2] = E[S_n^2] = 1$

- $\theta_q := \tau_q - q/n$ is an $\mathcal{F}_{\tau_q}$-martingale:

- $p < 2 \Rightarrow p' > 2$. Here $p' = 4$
Skorohod embedding and CLT

- Let $B$ be a BM and $\mathcal{F}$ its natural filtration. If $S_q := \sum_{k=1}^q \frac{Y_k''}{\rho \sqrt{n}}$
  - $\exists$ an increasing sequence $\tau_q$ of $\mathcal{F}$-stopping times.
  - $\exists$ an $\mathcal{F}_\infty$-measurable r.v. $\tilde{L}_{n+1}$
  - s. th. $[(S_1, \ldots, S_n, L_{n+1})] = [(B_{\tau_1}, \ldots, B_{\tau_n}, \tilde{L}_{n+1})]$.

- $E[\tau_n] = E[B_{\tau_n}^2] = E[S_n^2] = 1$

- $\theta_q := \tau_q - q/n$ is an $\mathcal{F}_{\tau_q}$-martingale:

- $p < 2 \Rightarrow p' > 2$. Here $p' = 4$
  - $E[(\theta_q - \theta_{q-1})^2] \leq E[(\tau_q - \tau_{q-1})^2] \leq E[(Y_q''/\rho \sqrt{n})^4] \leq \tilde{K}/n^2$
  - $C_4 E[(B_{\tau_q} - B_{\tau_{q-1}})^4] = C_4 E[(Y_q''/\rho \sqrt{n})^4] \leq \tilde{K}/n^2$
Skorohod embedding and CLT

- Let $B$ be a BM and $\mathcal{F}$ its natural filtration. If $S_q := \sum_{k=1}^{q} Y''_k / \rho \sqrt{n}$

$\exists$ an increasing sequence $\tau_q$ of $\mathcal{F}$-stopping times.

$\exists$ an $\mathcal{F}_\infty$-measurable r.v. $\tilde{L}_{n+1}$

s. th. $[(S_1, \ldots, S_n, L_{n+1})] = [(B_{\tau_1}, \ldots, B_{\tau_n}, \tilde{L}_{n+1})]$.

- $E[\tau_n] = E[B_{\tau_n}^2] = E[S_n^2] = 1$

- $\theta_q := \tau_q - q/n$ is an $\mathcal{F}_{\tau_q}$-martingale:

- $p < 2 \Rightarrow p' > 2$. Here $p' = 4$

$E[(\theta_q - \theta_{q-1})^2] \leq E[(\tau_q - \tau_{q-1})^2] \leq C_4 E[(B_{\tau_q} - B_{\tau_{q-1}})^4] = C_4 E[(Y''_q / \rho \sqrt{n})^4] \leq \tilde{K}/n^2$

- $\text{var}(\tau_n) = \text{var}(\theta_n) \leq \tilde{K}/n$
Skorohod embedding and CLT

- Let $B$ be a BM and $\mathcal{F}$ its natural filtration.
  
  If $S_q := \frac{\sum_{k=1}^{q} Y_k}{\rho \sqrt{n}}$

  $\exists$ an increasing sequence $\tau_q$ of $\mathcal{F}$-stopping times.

  $\exists$ an $\mathcal{F}_\infty$-measurable r.v. $\tilde{L}_{n+1}$

  s. th. $[\langle S_1, \ldots, S_n, L_{n+1} \rangle] = [\langle B_{\tau_1}, \ldots, B_{\tau_n}, \tilde{L}_{n+1} \rangle]$.

- $E[\tau_n] = E[B_{\tau_n}^2] = E[S_n^2] = 1$

- $\theta_q := \tau_q - q/n$ is an $\mathcal{F}_{\tau_q}$-martingale:

- $p < 2 \Rightarrow p' > 2$. Here $p' = 4$

- $\text{var}(\tau_n) = \text{var}(\theta_n) \leq \tilde{K}/n$

- $E[(B_{\tau_n} - B_1)^2] = \|\tau_n - 1\|_{L^1} \leq \sqrt{\text{var}(\tau_n)} \leq \sqrt{\tilde{K}/n}$