A Study of Approval Voting on Large Poisson Games*

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Abstract

In a focal equilibrium of a Large Poisson voting game, voters consider that the most likely pivot outcome occurs between some pair of candidates. A voting rule \( V \) is less focally manipulable than a voting rule \( V' \) if the number of focal equilibria that exist under \( V \) is lower or equal to the number of focal equilibria that exist under \( V' \). We show that Approval Voting is less focally manipulable than Plurality Voting. We also provide two examples that shed some light on the structure of this equilibrium refinement. The first one shows that there can exist focal equilibria where the Condorcet Winner gets no vote and that sincere behavior is not guaranteed under Approval Voting. In the second one, the Condorcet Winner is not the Winner of the election in any of the equilibria of the Large Poisson game under Approval Voting.

KEYWORDS: Approval voting, Poisson Games, Plurality voting, Condorcet Winner.

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1 Introduction

The role of information on voters’ behavior and therefore its impact on electoral outcomes is a phenomenon not very much studied in game-theoretical models with strategic voters. It seems quite intuitive that the wasted-vote effect (not voting for a candidate that you prefer because you think he has no chance of winning the election) has a non-negligible weight on the result of an election. Plurality Voting, one of the most used voting rules,  

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allows the voter to vote for at most one candidate and thus seems to be quite vulnerable to this wasted-vote effect. The proponents of Approval Voting or AV (allowing the voter to vote for as many candidates as he wishes) have often suggested that it helps to mitigate the effects of this informational effect. One possible way of modeling this effect in a game-theoretical framework is by introducing an uncertainty over the chances of winning the election of each candidate, the underlying intuition for Large Elections models that this work analyses. It can be argued that these prior beliefs can be simply achieved by means of a typical Bayes-Nash equilibrium. However, these Large Elections models have an important advantage. There is a common knowledge probability over the different outcomes which is endogenously determined by voters' actions. Then, a voter's action uniquely depends on his private information, simplifying the analysis of the voting game.

To our knowledge, there exists two main Large Elections models: Myerson [4, 5, 6]'s Population Uncertainty model and Laslier [3]'s Score Uncertainty one. Whereas the latter is based on the introduction of an uncertainty over the scores of candidates (by assuming that there exists a small probability each ballot is not recorded), the former, also known as Large Poisson Games, introduces an uncertainty over the total number of voters in the election. Indeed, it assumes that the total number of voters in the game is not constant and is drawn from a Poisson distribution of a given parameter $n$, the expected size of the population.

One of the objectives of Large Elections models is to study of the role of information manipulation and the robustness of voting rules with respect to it, a different approach to the standard Arrowian manipulation (manipulating by voting). Myerson [6] identifies a series of drawbacks for voting rules like Plurality voting associated to this manipulation of information in Large Poisson Games. Plurality voting is too vulnerable to this type of manipulation as once the attention of voters is focused on a pair of candidates, this pair becomes the most likely one to be in contention for victory (the self-fulfilling property of Plurality voting). Myerson [6] draws a positive conclusion on some simple Poisson voting games over the properties of AV. Indeed, in these examples AV is more robust to this manipulation of information than other voting rules and leads to “better” preference aggregation by its higher degree of flexibility.

The strength of this conclusion has been weakened by Nuñez [8] that proves that AV does not always satisfy Condorcet Consistency in Large Poisson Games. Indeed, Nuñez [8] provides an example where a candidate who is ranked first by more than half of the population (and thus the Condorcet Winner) is not the Winner of the election in equilibrium. However, in the previously mentioned situation, there also exist equilibria where the Condorcet Winner wins the election. As argued by Myerson and Weber [7] the multiplicity of equilibria has a political significance. A large set of equilibria in an electoral
situation implies that informational issues have a great influence when determining the result of the election.

As of now, little is known regarding the size of the set of equilibria that can arise under AV. This work suggests the notion of focal manipulation, based on the number of focal equilibria, as a measure of the degree of manipulation through information of a voting rule. A similar concept of focal manipulation comes from Schelling [9] and was discussed by Myerson and Weber [7]. In a focal equilibrium, voters consider that the most likely pivot outcome occurs between some pair of candidates. Furthermore, this pivot is infinitely more probable than the rest of pivot outcomes. This focalisation of a voter’s information on this pair of candidates has a strong impact on his best response. A voting rule $V$ is focally manipulable if there exists different focal equilibria under $V$. Such a voting rule leads to different outcomes depending on the prior beliefs voters have with regards to the most likely pivot outcome. We may say that a voting rule $V$ is less focally manipulable than a voting rule $V'$ if the number of focal equilibria that exist under $V$ is lower or equal to the number of focal equilibria that exist under $V'$. Laslier [3] shows that in the Score Uncertainty model there exists a unique equilibrium which is focal under AV. In such an equilibrium, voters’ best responses are sincere and the Condorcet Winner wins the election whenever a Condorcet Winner exists.

We show that the set of focal equilibria under Approval Voting is a subset of the set of focal equilibria under Plurality Voting in Large Poisson Games. This result implies that AV satisfies a weaker version of the self-fulfilling property of Plurality Voting proved by Myerson [6]. Hence, we prove that AV is less focally manipulable than Plurality Voting so that Condorcet losers cannot be the winners of the election under AV. Besides, we characterize the two kind of focal equilibria that can arise on Poisson Games under AV. We divide them in two categories: the ones in which the likelihood of the close races have an intuitive ordering and the ones which do not. In an intuitive equilibrium, voters consider that the most probable close race occurs between the Front Runner and the Main Challenger whereas this is not the case in a non-intuitive one. The intuitive equilibria are shown to be less robust than the non-intuitive one to changes in the intensity of preferences (i.e. cardinality of utility).

Once we have shown that AV refines the set of focal equilibria with regards to Plurality Voting, the present work addresses the issue of what kind of equilibria remain in this refined set. We provide two examples that show that AV does not satisfy neither Condorcet Consistency nor sincerity, properties systematically analyzed in the debate over AV and that Plurality voting does not satisfy. A first example shows that sincere behavior is not always an equilibrium and that the Condorcet Winner can get no vote. A second example shows that it can be the case that, with three candidates, the Condorcet Winner is not
the winner of the election in any of the equilibria of the game.

This paper is structured as follows. Section 2 introduces the basic model. Section 3 presents the characterization of the set of focal equilibria under AV. Section 4 introduces the example where the Condorcet Winner gets no vote and finally, Section 5 discusses in detail the situation where the Condorcet Winner does not coincide with the Winner of the election in any of the large equilibria of the game and Section 6 concludes.

2 The model

2.1 The basic setting

A Poisson random variable $\mathcal{P}(n)$ is a discrete probability distribution that depends on a unique parameter which represents its mean. The probability that a Poisson random variable of parameter $n$ takes the value $l$, being $l$ a nonnegative integer is equal to

$$e^{-n\frac{n^l}{l!}}.$$

A Poisson voting Game of expected size $n$ is a game such that the actual number of voters taking part in the election is a random variable drawn from a Poisson distribution with mean $n$. This assumption represents the uncertainty faced by voters w.r.t. the number of voters that show up the day of the election. The probability distribution and its parameter $n$ are common knowledge.

Each voter has a type $t$ in set of types $T$ that defines his cardinal preferences over the set of candidates $K$. A voter’s payoff only depends on the candidate who is elected. The preferences of a voter with a type $t$ are denoted by $u_t = (u_t(k))_{k \in K}$. Thus, for a given $t$, $u_t(j) > u_t(k)$ implies that $t$-voters strictly prefer candidate $j$ to candidate $k$. Each voter’s type is independently drawn from $T$ according to the distribution of types denoted by $r = (r(t))_{t \in T}$.

A finite Poisson game of expected size $n$ is then represented by $(K, T, n, r, u)$. A large Poisson game simply represents the limit when $n$ tends towards infinity of a finite Poisson game of expected size $n$.

For any pair of candidates $k, j \in K$, let $T_{k,j} = \{t \in T \mid u_t(k) > u_t(j)\}$ be the set of preference types where candidate $k$ is strictly preferred to candidate $j$. The Condorcet Winner (C.W.) of the election is defined as:

\[1\]The distribution of types satisfies $r(t) > 0 \forall t \in T$ and $\sum_{t \in T} r(t) = 1$. Voters are therefore not allowed to abstain.
Definition 1. A candidate \( k \) is called the Condorcet Winner (C.W.) of the election if

\[
\sum_{t \in T_{k,j}} r(t) > 1/2 \ \forall \ j \in K, j \neq k.
\]

Similarly, a Condorcet Loser of the election is a candidate \( k \) such that \( \sum_{t \in T_{k,j}} r(t) < 1/2 \ \forall \ j \in K, j \neq k \).

In order to completely determine an election in a Poisson voting game, the voting rule remains to be specified. A voting rule \( V \) is simply defined by the finite set of possible ballots denoted by \( C \). Each ballot is a vector that denotes the number of points that the voter gives to each candidate. The vectors are summed and that the candidate that gets most points wins the election. Then, a Poisson Approval Voting game will be represented by \((K, T, C, n, r, u)\) in which \( C \subset K \) as an AV ballot simply consists of a subset of the set of candidates. A Plurality voting ballot simply specifies the candidate who is approved of by the voter.

Whereas the meaning of sincere ballot is clear in a rule like Plurality voting (a sincere ballot simply votes for the preferred candidate of the voter), it is not that simple in a rule like AV where the voter can vote for different candidates. The definition of sincere AV ballot we adopt is the one stated by Brams [2] that can be defined as follows.

Definition 2 (Sincerity). An AV ballot is sincere if, given the lowest-preferred candidate \( k \) that a voter approves of, he also approves of all candidates he prefers to \( k \).

With such a definition of sincerity, there is not anymore a dichotomy between strategic voting and sincere voting. Indeed, this definition allows for several sincere AV ballots.

As shown by Myerson [4], assuming a Poisson population has two main advantages: common public information and independence of actions.\(^2\) Indeed, as a consequence of the common public information property concerning the probabilities over the outcomes of the election, voters’ strategies uniquely depend on their type which summarizes their private information. Indeed, we represent voters’ actions by the strategy function \( \sigma(c \mid t) \) \(^3\) which is a function from \( T \) into \( \Delta(C) \) the set of probability distributions over \( C \). Formally, we write

\[
\sigma : T \rightarrow \Delta(C)
\]

\[
t \mapsto \sigma(. \mid t).
\]

A voter with type \( t \) chooses ballot \( c \) with probability \( \sigma(c \mid t) \). Then, given the distribution of voters \( r \) and the strategy function \( \sigma(. \mid t) \), we define for each \( c \in C \) the vote

\(^2\)For a detailed description of both properties, the reader can refer to Myerson [4] and Nuñez [8].
\(^3\)The strategy function satisfies \( \sigma(c \mid t) \geq 0 \ \forall \ c \in C \) and \( \sum_{d \in C} \sigma(d \mid t) = 1 \).
distribution $\tau = (\tau(c))_{c \in C}$ as

$$\tau(c) = \sum_{t \in T} \tau(t)\sigma(c \mid t).$$

The vote distribution $\tau$ represents the share of votes each ballot gets. We denote by $x(c)$ the Poisson random variable with parameter $n\tau(c)$ that describes the number of voters $x(c)$ who choose ballot $c$. Furthermore the vote profile $x = (x(c))_{c \in C}$ is a vector of length $C$ of independent random variables (due to the independent actions property).

The set of electoral outcomes given ballot set $C$ is denoted by $B(C)$, where

$$B(C) = \{b \in \mathbb{R}^C \mid b(c) \text{ is a non-negative integer for all } c \in C\}$$

We denote by $b \in B(C)$ a vector of length $C$ of non-negative integer numbers. Each component $b(c)$ of vector $b$ accounts for the number of voters who vote for ballot $c$. The subsets of $B(C)$ will be denoted by capital letters $B \in B(C)$.

Given the vote profile $x$, the (common knowledge) probability that the outcome is equal to a vector $b \in B(C)$ is such that

$$P[x = b \mid n\tau] = P[(\bigcap_{c \in C} x(c) = b(c)) \mid n\tau]$$

$$= \prod_{c \in C} P[x(c) = b(c) \mid n\tau]$$

$$= \prod_{c \in C} \left(\frac{e^{-n\tau(c)}(n\tau(c))^{b(c)}}{b(c)!}\right).$$

For ease of notation, we refer to $P[x = b \mid n\tau]$ by $P[x = b]$. We will be mainly interested in computing the probabilities of subsets of $B(C)$ rather than probabilities of vectors themselves, as for instance the probability of two given ballots get the same number of votes. Given the vote profile $x$, we write that the probability of the outcome $B \in B(C)$ is

$$P[x = B] = \sum_{b \in B} P[x = b].$$

Let $C_k$ denote the set of ballots in which candidate $k$ is approved. Given the vote profile $x$, the score distribution $\rho = (\rho(k))_{k \in K}$ describes the share of votes that each candidate gets. For each $k \in K$,

$$\rho(k) = \sum_{c \in C_k} \tau(c).$$

It follows that the number of voters that vote for a candidate $k$ is drawn from a Poisson random variable with mean $n\rho(k)$. Given the score distribution, we define the score profile

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4In probabilistic terminology, an outcome is usually referred as an event or realization of a random variable, i.e. the value that is actually observed.
s = (s(k))_{k \in K} describes the number of voters who vote for each candidate k with

\[ s(k) = \sum_{c \in C_k} x(c) \sim \mathcal{P}(n\rho(k)) \]

Given that under AV voters can vote for several candidates, it is not generically true that the score profile s is a vector of independent random variables. As will be shown this is an important property of AV on Poisson games. Indeed, due to this correlation between the candidate scores, this rule can lead to equilibria in which the Condorcet Winner does not win the election.

Given an outcome \( B \in B(C) \), let \( M(B) = \arg \max_{j \in K} \rho(j) \) denote the set of candidates with the most points. Assuming a fair toss of a coin, the probability of candidate \( k \) winning the election given the vector \( B \in B(C) \) is

\[ Q[k \mid B] = \begin{cases} 1/\#(M(B)) & \text{if } k \in M(B) \\ 0 & \text{if } k \notin M(B). \end{cases} \]

For any outcome \( B \in B(C) \) and any ballot \( c \in C \), we let \( B + \{c\} \) denote the outcome such that one ballot \( c \) is added. That is, we write that the outcome \( D \in B(C) \) is such that

\[ D = B + \{c\} = \{d \in D \subset B(C) \mid d(c) = b(c) + c(c) \text{ for any } b \in B \subset B(C), c \in C\}. \]

Thus, given the vote profile \( x \), a voter with type \( t \) casts the ballot \( c \) that maximizes his expected utility

\[ E[c \mid n\tau] = \sum_{B \in B(C)} P[x = B] \sum_{k \in K} Q[k \mid x = B + \{c\}] u_t(k). \]

### 2.2 Focal equilibrium

We refer to \( \{\sigma, \tau\} \) as an equilibrium of the finite Poisson voting game \((K, T, C, n, r, u)\) if \( \tau \) is the vote distribution corresponding to the strategy function \( \sigma \) and, for each \( c \in C \) and each \( t \in T \),

\[ \sigma(c \mid t) > 0 \implies c \in \arg \max_{d \in C} E[d \mid n\tau]. \]

Nevertheless, as the focus of this work is on elections with a large number of voters, one shall look at the limits of equilibria as the expected number of voters \( n \) tends to infinity. Thus, we refer to a large equilibrium sequence of \((K, T, C, r, u)\) to denote any equilibria sequence \( \{(\sigma_n, \tau_n)\}_{n \to \infty} \) of the finite voting games \((K, T, C, n, r, u)\) such that the vectors \((\sigma_n, \tau_n)\) are convergent to some limit \((\sigma, \tau)\) as \( n \to \infty \) in the sequence. We refer to this limit \((\sigma, \tau)\) as a large equilibrium of \((K, T, C, r, u)\). Furthermore, we refer to a sequence of outcomes in \( B(C) \) by \( \{B_n\}_{n \to \infty} \). The limit \( B \) of a sequence of outcomes \( \{B_n\}_{n \to \infty} \) in \( B(C) \) is an outcome and so it is a subset of \( B(C) \).
Following Myerson [6], we assume that each voter determines which ballot he casts by maximizing his expected utility. As voters are instrumentally motivated, they care only about the influence of their own vote in determining the Winner’s identity. As usual in voting environments with a large number of voters, a voter’s action has a negligible impact on most of the possible outcomes of the election. Indeed, it has some impact only if there is some set of candidates involved in a close race for first place where one ballot could pivotally change the result of the election: a pivot.

**Definition 3.** Given the score profile $x$ and a subset $Y$ of the set of candidates $K$, an outcome $B \in B(C)$ is a pivot($Y$) if and only if:

$$\forall y \in Y, \ s(y) \geq \max_{k \in K} s(k) - 1$$

$$\forall k \not\in Y, \ s(k) < \max_{k \in K} s(k) - 2.$$  

The set of all pivot outcomes is denoted by $\Sigma(C) \subset B(C)$, where

$$\Sigma(C) = \{ B \in B(C) \mid \exists Y \subset K, B = \text{pivot}(Y) \}.$$  

Besides, the set of all pivot outcomes in which candidate $k$ is involved is denoted by $\Sigma(C, k) \subset \Sigma(C)$, where

$$\Sigma(C, k) = \{ B \in \Sigma(C) \mid \exists Y \subset K \text{ s.t. } k \in Y \text{ and } B = \text{pivot}(Y) \}.$$  

Thus, given the vote profile $\tau$, the expected utility for a voter with preference ordering $t$ of casting ballot $c$ is such that

$$E[c \mid n\tau] = \sum_{B \in B(C)} P[x = B] \sum_{k \in K} Q[k \mid x = B + \{c\}] u_t(k)$$

$$= \sum_{B \in \Sigma(C)} P[x = B] \sum_{k \in K} Q[k \mid x = B + \{c\}] u_t(k).$$  

Indeed, in any outcome $B$ which is not a pivot, a single vote has a negligible impact on the Winner’s identity. Given this expected utility, a voter’s decision problem can be described in a simple way. Let $c$ and $c'$ be two ballots that only differ by one extra candidate $k$: $c' = c \cup k$. In order to evaluate which of the ballots the $t$-voter casts, he computes the sign of the following expression

$$\Delta = E[c' \mid n\tau] - E[c \mid n\tau]$$

$$= \sum_{B \in \Sigma(C)} P[x = B] \sum_{k \in K} [Q[k \mid x = B + \{c'\}] - Q[k \mid x = B + \{c\}]] u_t(k)$$

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The sum $\Delta$ simply represents the effect of adding candidate $k$ to his ballot in his expected utility. However, adding this extra candidate to his ballot can only have an impact in the cases where this candidate is involved in a pivot. Therefore, $\Delta$ can be rewritten as follows:

$$\sum_{B \in \Sigma(C,k)} P[x = B] \sum_{k \in K} \left[ Q[k \mid x = B + \{c\}] - Q[k \mid x = B + \{c\}] \right] u_t(k)$$

Then, if there exists a pivot $\text{pivot}(Y) \in \Sigma(C,k)$ where candidate $k$ is involved which probability becomes infinitely more likely as $n$ tends towards infinity than every other pivot $b \in \Sigma(C,k)$, one can factor out by this pivot. Indeed, let us assume that every pivot $B$ where candidate $k$ is involved becomes infinitely less likely than pivot $\text{pivot}(Y)$ as the expected number of voters $n$ tends towards infinity.

$$\lim_{n \to \infty} \frac{P[x = B]}{P[x = \text{pivot}(Y)]} = 0 \text{ for all } B \in \Sigma(C,k).$$

Given this focalisation of voters’ attention on the outcome $\text{pivot}(Y)$, a voter’s decision (the sign of $\Delta$) is reduced to evaluating which ballot maximizes his expected utility in case of a $\text{pivot}(Y)$,

$$\text{sign}(\Delta) = \text{sign} \left( \sum_{k \in K} \left[ Q[k \mid x = \text{pivot}(Y) + \{c\}] - Q[k \mid x = \text{pivot}(Y) + \{c\}] \right] u_t(k) \right).$$

Repeating the previous procedure, one can deduce the best response for every voter in the election. However in order to set up such a procedure, we need to introduce some mathematical tools. The probability of any pivot outcome generally tends to zero as the expected population $n$ becomes large. We can still compare their likelihood by comparing the rates at which their probabilities tend to zero. These rates can be measured by a concept of magnitude, defined as follows.

Given a large equilibrium sequence $\{\sigma_n, \tau_n\}_{n \to \infty}$, the magnitude $\mu[B]$ of an outcome $B \in B(C)$ is such that

$$\mu[B] = \lim_{n \to \infty} \frac{1}{n} \log P[x = B]$$

Notice that the magnitude of an outcome must be inferior or equal to zero, since the logarithm of a probability is never positive. The main advantage of using magnitudes is to have an analytical way to compare likelihoods of outcomes rather than estimations. If one can show that a pivot between one pair of candidates has a magnitude that is strictly greater than the magnitude of a pivot between another pair of candidates, then the former becomes infinitely more likely as the expected number of voters $n$ tends towards infinity.

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5See Myerson [4, 5] and Nuñez [8] for examples that show the problems of working directly with probabilities of outcomes.
That is to say, given two subsets $Y$ and $Y'$ of the set of candidates $K$, for any pair of pivot outcomes $\text{pivot}(Y)$ and $\text{pivot}(Y') \in B(C)$, if

$$\mu[\text{pivot}(Y)] > \mu[\text{pivot}(Y')] ,$$

then we know that,

$$\lim_{n \to \infty} \frac{P[x = \text{pivot}(Y)']}{P[x = \text{pivot}(Y)]} = 0 .$$

Therefore, if given the vote profile $x$, there exists a strict ordering of the magnitudes of the pivot outcomes, we can ensure the existence of a unique best response. Thus, it seems clear that the concept of magnitude is quite useful in this probabilistic environment. Using this concept, we define the concept of focal equilibrium.

We refer to focal equilibrium as a large equilibrium $(\sigma, \tau)$ of the game $(K, T, C, r, u)$ in which there is a pair of candidates $k$ and $j$ such that the probability of the outcome $\text{pivot}(k,j)$ has a strictly higher magnitude than the magnitude of the pivot outcomes between any other subset of candidates. Formally,

**Definition 4.** The large equilibrium $(\sigma, \tau)$ of the game $(K, T, C, r, u)$ is a focal equilibrium if there exists a pivot $(k,j) \subset \Sigma(C)$ for some $k$ and $j \in K$ such that

$$\mu[\text{pivot}(k,j)] > \mu[B] \forall B \in \Sigma(C) .$$

Laslier [3] shows that in the Score Uncertainty model there exists a unique focal equilibrium in which Condorcet Consistency is satisfied. Formally, Laslier’s result can be stated as follows.

**Theorem 1** (Laslier [3]). In the Score Uncertainty model, provided that every candidate has a strictly positive share of votes, there exists a unique focal equilibrium under Approval Voting that satisfies the following properties:

1. The Condorcet Winner wins the election whenever it exists.
2. Voters’ best responses are sincere in equilibrium.
3. The magnitudes of two-candidate pivot outcomes are strictly ordered.
4. The candidates involved in the most probable pivot are the Front Runner and the Main Challenger.

In such a model, the set of focal equilibria under Approval Voting has very interesting properties. Thus, it seems relevant to understand how is this set on Large Poisson Games. Whereas the Condorcet Consistency was shown not to hold by Nuñez [8], this work studies the whole structure of focal equilibria.
This approach is slightly different from the one used on discriminatory equilibria of Myerson [5] as we now prove. In order to describe a discriminatory equilibrium, we need first to introduce the notions of serious pivot and candidate. Indeed, for any two candidates \(k\) and \(j\), we may say that the outcome \(\text{pivot}(k, j)\) is serious in a large equilibrium sequence if and only

\[
\limsup_{n \to \infty} \frac{P[x = \text{pivot}(k, j)]}{P[x = \Sigma(C)]} > 0.
\]

In a large equilibrium sequence, we may say that a candidate \(k\) is serious if and only if there is some other candidate \(j\) such that the outcome \(\text{pivot}(k, j)\) is serious. Then, according to Myerson’s terminology we say that a large equilibrium is discriminatory if and only if there is a candidate in \(K\) who is not serious.

In a three candidates situation (for instance \(k\), \(j\) and \(l\)) both approaches are identical. Formally, the situations such as

\[
\mu[\text{pivot}(k, j)] = \mu[\text{pivot}(k, l)] > \mu[\text{pivot}(j, l)],
\]

and its permutations are excluded as neither candidate \(j\) nor candidate \(k\) can be considered as non-serious candidates. They are not anymore equivalent in situations with more than three candidates.

One of the objectives of Large Poisson Games is to study of the role of information manipulation and the robustness of voting rules with respect to it, a different approach to the standard Arrowian manipulation (manipulating by voting). Using the focal equilibrium approach, we can measure how much manipulable through the information a voting rule can be. We define the concept of focal manipulation as follows.

**Definition 5.** A voting rule \(V\) is focally manipulable in a Large Poisson Game if there exists different focal equilibria \((\sigma, \tau)\) under \(V\).

In other words, a voting rule which is focally manipulable leads to different outcomes as a function of the prior beliefs voters have with regards to the pivot outcome with the highest magnitude. As shown by Myerson [6], Plurality Voting and voting rules that are similar to Plurality (in the sense that a voter rewards much more his preferred candidate with respect to the other candidates) tend to have many focal equilibria. This can be a major drawback of Plurality Voting as it leads to trivial equilibria where almost\(^6\) any candidate can win. On the contrary, Negative voting and similar voting rules have a tendency to create too few of these focal equilibria. It seems desirable that in a large Poisson voting game in which the expected number of voters tends towards infinity there is a pivot

\(^6\)We mean by “almost any candidate” a Condorcet Loser. Indeed, a Condorcet Loser is less preferred than any other candidate in pairwise comparisons. As in a focal equilibrium voters focus their attention in a pair of candidates, a Condorcet Loser cannot win the election under Plurality Voting.
outcome between some pair of candidates that is infinitely more likely than the others. For instance, it seems particularly interesting in a game with three candidates in which one of them who is universally disliked by all voters (as in the One Bad Apple example of Myerson [6]). In order to compare voting rules by their degree of focal manipulation, we may say that

**Definition 6.** A voting rule \( V \) is less focally manipulable than a voting rule \( V' \) if the number of focal equilibria \((\sigma, \tau)\) that exist under \( V \) is lower or equal to the number of focal equilibria that exist under \( V' \).

As we will see throughout, Approval Voting is less focally manipulable than Plurality Voting. However, Approval Voting can uniquely lead to focal equilibria which are not Condorcet Consistent (see Section 5).

### 2.3 Magnitudes of pivot outcomes

In order to compute the magnitude of pivot outcomes, we need to introduce two results that will be useful throughout: the Dual Magnitude Theorem and the Magnitude Equivalence Theorem. The Dual Magnitude Theorem or \( DMT \) was introduced by Myerson [5]. It states a method to compute the magnitude of outcomes that can be defined by a series of linear inequalities involving the vote profile \( x = (x(c))_{c \in C} \).

**Theorem 2** (Dual Magnitude Theorem, Myerson [5]). *Given the vote profile \( x \), let \( B \in B(C) \) be an outcome defined by*

\[
B = \{ \sum_{c \in C} a_k(c)x(c) \geq 0 \forall k \in J \},
\]

*in which \( J \) is a finite set and parameters \( a_k(c) \) are given for every \( k \in J \) and \( c \in C \). Suppose that \( \lambda \in \mathbb{R}^C \) is an optimal solution to the problem*

\[
\min_{\lambda} \sum_{c \in C} \tau(c)(\exp(\sum_k \lambda_k a_k(c)) - 1) \quad \text{s.t.} \lambda_k \geq 0, \forall k \in J. \quad (F)
\]

*Then the optimal value of the objective function \( (F) \) coincides with the magnitude \( \mu[B] \) of the outcome \( B \in B(C) \) and the limits of the c-offset ratios associated are such that*

\[
\alpha(c) = \exp(\sum_k \lambda_k a_k(c)), \text{ for all } c \in C.
\]

This technical result states that the magnitude of an outcome defined as a cone (series of linear inequalities) coincides with the optimal value of a simple minimization problem. However, as argued elsewhere (see Nuñez [8]) a pivot outcome cannot be generally defined by a series of linear inequalities implying the vote profile \( x \). Indeed, a pivot outcome has a
more complex geometrical structure that precludes from this simple definition. In order to overcome this problem, the Magnitude Equivalence Theorem or MET (Núñez [8]) shows that for every outcome $\text{pivot}(Y)$ there exists an outcome with the same magnitude than can be defined by a series of linear inequalities involving the vote profile $x$. In other words, for every $\text{pivot}(Y)$ there exists an outcome $B$ defined by

$$B = \{ \sum_{c \in C} a_k(c)x(c) \geq 0 \forall k \in J \},$$

such that both magnitudes coincide, i.e. $\mu[B] = \mu[\text{pivot}(Y)]$.

**Theorem 3** (Magnitude Equivalence Theorem, Núñez [8]). *Let $Y$ be a subset of $K$ and $\text{pivot}(Y)$ be its associated pivot outcome. Given a large equilibrium sequence $\{\sigma_n, \tau_n\}_{n \to \infty}$, we can write $\mu[\text{pivot}(Y)] = \mu[B]$, for some outcome $B \in B(C)$ defined by

$$B = \{ s(k) = s(l) \forall k, l \in Y \} \cap \{ s(k) \geq s(l) \forall k \in Y \text{ and } l \in K \setminus Y \}. $$

Once the main model and the mathematical tools have been discussed, we now present the characterization of Focal Equilibria under $AV$.

## 3 Characterization of Focal Equilibria under $AV$

As previously shown by Myerson [6], there does not exist focal equilibria in three candidates elections\(^7\) for a large class of voting rules that include Borda count and negative voting. On the contrary, there is another class of voting rules (among which Plurality Voting is included) in which there are too many focal equilibria. Myerson [6] suggests then that $AV$ is a good voting rule as it induces a balance between these two classes of voting rules. A simple example is now presented that shows how is a focal equilibria in Poisson Approval Voting game. In this example, there is a unique focal equilibrium in which preferences are correctly aggregated by $AV$.

**Example 1.** *Focal Equilibria on an $AV$ game.*

Let us consider a Large Poisson Voting game where there are three candidates $K = \{a, b, c\}$

\(^7\)Let us remark that in a three candidates elections a discriminatory equilibrium is a focal equilibrium as previously argued.
and that voters’ preferences are such that

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$c$</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>$a$</td>
<td></td>
</tr>
</tbody>
</table>

Voters with type $t_1$ strictly prefer candidate $a$ to candidate $b$ and candidate $b$ to candidate $c$ and similarly for $t_2$-voters. The expected type distribution $r(t)$ is equal to $r(t) = (r(t_1), r(t_2))$ with $r(t_1) > r(t_2)$.

In this Poisson Voting game, there is a trivial focal equilibrium under Approval voting. Indeed, let us assume that $t_1$-voters vote for candidate $a$ and $t_2$-voters vote for candidate $c$, i.e. the strategy function $\sigma$ satisfies:

$$\sigma(a \mid t_1) = 1 \text{ and } \sigma(c \mid t_2) = 1.$$ 

Given these strategies, the pivot outcome between candidates $a$ and $c$ becomes infinitely more likely than the other pivot outcomes as $n \to \infty$. Formally, the magnitudes of the pivot outcomes satisfy

$$\mu[\text{pivot}(a,c)] = \mu[s(a) = s(c)] = -\left(\sqrt{r(t_1)} - \sqrt{r(t_2)}\right)^2 > \mu(\text{pivot}(a,b)) = \mu(\text{pivot}(b,c)) = -1.$$ 

In such a focal equilibrium, no voter votes for candidate $b$. Indeed, as no voter votes for candidate $b$, voting for this candidate has an impact on the outcome of the election if and only if the number of voters in the election is inferior to 2. However, in such an outcome, voters will vote just for their preferred candidate ($a$ or $c$). Approval Voting correctly aggregates preferences in this focal equilibrium and the Condorcet Winner, candidate $a$, wins the election.

In this game, this trivial focal equilibrium under $AV$ is unique. Indeed, there are no focal equilibria in which candidate $b$ is involved in the pivot with the highest magnitude. Let us first assume that voters anticipate that the magnitude $\text{pivot}(a,b)$ is higher than the magnitudes of other pivot outcomes. Then $t_1$-voters will cast ballot $a$ (they vote for their preferred alternative $a$ and against candidate $b$ as the most probable pivot in which candidate $b$ is involved is against candidate $a$) and $t_2$-voters will cast ballot $b,c$ (as the most probable pivot involving candidate $b$ is against candidate $a$, their less preferred one). But then the outcome $\text{pivot}(a,c)$ is equally likely than $\text{pivot}(a,b)$ showing that this is not a focal equilibrium. One can easily show repeating the same kind of arguments that there does not exist another focal equilibrium in this $AV$ game.

Finally, it should be remarked that there exists a large equilibrium in which voters anticipate that $\text{pivot}(a,b)$, $\text{pivot}(a,c)$ and $\text{pivot}(b,c)$ have the same magnitude. Indeed, taken into account that voters’ best responses are such that
\[ \sigma(a \mid t_1) = 1 \text{ and } \sigma(b, c \mid t_2) = 1. \]

Indeed, given this strategy function, the magnitude of the outcome \( \text{pivot}(a, b) \) is equal to the magnitude of the outcome \( \text{pivot}(a, c) \),
\[
\mu[\text{pivot}(a, b)] = \mu[x(a) = x(b, c)] = -\left(\sqrt{r(t_1)} - \sqrt{r(t_2)}\right)^2 = \mu[\text{pivot}(a, c)] = -1.
\]

Besides, the magnitude of the outcome \( \text{pivot}(b, c) \) satisfies
\[
\mu[\text{pivot}(b, c)] = \mu[x(b, c) \geq x(a)] = -\left(\sqrt{r(t_1)} - \sqrt{r(t_2)}\right)^2 \text{ as } r(t_1) > r(t_2).
\]

This large equilibrium is the limit of a large equilibrium sequence of the type
\[
\sigma_n(a \mid t_1) = 1, \quad \sigma_n(b, c \mid t_2) = 1 - \varepsilon_n \quad \text{and} \quad \sigma_n(b \mid t_2) = \varepsilon_n,
\]
for some \( \varepsilon_n \to 0 \) whenever \( n \) tends towards infinity. In this large equilibrium sequence, the probability \( P[x = \text{pivot}(a, b)] \) is higher than the probability \( P[x = \text{pivot}(a, c)] \) even if they both have the same magnitude.

In this Poisson voting game, there exists three focal equilibria under Plurality voting. Whenever voters consider that there is a pivot outcome involving a pair of candidates with a strictly higher magnitude than the other pivot outcomes, voters only vote for one of these candidates. Therefore, these strategies induce a focal equilibrium.

Next theorem characterizes the focal equilibria that arise under AV in Poisson voting games.

**Theorem 4.** In a Large Poisson Approval voting game, whenever there exists a focal equilibrium such that the \( \text{pivot}(k, j) \) is the most probable pivot outcome, then the Winner of the election is either candidate \( k \) or candidate \( j \), provided that neither \( k \) nor \( j \) is expected to be unanimously preferred.

**Proof.** Let us pick a pair of candidates \( k \) and \( j \in K \). Let us assume that there exists a focal equilibrium such that \( \text{pivot}(k, j) \) is the pivot outcome with the highest magnitude and where the Winner of the election is candidate \( l \) \( (l \neq k, j) \).

As neither \( k \) nor \( j \) is expected to be unanimously preferred, every voter votes for either candidate \( k \) or \( j \) and no voter votes for both of them. In other words, voters choose among the following ballots: \( k, j, k, l \) and \( j, l \). Therefore, the vote profile \( x = (x(c))_{c \in C} \) is the following vector of random variables
\[
x(k) \sim \mathcal{P}(n\tau(k)), \quad x(j) \sim \mathcal{P}(n\tau(j)), \quad x(k, l) \sim \mathcal{P}(n\tau(k, l)) \quad \text{and} \quad x(j, l) \sim \mathcal{P}(n\tau(j, l)).
\]
Let us assume that some candidate $l$ has a expected score $\rho(l)$ strictly higher than the expected scores $\rho(k)$ and $\rho(j)$ of candidates $k$ and $j$. Therefore, the vote distribution $\tau = (\tau(c))_{c \in C}$ satisfies
\[
\tau(k) < \tau(j, l) \text{ and } \tau(j) < \tau(k, l),
\]
in order to ensure that $\rho(k) = \tau(k) + \tau(k, l) < \tau(j, l) + \tau(k, l) = \rho(l)$ and $\rho(j) = \tau(j) + \tau(j, l) < \tau(j, l) + \tau(k, l) = \rho(l)$. Given the score distribution, the score profile $s = (s(k))_{k \in K}$ is such that
\[
s(k) \sim \mathcal{P}(n(\tau(k) + \tau(k, l))), \quad s(j) \sim \mathcal{P}(n(\tau(j) + \tau(j, l)))
\]
and $s(l) \sim \mathcal{P}(n(\tau(j, l) + \tau(k, l)))$.

Following the $MET$, we can write that the outcome $\text{pivot}(k, j)$ has the following magnitude
\[
\mu[\text{pivot}(k, j)] = \mu[s(k) = s(j) \geq s(l)].
\]

Then, applying the $DMT$, we can write that the magnitude $\mu[\text{pivot}(k, j)]$ is equal to the optimal value of the following constrained minimization problem:
\[
\min_{\lambda} \tau(k) \exp[\lambda_1 - \lambda_2 + \lambda_3] + \tau(j) \exp[-\lambda_1 + \lambda_2] + \tau(k, l) \exp[\lambda_1 - \lambda_2] + \tau(j, l) \exp[-\lambda_1 + \lambda_2 - \lambda_3] - 1,
\]
such that $\lambda_i \geq 0 \forall i$. Then, given that $\tau(k) < \tau(j, l)$ and $\tau(j) < \tau(k, l)$, this minimization problem yields to the optimal value
\[
\mu[\text{pivot}(k, j)] = -((\sqrt{\tau(k)} - \sqrt{\tau(j, l)})^2 - (\sqrt{\tau(j)} - \sqrt{\tau(k, l)})^2).
\]

Similarly, the magnitude of the $\text{pivot}(k, l)$ coincides with the magnitude of the outcome $[s(k) = s(l) \geq s(j)]$ as stated by the $MET$. So, applying the $DMT$, the magnitude of this pivot outcome coincides with the optimal value of the following constrained optimization problem
\[
\min_{\lambda} \tau(k) \exp[\lambda_1 - \lambda_2 + \lambda_3] + \tau(j) \exp[-\lambda_3] + \tau(k, l) \exp[\lambda_3] + \tau(j, l) \exp[-\lambda_1 + \lambda_2 - \lambda_3] - 1.
\]
such that $\lambda_i \geq 0 \forall i$. This minimization problem yields to the following solution
\[
\mu[\text{pivot}(k, l)] = -((\sqrt{\tau(k)} - \sqrt{\tau(j, l)})^2 - (\sqrt{\tau(j)} - \sqrt{\tau(k, l)})^2) > \mu[\text{pivot}(k, j)],
\]
as $\tau(j) < \tau(k, l)$ which contradicts $\text{pivot}(k, j)$ being the most probable pivot and so implies that this is not a focal equilibrium.

**Corollary 1.** Approval Voting does not implement a Condorcet loser as the Winner of the election in a focal equilibrium.
Corollary 2. Whenever there exists a focal equilibrium in which the pivot with the highest magnitude includes the Condorcet Winner, the Condorcet Winner wins the election.

Corollary 3. Approval Voting is less focally manipulable than Plurality Voting

Proof. As previously mentioned, Plurality Voting has a “self-fulfilling” property in Large Poisson Games characterized by Myerson [6]. The main intuition for such a property can be summarized as follows. Let us pick a pair of candidates $k$ and $j$. In a Poisson Plurality voting game, there exists a focal equilibrium in which the outcome $pivot(k,j)$ has the highest magnitude under Plurality Voting for any pair of candidates $k$ and $j$, provided that neither $k$ nor $j$ is expected to be unanimously preferred. Indeed, with such a voting rule, voters are constrained to vote for a single candidate. Voters’ best responses are either to vote for candidate $k$ or for candidate $j$ as $pivot(k,j)$ becomes infinitely more likely than the other pivot outcomes when the expected number of voters tends towards infinity. Therefore, the most probable pivot outcome is $pivot(k,j)$ which shows that this is an equilibrium.

On the contrary, the existence of a focal equilibrium in which $pivot(k,j)$ has the highest magnitude is not ensured under Approval Voting. Indeed, voters’ best responses are either to vote for candidate $k$ or for candidate $j$ and no voter votes for both of them. Furthermore, each voter votes for a subset of the set of candidates he prefers to either candidate $k$ or candidate $j$. If there exists a candidate $l$ that gets a expected score $\rho(l)$ which is strictly higher than $\rho(k)$ and $\rho(j)$, this is not an equilibrium as shown by Theorem 4. In other words, whenever there exists a focal equilibrium under Approval voting in which the outcome $pivot(k,j)$, the Winner of the election coincides with the Winner of the election under Plurality voting in the same type of equilibrium. As the existence of these focal equilibria is not ensured under $AV$, it follows that the set of focal equilibria under $AV$ is a (not necessarily strict) subset of focal equilibria under Plurality voting.

Previous results characterize the structure of focal equilibria on Large Poisson games. We divide them on two main categories: intuitive and non-intuitive ones. In an intuitive equilibrium, the Front Runner and the Main Challenger are the candidates who are involved in the most probable pivot. On the contrary, non-intuitive equilibria are characterized by the fact that the most probable pivot does not include the Main challenger.

Theorem 5. [Intuitive Equilibrium] Let $\{\sigma, \tau\}$ be a focal equilibrium such that $pivot(k,j)$ is the most probable pivot outcome. Whenever the score distribution $\rho = (\rho(k))_{k \in K}$ satisfies

$$\rho(k) > \rho(j) > \rho(l),$$

and the magnitudes of pivot outcomes satisfy the following inequalities:

$$\mu[pivot(k,j)] > \mu(pivot(k,l)] = \mu[pivot(j,l)],$$
for any candidate \( l \) different from \( k \) and \( j \). Besides, an intuitive equilibrium is a focal equilibrium in which previous inequalities hold.

Proof. Let \( \{\sigma, \tau\} \) be a focal equilibrium such that \( \text{pivot}(k, j) \) is the most probable pivot outcome. For a given candidate \( l \in K \) such that \( l \neq k, j \), voters choose among the following ballots: \( k, j, k, l \), and \( j, l \). Therefore, the vote profile \( x = (x(c))_{c \in C} \) is the following vector of random variables

\[
x(k) \sim \mathcal{P}(n\tau(k)), \quad x(j) \sim \mathcal{P}(n\tau(j)), \quad x(k, l) \sim \mathcal{P}(n\tau(k, l)) \quad \text{and} \quad x(j, l) \sim \mathcal{P}(n\tau(j, l)).
\]

Let us assume that the score distribution \( \rho = (\rho(k))_{k \in K} \) satisfies \( \rho(k) > \rho(j) > \rho(l) \). This assumption over the score distribution implies that the vote distribution \( \tau = (\tau(c))_{c \in C} \) satisfies

\[
\rho(k) = \tau(k) + \tau(k, l) > \rho(j) = \tau(j) + \tau(j, l) > \rho(l) = \tau(k, l) + \tau(j, l)
\]

\[
\Rightarrow \tau(k) > \tau(j, l) \quad \text{and} \quad \tau(j) > \tau(k, l).
\]

Given the score distribution, the score profile \( s = (s(k))_{k \in K} \) is such that

\[
s(k) \sim \mathcal{P}(n(\tau(k) + \tau(k, l))), \quad s(j) \sim \mathcal{P}(n(\tau(j) + \tau(j, l)))
\]

and \( s(l) \sim \mathcal{P}(n(\tau(j, l) + \tau(k, l))) \).

Let us first assume that the probability distribution over the pivot outcomes satisfies

\[
\mu[\text{pivot}(k, j)] > \mu(\text{pivot}(k, l)) > \mu(\text{pivot}(j, l)).
\]

In response to this information, we can determine the types that vote for the different ballots in the election. Indeed, the voters whose type satisfies \( u_t(k) > u_t(j) > u_t(l) \) cast ballot \( k \) as a best response. A \( t \)-voter does not vote for candidate \( j \) as the most probable race where candidate \( j \) is involved is against candidate \( k \). Similarly, one can show that voters’ best responses are the ones summarized by the following table.

<table>
<thead>
<tr>
<th>Voter’s type</th>
<th>Best Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_t(k) &gt; u_t(j) &gt; u_t(l), u_t(k) &gt; u_t(l) &gt; u_t(j) )</td>
<td>( k )</td>
</tr>
<tr>
<td>( u_t(l) &gt; u_t(k) &gt; u_t(j) )</td>
<td>( k, l )</td>
</tr>
<tr>
<td>( u_t(j) &gt; u_t(k) &gt; u_t(l) )</td>
<td>( j )</td>
</tr>
<tr>
<td>( u_t(j) &gt; u_t(l) &gt; u_t(k), u_t(l) &gt; u_t(j) &gt; u_t(k) )</td>
<td>( j, l )</td>
</tr>
</tbody>
</table>

Given this voter’s best responses, the DMT and the MET entail that

\[
\mu[\text{pivot}(k, j)] > \mu(\text{pivot}(j, l)) = \mu[\text{pivot}(k, l)].
\]

This is a contradiction as we have assumed that the magnitudes of the pivot outcomes are strictly ordered.
Similarly, one can show that if we assume that the probability distribution over the pivot outcomes satisfies
\[
\mu[pivot(k, j)] > \mu[pivot(j, l)] > \mu[pivot(k, l)]
\]
a similar reasoning applies. In other words, whenever the score distribution \( \rho = (\rho(k))_{k \in K} \) satisfies
\[
\rho(k) > \rho(j) > \rho(l),
\]
the magnitudes of pivot outcomes are not strictly ordered in an intuitive equilibrium. □

This lack of strict ordering on the magnitudes of pivot outcomes has an important consequence as far as the characterization and the stability of these equilibria is concerned. Indeed, the existence of equalities between the magnitudes of pivot outcomes in an equilibrium implies the utility levels play an important role in decision process. When there exists a strict ordering of these magnitudes, the decision process is simpler and does not depend on the intensity of preferences. This affects the stability of intuitive equilibria and leads as will be shown throughout to situations in which the unique focal equilibrium does not select the Condorcet Winner.

**Theorem 6. [Non-Intuitive Equilibrium]** Let \( \{\sigma, \tau\} \) be a focal equilibrium such that \( pivot(k, j) \) is the most probable pivot outcome. Whenever the score distribution \( \rho = (\rho(k))_{k \in K} \) satisfies
\[
\rho(k) > \rho(l) > \rho(j),
\]
for some candidate \( l \) different from \( k \) and \( j \) then the magnitudes of pivot outcomes satisfy the following inequalities:
\[
\mu[pivot(k, j)] > \mu[pivot(k, l)] > \mu[pivot(j, l)].
\]

Besides, a non-intuitive equilibrium is a focal equilibrium in which previous inequalities hold.

**Proof.** The logic of the proof is similar to the one exposed for Theorem 5. Therefore, its details are omitted. □

Once we have shown that \( AV \) implies a refinement on the set of focal equilibria when compared with \( PV \), we still need to understand how this refinement is. Indeed, the fact that the set is refined precludes Condorcet losers to win the election but not the failure of Condorcet Consistency. Indeed, Nuñez [8] shows that Condorcet Consistency is not guaranteed even when the Condorcet Winner is the preferred candidate for more than half of the population. Next sections deal with the structure of this refinement, showing that it may be very extreme. We first show that \( AV \) can prevent the Condorcet Winner
to get a single vote at equilibrium. Then, we prove that it can the case that the Condorcet Winner is not the Winner of the election at any large equilibrium of the game.

4 Sincerity of Approval Voting

This section discusses in detail an example of Large Poisson game where there exists a focal equilibrium under Approval Voting in which neither sincere behavior of the voters nor Condorcet Consistency are satisfied. Besides, in this equilibrium, the Condorcet Winner gets no vote. In this situation, the Condorcet Winner is “not-so-desirable” in the sense that no voter ranks him first. Indeed, there is a huge utility difference between the preferred candidate of every type of voter and the Condorcet Winner. The equilibrium is quite stable as we do not specify the type distribution, only some general conditions for it to hold.

Let us consider a Large Poisson Approval Voting game where there are four candidates $K = \{a, b, c, d\}$ and three types of voters $T = \{t_1, t_2, t_3\}$. The voters’ types are as follows

<table>
<thead>
<tr>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$u_t(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
<td>1000</td>
</tr>
<tr>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>3</td>
</tr>
<tr>
<td>$b$</td>
<td>$c$</td>
<td>$b$</td>
<td>2</td>
</tr>
<tr>
<td>$c$</td>
<td>$a$</td>
<td>$a$</td>
<td>1</td>
</tr>
</tbody>
</table>

in which the fourth column of the previous matrix implies that each voter associates a utility level of 1000 to his preferred candidate, a utility level of 3 to his second preferred candidate and so on. Besides the distribution of types satisfies

$$r(t_1) < r(t_2) < r(t_3) < \frac{1}{2}.$$ 

It is easy to see that candidate $d$ is the Condorcet Winner as,

$$\sum_{t \in T_{d,a}} r(t) = r(t_2) + r(t_3) > 1/2,$$

$$\sum_{t \in T_{d,b}} r(t) = r(t_1) + r(t_3) > 1/2,$$

$$\sum_{t \in T_{d,c}} r(t) = r(t_1) + r(t_2) > 1/2.$$

Under Approval Voting, we claim that there exists a focal equilibrium $(\sigma, \tau)$ of the game $(K, T, C, r, u)$ in which the C.W. does not get any vote and voters’ best responses are not sincere. In such a focal equilibrium, the strategy function $\sigma(. | t)$ satisfies:

$$\sigma(a, b | t_1) = \sigma(b | t_2) = \sigma(c | t_3) = 1.$$
Therefore, the vote distribution is such that
\[ \tau(a, b) = r(t_1), \quad \tau(b) = r(t_2), \quad \tau(c) = r(t_3). \]

Given the vote distribution, the vote profile \( x = (x(c))_{c \in C} \) is the following vector of random variables
\[ x(a, b) \sim \mathcal{P}(nr(t_1)), \quad x(b) \sim \mathcal{P}(nr(t_2)) \quad \text{and} \quad x(c) \sim \mathcal{P}(nr(t_3)). \]

In such an equilibrium, the score distribution \( \rho = (\rho(k))_{k \in K} \) is such that
\[ \rho(a) = \tau(a, b) = r(t_1), \quad \rho(b) = \tau(a, b) + \tau(b) = r(t_1) + r(t_2) \quad \text{and} \quad \rho(c) = r(t_3). \]

As the score distribution shows, the Winner of the election is candidate \( b \) and candidate \( d \) (the C.W.) gets no vote in equilibrium. Finally, given the score distribution, the score profile \( s = (s(k))_{k \in K} \) is such that
\[ s(a) = x(a, b) \sim \mathcal{P}(r(t_1)n), \quad s(b) = x(a, b) + x(b) \sim \mathcal{P}((r(t_1) + r(t_2))n) \quad \text{and} \quad s(c) = x(c) \sim \mathcal{P}(r(t_3)n). \]

Let us now show that \((\sigma, \tau)\) is a focal equilibrium. In order to do so, we need to prove that the vote distribution induces a probability distribution over the pivot outcomes in the election such \((\sigma, \tau)\) is a best response for all voters.

**Magnitude of a pivot between candidates \( a \) and \( b \).**

The \( MET \) states that the magnitude of the pivot between candidates \( a \) and \( b \) coincides with the magnitude of the outcome \( \{ s(a) = s(b) \geq s(c) \} \), i.e.
\[ \mu[pivot(a, b)] = \mu[\{ s(a) = s(b) \geq s(c) \}] \]

Therefore, the \( DMT \) implies that this magnitude is equal to the solution of the following optimisation problem.
\[ \mu[\{ s(a) = s(b) \geq s(c) \}] = \min_{\lambda} \tau(a, b)\exp[\lambda_3] + \tau(b)\exp[-\lambda_1 + \lambda_2] + \tau(c)\exp[-\lambda_3] - 1 \]
such that \( \lambda_i \geq 0 \ \forall \ i \). The solution of this constrained minimization problem entails that the magnitude of the pivot outcome between candidates \( a \) and \( b \) satisfies
\[ \mu[pivot(a, b)] = -r(t_2) - (\sqrt{r(t_1)} - \sqrt{r(t_3)})^2. \]

**Magnitude of a pivot between candidates \( a \) and \( c \).**

Combining the \( MET \) and the \( DMT \), the magnitude of a pivot between candidates \( a \) and \( c \) is equal to
\[ \mu[pivot(a, c)] = \mu[\{ s(a) = s(c) \geq s(b) \}] = -r(t_2) - (\sqrt{r(t_1)} - \sqrt{r(t_3)})^2 = \mu[pivot(a, b)]. \]
Magnitude of a pivot between candidates $b$ and $b$.

Combining the MET and the DMT, the magnitude of a pivot between candidates $b$ and $c$ is equal to

$$
\mu[\text{pivot}(b, c)] = \mu[\{s(b) = s(c) \geq s(a)\}] = - \left(\sqrt{r(t_1)} + r(t_2) - \sqrt{r(t_3)}\right)^2.
$$

Therefore, the magnitudes of the pivot outcomes are ordered as follows:

$$
\mu[\text{pivot}(b, c)] > \mu[\text{pivot}(a, b)] = \mu[\text{pivot}(a, c)]
$$

Taking into account the ordering of the magnitudes, one can determine the ballot that each voter of a given type chooses. In particular, it is important to be clarify why $t_1$-voters do not vote for candidate $d$, a candidate they prefer to candidate $b$. As we assume no one votes for candidate $d$, the only situation where voting for candidate $d$ is pivotal is when only a single voter votes. In this case, it is never optimal to vote for candidate $d$. Indeed, whenever voting for $d$ pivotally changes the outcome of the election it lowers the probability of winning of the best-ranked candidates ($a$, $b$ or $c$). Thus, no voter rationally votes for candidate $d$. Formally, the expected utility of casting ballot $\{a, b\}$ is strictly higher than the expected utility for $t_1$ voters of casting ballot $\{a, b, d\}$. Indeed, we can write

$$
\Delta = E[a, b \mid n \tau] - E[a, b, d \mid n \tau]
$$

However, the outcomes where adding candidate $d$ has an impact in the outcome are the ones where the score of each candidate is of at most equal to one (as no voter votes for candidate $d$). Among these ones, there are two outcomes with positive probability and where switching from ballot $a, b$ to ballot $a, b, d$ makes a change in the expected utility: $(0, 0, 0, 0)$ and $(0, 0, 1, 0)$ where each coordinate stands for the number of votes each candidate gets. Thus, we can rewrite the difference of expected utility as follows:

$$
\Delta = P[x = (0, 0, 0, 0)]\left(\frac{u_{t_1}(a) + u_{t_1}(b)}{2} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(d)}{3}\right)
$$

+ $P[x = (0, 0, 1, 0)]\left(\frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c)}{3} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c) + u_{t_1}(d)}{4}\right)$

Then, the effect of switching from ballot $\{a, b, d\}$ to ballot $\{a, b\}$ is located in two outcomes. Furthermore, in both outcomes, we have that

$$
\left(\frac{u_{t_1}(a) + u_{t_1}(b)}{2} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(d)}{3}\right) = \left(\frac{1002}{2} - \frac{1005}{3}\right) > 0,
$$

and

$$
\left(\frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c)}{3} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c) + u_{t_1}(d)}{4}\right) = \left(\frac{1003}{3} - \frac{1006}{4}\right) > 0.
$$
Switching from ballot $a,b,d$ to ballot $a,b$ yields to a strictly positive gain in expected utility for $t_1$-voters.

Repeating similar arguments for the different ballots yields to the conclusion that is a strict best response not to vote for candidate $d$ for $t_1$-voters. Furthermore, when a $t_1$ voter decides between casting ballot $a$ and ballot $a,b$, he takes into account the influence of adding candidate $b$. In order to do so, he cares about the most probable pivot outcome involving candidate $b$: in this case, the one involving candidates $b$ and $c$. Therefore, as a $t_1$-voter prefers candidate $b$ rather than candidate $c$, he casts ballot $a,b$. Similarly, one can show that the strategy function $\sigma$ is a best response to the information and so this is an equilibrium.

The previous example shows that AV does not generically satisfy neither sincerity nor the Condorcet Winner getting a strictly positive share of votes on Large Poisson Games. Formally, we write that:

**Theorem 7.** On Large Poisson Games, sincere behavior of voters is not guaranteed under Approval Voting in equilibrium.

**Theorem 8.** On Large Poisson Games, the Condorcet Winner can get no vote under Approval Voting in equilibrium.

## 5 No large equilibrium is Condorcet Consistent

This section discusses in detail a Large Poisson Approval Voting game where the Condorcet Winner is not the Winner of the election in any of the two large equilibria. The first equilibrium is a focal one in which the Condorcet Winner does not win the election. In the second large equilibrium, all candidates get the same expected share of votes.

There are three candidates ($a$, $b$ and $c$) and candidate $a$ is the Condorcet Winner. In such a situation, there exists three possible focal equilibria. As stated by Corollary 2, whenever there exists a focal equilibrium in which candidate $a$ is involved in the pivot outcome with the highest magnitude then candidate $a$ wins the election. In the situation described, whenever candidate $a$ is involved in the pivot with the highest magnitude, the candidate who is not involved in such a pivot gets the highest expected score. Therefore, there does not exist such a focal equilibrium as shown by Theorem 4. Furthermore, there exists another large equilibrium, involving mixed strategies, in which the three candidates get the same expected share of votes.

Let us consider a Large Poisson Approval Voting game where there are three candidates
$K = \{a, b, c\}$ and that voters’ preferences are such that

\[
\begin{array}{cccccc}
t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\
a & a & b & b & c & c \\
b & c & a & c & a & b \\
c & b & c & a & b & a \\
\end{array}
\]

with the cardinal utilities associated to each candidate described as follows

\[
\begin{array}{cccccc}
u_{t_1} & u_{t_2} & u_{t_3} & u_{t_4} & u_{t_5} & u_{t_6} \\
100 & 100 & 100 & 100 & 100 & 100 \\
99 & 99 & 1 & 99 & 1 & 99 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

in which $u_{t_1}(a) = 100, u_{t_1}(b) = 99, u_{t_1}(c) = 0$ and similarly for the other types of voters. Besides, the distribution of types satisfies

\[
r(t_1) = 0.2, \ r(t_2) = 0.24, \ r(t_3) = 0.1, \ r(t_4) = 0.25, \ r(t_5) = 0.15, \text{ and } r(t_6) = 0.06.
\]

Given this distribution of preferences, candidate $a$ is the Condorcet Winner as

\[
\sum_{t \in T_{a,b}} = r(t_1) + r(t_2) + r(t_5) > 1/2, \\
\sum_{t \in T_{a,c}} = r(t_1) + r(t_2) + r(t_3) > 1/2.
\]

We claim that there is a unique focal equilibrium $\{\sigma, \tau\}$ in which the Winner of the election does not coincide with the C.W. In this focal equilibrium, the strategy function $\sigma$ satisfies

\[
\sigma(a, b \mid t_1) = \sigma(a, c \mid t_2) = \sigma(b \mid t_3) = 1 \text{ and } \sigma(b \mid t_4) = \sigma(c \mid t_5) = \sigma(c \mid t_6) = 1.
\]

and then vote distribution $\tau = (\tau(c))_{c \in C}$ is such that

\[
\tau(a, b) = r(t_1), \quad \tau(b) = r(t_3) + r(t_4), \quad \tau(a, c) = r(t_2), \text{ and } \tau(c) = r(t_5) + r(t_6).
\]

Given the vote distribution, the vote profile $x = (x(c))_{c \in C}$ is the following vector of random variables

\[
x(a, b) \sim \mathcal{P}(0.2n), \ x(b) \sim \mathcal{P}(0.35n), \ x(a, c) \sim \mathcal{P}(0.24n) \text{ and } x(c) \sim \mathcal{P}(0.21n).
\]
The expected share of votes for each candidate represented by the score distribution \( \rho = (\rho(k))_{k \in K} \) is such that

\[
\rho(b) = \tau(a, b) + \tau(b) = 0.55, \quad \rho(c) = \tau(a, c) + \tau(c) = 0.45
\]

and \( \rho(a) = \tau(a, b) + \tau(a, c) = 0.44 \),

which implies that \( b \) is the Winner of the election. Given the score distribution, the score profile \( s = (s(k))_{k \in K} \) is such that

\[
s(b) = x(a, b) + x(b) \sim \mathcal{P}(0.55n), \quad s(c) = x(a, c) + x(c) \sim \mathcal{P}(0.45n)
\]

and \( s(a) = x(a, b) + x(a, c) \sim \mathcal{P}(0.44n) \).

Finally, the strategy function \( \sigma \) implies the following ordering of the magnitudes of pivot outcomes

\[
\mu[pivot(b, c)] = -0.005 > \mu[pivot(a, b)] = \mu[pivot(a, c)] = -0.01.
\]

Best responses for voters with types \( t_1, t_2, t_4 \) and \( t_6 \) are directly implied by the fact that \( pivot(b, c) \) has the highest magnitude:

\[
\sigma(a, b \mid t_1) = \sigma(a, c \mid t_2) = 1 \quad \text{and} \quad \sigma(b \mid t_4) = \sigma(c \mid t_6) = 1.
\]

However, the decision of voters with type \( t_3 \) and \( t_5 \) remains to be clarified. Their decision depends on the intensity of their preferences as the probabilities of the pivot outcomes are “not too different”, in the sense that they have the same magnitude. As we show, these voters only vote for their preferred candidate as there is a huge utility difference between their second and their preferred candidate.

Let us describe their decision problem in detail. A \( t_3 \)-voter decides between casting ballot \( b \) and ballot \( a, b \). In order to determine which of the ballots maximizes his expected utility he computes the sign of the following expression:

\[
\Delta = E[b \mid n\tau] - E[a, b \mid n\tau].
\]

As voters are instrumental, their expected utility uniquely depends on the pivot outcomes which implies:

\[
\Delta = (\frac{u_{t_3}(b) - u_{t_3}(a)}{2})P[x = pivot(a, b)] + (\frac{u_{t_3}(b) + u_{t_3}(c) - 2u_{t_3}(a)}{6})P[x = pivot(a, b, c)]
\]

\[
+ (\frac{u_{t_3}(c) - u_{t_3}(a)}{2})P[x = pivot(a, c)].
\]

Voters with type \( t_3 \) vote only for \( b \) rather than for \( a \) and \( b \) if and only if \( \Delta > 0 \). Replacing the utility values in the expression \( \Delta \), we can write that the condition \( \Delta > 0 \) is
equivalent to
\[
\frac{99}{2} P[x = pivot(a, b)] + \frac{98}{6} P[x = pivot(a, b, c)] > \frac{1}{2} P[x = pivot(a, c)]
\]
\[
\iff 297 P[x = pivot(a, b)] + 98 P[x = pivot(a, b, c)] > 3 P[x = pivot(a, c)].
\]

In order to show that this condition holds, we need to estimate the probabilities of the three pivot outcomes as their magnitudes coincide. As Myerson [5] remarks, these pivot-probability formulas can be derived from mathematical formulas involving Bessel functions. When the number of votes for two given ballots \(c\) and \(c'\) are two independent Poisson random variables with means \(n\tau(c)\) and \(n\tau(c')\), respectively, the probability that ballot \(c\) gets exactly \(k\) more votes than ballot \(c'\) is
\[
e^{-n(\tau(c)+\tau(c'))} \left(\frac{\tau(c)}{\tau(c')}\right)^{k/2} I_k \left(2n\sqrt{\tau(c)\tau(c')}\right)
\]
where \(I_k\) is a modified Bessel function; see formula 9.6.10 in Abramowitz and Stegun [1].

Using estimations based on these formulas (included in the Appendix), we show that the sign of \(\Delta\) is positive. The following matrix presents the probabilities of the pivot outcomes in which the parameter \(n\) represents the expected number of voters in the game.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(P[x = pivot(a, b)])</th>
<th>(P[x = pivot(a, b, c)])</th>
<th>(P[x = pivot(a, c)])</th>
<th>(\text{sign}(\Delta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.0138</td>
<td>0.0031</td>
<td>0.0063</td>
<td>+</td>
</tr>
<tr>
<td>1000</td>
<td>2.94 \times 10^{-7}</td>
<td>2.49 \times 10^{-8}</td>
<td>7.07 \times 10^{-7}</td>
<td>+</td>
</tr>
<tr>
<td>10000</td>
<td>6.59 \times 10^{-49}</td>
<td>3.95 \times 10^{-50}</td>
<td>9.14 \times 10^{-50}</td>
<td>+</td>
</tr>
<tr>
<td>100000</td>
<td>6.22 \times 10^{-459}</td>
<td>2.34 \times 10^{-460}</td>
<td>7.06 \times 10^{-460}</td>
<td>+</td>
</tr>
<tr>
<td>1000000</td>
<td>4.60 \times 10^{-4551}</td>
<td>1.68 \times 10^{-4552}</td>
<td>5.06 \times 10^{-4552}</td>
<td>+</td>
</tr>
</tbody>
</table>

These estimations entail that \(t_3\)-voters’ best response is to vote only for their preferred candidate. Similarly, it can be shown that \(t_5\)-voters’ best response is to vote only for their preferred candidate, which shows that the pair \(\{\sigma, \tau\}\) is a focal equilibrium.

Let us now assume that there exists a focal equilibrium \(\{\sigma, \tau\}\) in which the most probable pivot outcome is \(pivot(a, b)\). Then, by Corollary 2, if there exists a focal equilibrium in which the Condorcet Winner (candidate \(a\)) is included in the pivot with the highest magnitude, then he must be the Winner of the election. Let us assume that the magnitudes of pivot outcomes are ordered as follows
\[
\mu[pivot(a, b)] > \mu[pivot(a, c)] = \mu[pivot(b, c)].
\]
Given this ordering of the magnitudes, the strategy function \(\sigma\) satisfies:
\[
\sigma(a \mid t_1) = \sigma(a, c \mid t_2) = \sigma(b \mid t_3) = 1 \text{ and } \sigma(b, c \mid t_4) = \sigma(a, c \mid t_5) = \sigma(b, c \mid t_6) = 1.
\]

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Both \( t_2 \) and \( t_4 \)-voters vote for candidate \( c \) as the pivot outcome between candidates \( a \) and \( c \) has the same magnitude than the pivot between candidates \( b \) and \( c \). Indeed, by similar reasonings to the ones involving estimations previously presented, it can be shown that both types of voters vote for their second preferred candidate. Therefore, the vote distribution \( \tau \) is such that

\[
\tau(a) = 0.2, \quad \tau(b) = 0.1, \quad \tau(a,c) = 0.39, \quad \text{and} \quad \tau(b,c) = 0.31,
\]

which implies that the score distribution \( \rho = (\rho(k))_{k \in K} \) satisfies

\[
\rho(c) = 0.7 > \rho(a) = 0.59 > \rho(b) = 0.41.
\]

Hence, candidate \( a \) is not the Winner of the election which is a contradiction as candidate \( a \) is involved in the pivot with the highest magnitude. Furthermore, assuming a strict ordering of the magnitudes leads to the same type of contradiction. Hence, there does not exist a focal equilibrium in which the outcome \( \text{pivot}(a,b) \) has the highest magnitude.

Similarly, if we assume that the most probable pivot outcome in a large equilibrium is \( \text{pivot}(a,c) \), we can show that candidate \( b \) is expected to have a strictly higher share of votes than candidate \( a \), contradicting the fact that there exists such a focal equilibrium.

### 5.1 Large Non-focal Equilibria

We have previously shown that there does not exist a focal equilibrium in which the Condorcet Winner \( a \) can win the election. We now focus on the rest of large equilibria of this game and show that the Condorcet Winner is not the Winner at any of them.

Let us first assume that the magnitudes of the pivot outcomes are ordered as follows

\[
\mu[\text{pivot}(a,b)] = \mu[\text{pivot}(a,c)] > \mu[\text{pivot}(b,c)].
\]

Given this ordering of the magnitudes, the strategy function \( \sigma \) satisfies

\[
\sigma(a \mid t_1) = \sigma(a \mid t_2) = 1 \quad \text{and} \quad \sigma(b,c \mid t_4) = \sigma(b,c \mid t_6) = 1.
\]

The voters with type \( t_1 \) and \( t_2 \) do not vote for their respective second preferred candidates \( b \) and \( c \) as the most probable pivot in which they are involved is against candidate \( a \). Similarly, voters with type \( t_4 \) and \( t_6 \) vote for their respective second best candidates \( b \) and \( c \) as the most probable pivot in which they are involved is against candidate \( a \). The previous ordering of magnitudes does not specify enough information to determine \( t_3 \) and \( t_5 \)-voters’ best responses. It can be the case that such a voter plays a mixed strategy over a ballot representing their preferred candidate and a ballot involving his two
preferred candidate or that he plays a pure strategy and votes uniquely for his preferred candidate. This decision depends on the probabilities of the pivot outcomes. Formally, the best-response for these types of voters is equal to:

\[ \sigma(b \mid t_3) = p, \quad \sigma(a, b \mid t_3) = 1 - p \text{ and} \]
\[ \sigma(c \mid t_5) = q, \quad \sigma(a, c \mid t_5) = 1 - q \text{ for some } p \text{ and } q \text{ s.t. } 0 \leq p, q \leq 1. \]

Given the strategy function \( \sigma \), the vote distribution \( \tau \) is such that

\[ \tau(a) = r(t_1) + r(t_2), \quad \tau(b) = pr(t_3), \quad \tau(c) = qr(t_5), \]
\[ \tau(a, b) = (1 - p)r(t_3), \quad \tau(a, c) = (1 - q)r(t_5) \text{ and } \tau(b, c) = r(t_4) + r(t_6). \]

In order to prove that the pair \( \{\sigma, \tau\} \) is not a large equilibrium, we need to show that voters’ best responses cannot imply a magnitude ordering equal to

\[ \mu[pivot(a, b)] = \mu[pivot(a, c)] > \mu[pivot(b, c)]. \]

Indeed, given the pair \( \{\sigma, \tau\} \), the expected share of votes for each candidate represented by the score distribution \( \rho = (\rho(k))_{k \in K} \) is such that

\[ \rho(a) = \tau(a) + \tau(a, b) + \tau(a, c), \quad \rho(b) = \tau(b) + \tau(a, b) + \tau(b, c) \]
\[ \text{and } \rho(c) = \tau(c) + \tau(a, c) + \tau(b, c). \]

Furthermore, the MET and the DMT entail that the magnitude of the outcome \( pivot(a, b) \) coincides with the optimal value of

\[ \tau(a) \exp[\lambda_1 - \lambda_2 + \lambda_3] + \tau(a, b) \exp[\lambda_3] + \tau(b) \exp[-\lambda_1 + \lambda_2] \]
\[ + \tau(b, c) \exp[-\lambda_1 + \lambda_2 - \lambda_3 + \tau(c)] \exp[-\lambda_3] + \tau(a, c) \exp[\lambda_1 - \lambda_2] \]

such that \( \lambda_i \geq 0 \forall i \). Similarly, the magnitude of the outcome \( pivot(a, c) \) coincides with the optimal value of

\[ \tau(a) \exp[\lambda_1 - \lambda_2 + \lambda_3] + \tau(a, b) \exp[\lambda_1 - \lambda_2] + \tau(b) \exp[-\lambda_3] \]
\[ + \tau(b, c) \exp[-\lambda_1 + \lambda_2 - \lambda_3 + \tau(c)] \exp[\lambda_1 - \lambda_2] + \tau(a, c) \exp[\lambda_3] \]

such that \( \lambda_i \geq 0 \forall i \). Both optimal values coincide and are higher than \( \mu[pivot(b, c)] \) if and only if

\[ \tau(a, b) = \tau(a, c) \text{ and } \tau(b) = \tau(c). \]

The previous equality implies that the expected score of candidate \( b \) is equal to the expected score of candidate \( c \), that is

\[ \tau(a, b) + \tau(b) = \tau(c) + \tau(a, c) \iff \rho(b) = \rho(c). \]
However, this is an contradiction as given the strategy function, the expected score of candidates $b$ is strictly lower than the expected score of candidate $c$,

\[ r(t_3) = 0.1 \text{ and } r(t_5) = 0.15 \]
\[ \implies \rho(b) = r(t_3) + r(t_4) + r(t_6) < \rho(c) = r(t_4) + r(t_5) + r(t_6). \]

Therefore, there does not exist such a large equilibrium. Similarly, one can prove that there does not exist large equilibria in this game such that the magnitudes are ordered as follows,

\[ \mu[pivot(i, j)] = \mu[pivot(i, k)] > \mu[pivot(j, k)]. \]

Finally, the case in which the three two-candidate pivot outcomes have the same magnitude remains to be addressed. However, in such a situation, the expected scores of the three candidates coincide. Indeed, the respective optimal values of the constrained minimization problems stated by the $MET$ and the $DMT$ coincide if and only if $\rho(a) = \rho(b) = \rho(c)$. Hence, in this Poisson Approval Voting game, there are two large equilibria: a focal one in which candidate $b$ wins the election and a non-focal one in which the three candidates get the same expected score.

### 6 Conclusion

This work analyses Large Poisson Games, one of the main models that have incorporated the role of information into large elections and tried to understand its strategic properties through the perspective of $AV$. The main focus is on focal equilibria that have been shown to lead to desirable outcomes in other Large election models.

This work states that $AV$ induces a refinement on the set of focal equilibria when compared with Plurality voting in this setting. This result implies that $AV$ is more robust with respect to focal manipulation. However, this work also shows that this refinement does not preclude paradoxical situations from arising. It could be the case that Condorcet Winner does not win the election in any large equilibrium of the game or that insincere behavior is an equilibrium under $AV$. The source of these undesirable outcomes seems to be the correlation between the scores of candidates that exists under $AV$. Voters vote given the scores of the candidates. However, on Large Poisson Games the number of voters who choose a given ballot is independent from the number of voters who choose another given ballot. This two-level system (the score and the vote profile) makes more difficult the computation of the probabilities of pivot outcomes and generates paradoxical equilibria.
References


7 Appendix

This appendix provides the formulas used for the estimations in the Poisson Approval Voting game in which there is a unique focal equilibrium. These estimations concern the decision problem of $t_3$-voters. As previously discussed, the decision of this type of voters boils down to the sign of the following expression.

$$\Delta = E[b \mid n\tau] - E[a, b \mid n\tau].$$

As voters are instrumental, their expected utility uniquely depends on the pivot outcomes which implies:

$$\Delta = \left(\frac{u_{t_3}(b) - u_{t_3}(a)}{2}\right)P[x = pivot(a, b)] + \left(\frac{u_{t_3}(b) + u_{t_3}(c) - 2u_{t_3}(a)}{6}\right)P[x = pivot(a, b, c)]$$

$$+ \left(\frac{u_{t_3}(c) - u_{t_3}(a)}{2}\right)P[x = pivot(a, c)].$$
in which

\[ P[x = \text{pivot}(a, b)] = P[x = (s(a) = s(b) \geq s(c) + 1)] + P[x = (s(a) = s(b) + 1 \geq s(c) + 1)] \]

\[ P[x = \text{pivot}(a, c)] = P[x = (s(a) + 1 = s(c) \geq s(b) + 2)] + P[x = (s(a) = s(c) \geq s(b) + 2)] \]

\[ P[x = \text{pivot}(a, b, c)] = P[x = (s(a) + 1 = s(b) + 1 = s(c))] + 2P[x = (s(a) = s(b) + 1 = s(c))]. \]

We estimate these probabilities using Modified Bessel functions. Indeed, when the number of votes for two given ballots \( c \) and \( c' \) are two independent Poisson random variables with means \( n\tau(c) \) and \( n\tau(c') \), respectively, the probability that ballot \( c \) gets exactly \( k \) more votes than ballot \( c' \) is

\[ e^{-n(\tau(c)+\tau(c'))} \left( \frac{\tau(c)}{\tau(c')} \right)^{k/2} I_k \left( 2n\sqrt{\tau(c)\tau(c')} \right) \]

where \( I_k \) is a modified Bessel function; see formula 9.6.10 in Abramowitz and Stegun [1].

Using such a result, we write that

\[ P[x = (s(a) = s(b) \geq s(c) + 1)] = P[(x(a, c) = x(b)) \cap (x(a, b) \geq x(c) + 1)] \]

\[ = P[(x(a, c) = x(b))]P[(x(a, b) \geq x(c) + 1)] \]

in which

\[ P[(x(a, c) = x(b))] = e^{-n(\tau(a,c)+\tau(b))} I_0 \left( 2n\sqrt{\tau(a,c)\tau(b)} \right), \]

and

\[ P[(x(a, b) \geq x(c) + 1)] = \sum_{k=1}^{\infty} e^{-n(\tau(a,b)+\tau(c))} \left( \frac{\tau(a,b)}{\tau(c)} \right)^{k/2} I_k \left( 2n\sqrt{\tau(a,b)\tau(c)} \right). \]

Similar reasonings lead us to write that the probabilities of the pivot outcomes are such that

\[ P[x = \text{pivot}(a, b)] = e^{-n} \left[ I_0 \left( 2n\sqrt{\tau(a,c)\tau(b)} \right) + \left( \frac{\tau(a,c)}{\tau(b)} \right)^{1/2} I_1 \left( 2n\sqrt{\tau(a,c)\tau(b)} \right) \right] \]

\[ \times \left[ \sum_{k=1}^{\infty} \left( \frac{\tau(a,b)}{\tau(c)} \right)^{k/2} I_k \left( 2n\sqrt{\tau(a,b)\tau(c)} \right) \right] \],

\[ P[x = \text{pivot}(a, c)] = e^{-n} \left[ I_0 \left( 2n\sqrt{\tau(a,b)\tau(c)} \right) + \left( \frac{\tau(c)}{\tau(a,b)} \right)^{1/2} I_1 \left( 2n\sqrt{\tau(a,b)\tau(c)} \right) \right] \]

\[ \times \left[ \sum_{k=2}^{\infty} \left( \frac{\tau(a,c)}{\tau(b)} \right)^{k/2} I_k \left( 2n\sqrt{\tau(a,c)\tau(b)} \right) \right] , \]

and

\[ P[x = \text{pivot}(a, b, c)] = e^{-n} \left[ I_0 \left( 2n\sqrt{\tau(a,b)\tau(c)} \right) \left( \frac{\tau(a,c)}{\tau(b)} \right)^{1/2} I_1 \left( 2n\sqrt{\tau(a,b)\tau(c)} \right) \right]. \]