Robust weighting schemes of multidimensional poverty attributes

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Abstract

This paper describes how to obtain a robust weighting scheme of well-being indicators to arrive at the least (highest) possible multidimensional poverty for a given population when we have a given set of pre-determined normative weights as a benchmark. In particular, we test whether the allocation of weights to each well-being dimension for assessing the poverty level of a given population are robust or whether different weighting schemes would have offered a highest (lowest) possible multidimensional poverty to that population. Identification of poor through the choice of dimension specific poverty lines and setting weights to different dimensions may lead to different poverty levels and a reversal of poverty assignments across populations. We offer a robust weighting scheme to the attributes of well-being which can equally well be applied to union, intersection or intermediate identification approaches when dealing with multidimensional indicators of poverty. We derive a robust weighting scheme for which multidimensional poverty is highest (lowest) for a given deprivation level. Moreover, different set of dimension specific poverty lines can be chosen where multidimensional poverty is driven by only a set of dimensions.

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where poverty lines above these levels can no longer be considered in the multidimensional poverty comparisons. We illustrate our methodology through multidimensional poverty analysis in different population groups in Kenya and Canada.

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1 Introduction

Traditionally, poverty is measured by income or consumption per person. Income or consumption per person cut-off values are determined and if a person is below the cut-off value, then that person is considered as poor. However, there is a widespread agreement that poverty is a multidimensional concept. In this respect, Sen’s (1985, 1987, 1992, 1993) writings played an important role to move forward and promote multidimensional poverty analysis. Sen (1987) argued that income itself cannot capture all possible dimensions of well-being. It is suggested that markets are imperfect, especially in developing countries and furthermore, many public goods such as education and health care as well as human rights notions for individuals are often offered outside the market place. Hence, income and/or consumption per person does not capture many important dimensions that lead people to poverty, such as housing, literacy, life expectancy and so on.

There has been a variety of ways to arrive at a measurement of multidimensional well-being of countries. One of the best-known examples is the Human Development Index (HDI) of the United Nations Development Programme (UNDP) (1990), which uses a weighted average of life expectancy, education level and GDP per capita for a given population. Moreover, the Human Poverty Index (HPI) is a measure that attempts to capture the some dimensions of poverty that exist in both poor and rich countries relying on fixed equal weights as well. The HPI-1-human poverty index for developing countries- measures human deprivations in the same three categories as HDI by longevity, knowledge and a decent standard of living and HPI-2, human poverty index for selected high-income OECD countries- includes in addition to the three dimensions in HPI-1, social exclusion.

More recently, the multidimensional poverty index (MPI) of the Oxford Poverty and Human Development Initiative (OPHI) and UNDP measures the intensity of poverty at the micro level (i.e., household and individuals in that household) in education, health outcomes, and standard of living. The MPI has three dimensions: health, education, and standard of living and each dimension has a different number of indicators. Multidimensional poverty measures of
developing countries are obtained by using the Alkire and Foster (2011), AF hereafter, methodology, whereby each dimension and each indicator within a dimension is equally weighted. A household is considered to be multidimensionally poor if the household is deprived in at least 30% of the weighted indicators.

Multidimensional poverty measurement requires two initial steps, that is identification and aggregation (Sen, 1976). In identification, one needs to determine the multidimensionally poor individual (or household), something that requires two cut-offs. Firstly, a dimension-specific poverty line is required to define the deprived individuals (households) in that particular dimension. The second cut-off determines the number of dimensions, that one has to be deprived to be considered as multidimensionally poor. For the second cut-off, Tsui (2002) and Bourguignon and Chakravarty (2003) choose a union approach, whereby any individual who is deprived in at least one dimension, is considered to determine the multidimensionally poor. On the other hand, the intersection approach states that any individual deprived in all dimensions is considered to be multidimensionally poor. AF suggested that when the number of dimensions is large, the union approach will often identify most of the population as being poor, whereas the intersection approach will often identify very few of the population as being poor. As a result, they provide a new class of multidimensional poverty measures based on the Foster-Greer-Thorbecke (FGT) family (Foster et al., 1984) by introducing an intermediate approach. Finally, given the dimension-specific poverty lines (i.e., first identification step), the cross dimension cut-off (i.e., second identification step) is determined after each dimension being assigned a set of weights prior to aggregation. Therefore, as in the case of unidimensional poverty measurement, there are various judgement calls that need to be made such as the selection of poverty dimensions, choice of aggregation methods, poverty lines for each dimension and the weights assigned to each dimension prior to the aggregation. In this paper, we mainly focus on the latter issue.

Traditionally, the above mentioned multidimensional poverty analysis of multiple attributes often assigns arbitrary weights to each attribute before aggregation. A serious shortcoming is that different weights given to each indicator prior to aggregation may lead to different multidimensional poverty levels and contradictory poverty rankings across populations. In this paper, we propose robust poverty frontiers with differential weights to each dimension which offers the highest (lowest) possible multidimensional poverty with respect to all possible weighting schemes on each attribute.

Our paper extends the application of stochastic dominance, SD hereafter, analysis to poverty comparisons and follows the idea first proposed by Atkinson (1987) and Foster and Shorrocks

\footnote{See Decanq and Lugo, 2013, for a detailed set of weights used in the multidimensional well-being composite indices, categorized under three groups: data-driven, hedonic and normative.}
(1988) for the one-dimensional case. The SD approach is widely used in poverty comparisons in the literature for both univariate and multivariate cases. In HDI, HPI and MPI, each dimension is attached an equal weight to give equal importance to each dimension (see Esposito and Chiappero-Martinetti, 2010, for discussion on hierarchical ordering of different multidimensional poverty dimensions through weight assignment). However, choosing a normalization procedure (i.e., transformation of dimensions) or dimension-specific poverty lines imposes implicit weights when constructing the overall index of well-being or poverty (see Ravallion, 1997; Noorbakhsh, 1998; Ravallion 2010; Decancq and Lugo, 2013)\textsuperscript{2}. One explanation is that the normalization procedure of the raw components in one dimension or setting dimension-specific poverty lines lower allows countries to achieve targeted goals relatively faster in some dimension than in the others. This is the optimistic poverty scenario, where the upper and lower bounds of the raw components in a given dimension are set in such a way as to reach a higher implicit weight in that dimension or a dimension-specific poverty line is chosen that is easier for individuals or households to reach. The opposite is true when the most pessimistic poverty scenario (highest possible poverty) is considered (e.g., the dimension-specific poverty lines may be high when compared to other well-being dimensions). Our paper is an extension of the stochastic dominance efficiency (SDE) approach which has been applied to optimal portfolio construction in finance. Scaillet and Topaloglou (2010), hereafter ST, use SDE tests that compare a given portfolio with an optimal diversified portfolio constructed from a set of assets through the use Kolmogorov-Smirnov type of test statistic. In the context of HDI, Pinar et al. (2013) apply the same methodology to obtain a best-case scenario weighted HDI.

In this paper, the weights derived from SDE analysis can be thought of as explicit weights that lead to the most optimistic poverty scenario (least poverty) or most pessimistic scenario (highest poverty). In this context, the dimension that gets relatively more weight is the one in which most people (household) in a given population realize higher relative levels of measured well-being (lower poverty) or lower levels of measured well-being (higher poverty). Therefore, SDE analysis sheds light on the weights given to the constituent well-being dimensions through their construction from raw components or through the identification of multidimensionally poor with the different sets of dimension-specific poverty lines and across dimensional weights. In this sense, SDE analysis can be considered as an assessment tool of the different dimensions that are being used for relative poverty comparisons.

Assigning weights to each dimension to arrive at the most optimistic best-case scenario (or most pessimistic worst-case scenario) that describes the level of poverty across people based on

\textsuperscript{2}See Decancq and Lugo, 2013, for how a choice of weighting schemes given to dimensions and transformation of each dimension affect the the trade-offs between dimensions in the composite wellbeing indices.
SD analysis has a number of advantages. Firstly, it provides a poverty (or well-being) measure resulting from the least variable combination of dimensions that maximizes the measured level of well-being (equivalently, minimizes the level of poverty) for a group of population. When the poverty measure (well-being index) is bounded (e.g., normalized poverty gaps and the weighted deprivation levels which are between 0 and 1), higher measured poverty levels for more people describe a distribution that is negatively skewed resulting in less variability across people. Secondly, economic theory is agnostic in terms of offering us strong guidance about the functional form of preferences and distributions of the different components of poverty, which makes SD attractive since it is nonparametric, in the sense that its criteria do not impose explicit functional form requirements on individual preferences or restrictions on the functional forms of probability distributions.

Our paper proposes answers to such questions: “Given the dimension-specific poverty lines, what is the weighting scheme of different dimensions where poverty takes its highest (lowest) level for a given weighted deprivation level when compared with all other possible weighting schemes?”; “Given the dimension-specific poverty lines, what is the weighting scheme of dimensions that will offer the highest (lowest) possible normalized poverty gaps and intensity of poverty?”; “If one were to choose different dimension-specific poverty lines, what is the dimension-specific poverty line that no matter the weighting scheme of dimensions, it offers the highest (lowest) possible multidimensional poverty” and so on.

In this paper, we provide two illustrative empirical applications which highlight the importance of the weighting scheme in both identification stages of the multidimensionally poor. In the first one, we will examine whether pre-determined equal weights assign the highest (lowest) possible multidimensional headcount ratio for all possible weighted sum of deprivations when compared with all possible weighting schemes by analyzing the distribution of MPI of the northeast region of Kenya in 2003. In this case, for a given weighted sum of deprivation, we will find the weighting scheme of dimensions which offers the highest (lowest) multidimensional headcount ratio. This weighting scheme captures information about the contribution of each dimension to the highest (lowest) multidimensional headcount ratio, something that is useful to policy makers in their quest to uncover the dimensions that contribute most (least) to the multidimensional headcount ratio.

The pre-determined dimension-specific poverty lines and weights to each dimension offer a FGT measure of average normalized poverty gaps for a given population. In this case, given the dimension-specific poverty lines, one can also obtain robust weighting schemes of different dimensions for a given normalized poverty gap, and poverty severity (i.e., population that has a higher normalized poverty gap) by employing different orders of stochastic dominance. For
example, let us consider two dimensions, income and education with poverty lines of $10,000 and 9 years of schooling respectively. If we follow an arbitrary aggregation approach, then the normalized poverty gap aggregation takes place by adding normalized poverty gaps in each dimension by using arbitrary assigned weights. In this paper, we can obtain robust weighting schemes that offer the highest (lowest) number of individuals who have a normalized poverty gap above a given normalized poverty gap for the population in question. Therefore, one can assess the importance of each dimension’s contribution to multidimensional poverty and policy makers can concentrate on improving these dimensions that contribute the most to the highest multidimensional poverty. In our second example, we will apply our methodology to individuals from different urban population sizes for different levels of poverty lines using Canadian data.

In the next section, we present a literature review of the different methods of multidimensional poverty measurement, and we offer the identification stages and multidimensional poverty measures. In section 3, we outline the proposed methodology that we will use. We also provide a characterization of the limiting distributions of the test statistics under the null hypothesis, asymptotic properties of the tests. In this section, we also provide estimation of robust poverty frontiers for different possible cases. In Section 4, we give details of bootstrapping methods used to compute p-values for testing SD of order $s$. Section 5 presents the mathematical formulations for numerical implementation of first-order SDE. In section 6 we present an application using real-world data from the northeastern region of Kenya collected from the Kenyan Demographic and Health Survey 2003 and Canadian poverty collected from Survey of Labour and Income Dynamics 2006. Finally section 7 concludes.

2 Identification and poverty measures

Initially, existing attempts to measure multidimensional poverty were to aggregate various attributes to a single index by assigning arbitrary weights to well-being dimensions. Bourguignon and Chakravarty (2003), BC hereafter, assign poverty lines for each dimension and identify a person poor if he/she falls below a poverty line at least in one dimension. Multidimensional poverty aggregation takes place after defining the poor and assigning different weights to each dimension. BC extend the FGT poverty measures to the multidimensional case where some parameters are set to capture risk aversion to poverty and substitutability or complementarity among dimensions.

When one considers the multidimensionally poor, there are different ways of identifying the poor. One way is the “union” approach where the person is considered multidimensionally poor if he/she is deprived at least in one dimension (see, e.g., Tsui, 2002). On the other hand,
the “intersection” approach considers a person as poor if he/she is deprived in all dimensions. AF stated that if the number of dimensions taken into consideration is too large, then both the “union” and “intersection” approaches will yield extreme cases. In the case of the “union” approach, if a person is deprived in only one dimension among many he/she will be considered as poor, even though that person may not be poor since he/she is not deprived in all other dimensions. As such the “union” approach will result in producing extreme multidimensional poverty levels. On the other hand, the “intersection” approach will result in lower levels of multidimensional poverty, since people deprived in nine dimensions among ten will be considered multidimensionally non-poor.

Assigning arbitrary or normative weights to the dimensions in the aggregation is the main focus of this paper, since having two valid assignments of weights to well-being dimensions could lead to different conclusions for a given population or poverty rankings across populations. Not only defining different poverty lines to each dimension (i.e., dimension-specific cut-offs) but also choosing different weights for each dimension would suggest a different set of poverty outcomes (i.e., different number of individuals that are classified as being poor, different set of normalized poverty gaps and intensity of poverty for a given population). For example, if one were to set different across dimension cut-off value than the pre-determined weighted sum of deprivation in MPI (e.g., 40% or 60% rather than 20% or 30%), this may suggest that multidimensional poverty is driven by some other dimensions and/or produce contradictory poverty rankings across different populations\(^3\). Moreover, normative weights prior to aggregation do assume the existence of substitution rates between the different well-being dimensions. Therefore, there has been extensions to the family of FGT poverty measures through a parameterization which accounts for different levels of substitution across different dimensions\(^4\).

AF propose a multidimensional poverty measurement that can be applied to the “intermediate” levels. Therefore, there are two cut-off values in AF approach: 1) \(l\), dimension cut-off values, below that level, a person is deprived in that dimension 2) \(k\), cut-off value for across dimensions, then the person is identified as multidimensionally poor if the person is deprived in more than \(k\) weighted sum of dimensions. One disadvantage of the AF approach is that after the definition of multidimensionally poor for a given \(l\) and \(k\), the individuals who are deprived less than \(k\) weighted sum of deprivation are censored before aggregation and assigned “0” level of deprivation. In this case, individuals that are deprived in some dimensions but less

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\(^3\)For example, BC use different weighting schemes to each dimension, whereas, Bennett and Mitra (2013) use different dimension specific weights to define the multidimensionally poor. The former study results in different poverty levels and the latter produces a reversal of poverty across populations.

\(^4\)See, for example, Tsui, 2002; Bourguignon and Chakravarty, 2003; Maasoumi and Lugo 2008; Alkire and Foster; 2011 for extensions of FGT family of poverty measures.
than pre-determined $k$ level are treated the same as individuals who experience no deprivation at all. As such, this censoring stage may alter results in the decomposition stage where each dimension’s contribution to total poverty is obtained (e.g., it is possible that all individuals who were deprived below $k$ are deprived in one dimension, and therefore after censoring the data, one may conclude that there is less of contribution of that dimension to overall poverty, yet this conclusion may be due to censoring). Moreover, the censoring stage also distorts the empirical distribution of deprivation levels of a given population. In our application, rather than censoring the deprivation levels, we offer the full empirical distribution of deprivation in a population to analyze the robust weighting schemes of dimensions at different sections of the empirical distribution.

UNDP and OPHI released the MPI in 2010 where the AF approach is applied to household surveys to obtain a MPI for 104 developing countries. The MPI is developed in the same spirit as the HDI, where there are three dimensions (education, health and standard of living), each dimension getting equal weight (i.e., $1/3$ each). Each dimension has also different number of indicators. For example, the education dimension consists of the years of schooling and child enrolment indicators, where each indicator is weighted equally (i.e., each indicator is contributing $1/6$ to the overall index). The health dimension has two indicators, child mortality and nutrition and each indicator is weighted equally (i.e., each indicator is contributing $1/6$ to the overall index). Finally, the standard of living dimension has six indicators: electricity, sanitation, drinking water, floor, cooking fuel and assets and each indicator is weighted equally (i.e., each indicator is contributing $1/18$ in arriving at the overall final index). Following the AF methodology, Alkire and Santos (2010) suggest that each indicator has a cut-off value where a person is deprived at that dimension if a household is below that cut-off value. Secondly, a person is considered multidimensionally poor if he/she is deprived in least 30% of the weighted indicators (see Alkire and Santos 2010 for details).

Weighting schemes assigned to each indicator and dimension in MPI is arbitrary and any weighting scheme other than equal weights may result in a different level of multidimensional poverty level for a given population. Moreover, the AF methodology censors the dimensions deprived by people who are not multidimensionally poor before aggregation across dimensions. If a person is deprived in only 10% or 20% of the weighted indicators, these deprived dimensions are not accounted for in the calculation of multidimensional poverty. It may be the case that the whole population is deprived in 10% or 20% of the weighted indicators, yet the AF method will end up assigning no multidimensional poverty to that specific population. In this paper, our contribution is to find robust weighting schemes that will produce the highest (lowest) possible multidimensional headcount ratio for a given weighted sum of deprivation rather than
relying on arbitrary weights given to each dimension before aggregation (see Tsui, 2002; Atkinson, 2003; Dutta et al., 2003; and Bossert et al., 2009 for different aggregation procedures). Hence, our paper will highlight the extent of poverty by offering the highest and lowest possible multidimensional poverty (i.e., lowest and highest possible well-being respectively) through the weighting scheme of each dimension, therefore inform policy makers of the dimensions which are mainly (least) responsible for multidimensional poverty.

In our paper we will apply SD to poverty comparisons in a multivariate setting. In this context, Davidson and Duclos (2000), DD hereafter, propose robust poverty lines for bivariate comparisons. In their work, for given income levels, they compare different populations and are able to ascertain whether one population is poorer than an other. However, DD looked at first and second order stochastic dominance using tests that rely on pairwise comparisons made at a fixed number of arbitrary chosen points. This is not a desirable feature since it introduces the possibility of test inconsistency. Barrett and Donald (2003), BD hereafter, on the other hand, offered consistent tests for SD that do not rely on fixed arbitrary points. Linton et al. (2005) propose a subsampling method which can deal with both dependent samples and dependent observations within samples. This is appropriate for conducting SD analysis with country panel data. The papers mentioned above applied SD to bivariate comparisons, whereas Duclos et al. (2006) propose multidimensional poverty frontiers for more than one dimension. In their work, one can find poverty frontiers for each dimension where one population is multidimensionally poorer than another one. For example, applying the Duclos et al. (2006) approach to multiple poverty dimensions, one can find poverty frontiers of each dimension that would conclude more poverty in population when compared with other. However, Duclos et al. (2006) did not allow for differential weights for each dimension. Moreover, in the same lines of DD, Duclos et al. (2006) apply tests that rely on pairwise comparisons made at a fixed number of arbitrary chosen points (e.g., Duclos et al. use a 20 by 20 grid spread evenly over the entire domain well-being dimensions). In a similar application, Bennett and Mitra (2013) propose robust poverty rankings of populations when different set of across dimensional cut-off values (i.e., second identification step, proportion of dimensions in which one must be deprived to be considered multidimensionally poor) are considered simultaneously by using a minimum p-value approach (see Bennett, 2010 for the minimum p-value approach). They provide comparisons of poverty for different groups of population and examined various identification steps (e.g., poverty cut-off for a given dimension or the dimensional weights) simultaneously.

In this paper, however, we will investigate whether the comparisons are robust to different weighting schemes assigned to each indicator prior to aggregation where we allow for full diversification of weights to each well-being dimension rather than normative weights. Moreover,
comparisons are made at all sections of the empirical distribution rather than relying on fixed set of arbitrarily chosen points. Finally, rather than the comparison of populations, we provide robust weighting scheme of multidimensional poverty levels of a given population. For example, given the dimension-specific cut-off values, we examine whether an arbitrary assigned weights to each dimension prior aggregation (e.g., equal weights) are robust or we can obtain set of weights that would offer the least and highest possible poverty levels for a given normalized poverty gap or severity (i.e., multidimensional poverty with FGT type of measures are tested). Furthermore, given the dimension-specific cut-offs, dimensional weights are also tested (i.e., AF type of multidimensional poverty measures are tested) for the ordinal or cardinal variables. In this case, we obtain weights for dimensions which will offer the least (highest) level of multidimensional poverty.

In the next subsection, we provide the formulation of multidimensional poverty through FGT family of poverty measures briefly and further sections will provide the SDE methodology to obtain the robust poverty frontiers.

2.1 Identification of poor and measuring multidimensional poverty

This section offers the definition and formulation of poor. Let us assume that there are $d$ dimensions of well-being. Let $Y = (Y_1, Y_2, ..., Y_d)$ denote a random population with joint distribution of achievements $F$. The well-being dimensions in $Y$ may be either ordinal or cardinal. Assume that some well-being dimensions being cardinal and some other being ordinal variables. Without loss of generality, all dimensions may be either ordinal or cardinal but let the first set of well-being dimensions, $0 \leq d_1 \leq d$ from random vector $Y$ are ordinal and remaining dimensions, $d - d_1$ are cardinal. For a given $0 < k \leq 1$ (level of weighted sum of deprived dimensions), given predetermined vector of poverty lines for each dimension, $l \in (0, l]^d$ and a vector of weights assigned to each dimension, denoted by $w$, we offer the multidimensional headcount ratio and generalized AF multidimensional measures following Bennett and Mitra (2013) as

$$H(l, k, w; F) = E_F \left[ \mathbb{I} \left( \sum_{j=1}^{d} w_j \mathbb{I}(Y_j \leq l_j) \geq k \right) \right],$$

and
\[ P_\alpha(l, k, w; F) = \mathbb{E}_F \left[ \mathbb{I} \left( \sum_{j=1}^d w_j \mathbb{I}(Y_j \leq l_j) \geq k \right) \left( \sum_{j=1}^{d_1} w_j \mathbb{I}(Y_j \leq l_j) \right) \right] \\
+ \mathbb{E}_F \left[ \mathbb{I} \left( \sum_{j=1}^d w_j \mathbb{I}(Y_j \leq l_j) \geq k \right) \left( \sum_{j=d_1+1}^d w_j \left( \frac{l_j - Y_j}{l_j} \right)^\alpha \mathbb{I}(Y_j \leq l_j) \right) \right]. \]

where \( \mathbb{I} \) denotes the indicator function (\( \mathbb{I}(Y_j \leq l_j) \geq k \) and \( \mathbb{I}(Y_j \leq l_j) \)) (Davidson and Duclos, 2000)\(^5\). Given the set of choices of \( l \) (i.e., dimension specific poverty lines), \( k \) (i.e., multidimensional dimension cut-off) and \( w \) (i.e., a vector of weights given to each dimension), an individual with observed vector of achievement \( Y = (Y_1, Y_2, ..., Y_d) \) is identified as multidimensionally poor if \( w_j \mathbb{I}(Y_j \leq l_j) \geq k \). Identification of multidimensionally poor individual involves a dual cut-off approach. The first step is to determine whether an individual is deprived in dimension \( j \) by comparing the level of achievement in that dimension to the corresponding poverty line (i.e., dimension-specific cut-off). The stage involves the identification whether an individual is deprived in weighted dimensions and considered as multidimensionally poor only if the weighted sum of the dimensions (i.e., sum of \( w \)) are at least equal to the multidimensional poverty threshold \( k \).

In this case, the use of equally weighted dimensions (e.g., HDI or MPI) suggest that the \( w \) equals to a vector of \( \frac{1}{d} \)'s when the weights to sum to 1. \( H(l, k, w; F) \) is the proportion of the population that is deprived in \( k \) proportion of dimensions if weights were standardized to sum to 1\(^6\). Whereas, the \( P_\alpha(l, k, w; F) \) measure severity of the poverty of the population. When \( \alpha = 1 \), it corresponds to the “poverty gap” and \( \alpha = 2 \) correspond to “poverty severity” (FGT-type measures). Greater the \( \alpha \) greater the emphasis being given to the poorest of the poor. Clearly, when \( \alpha = 0 \), the second term, \( P_\alpha(l, k, w; F) \), is the weighted sum of \( H(l, k, w; F) \).

For example, MPI has 3 dimensions (i.e., education, health and standard of living) and each dimension is given equal weights and its weights sum to 1 (i.e., \( w = (1/3, 1/3, 1/3) \)) and \( k = 0.3 \) (i.e., weighted sum is 30 percent or more of the dimensions). Furthermore, education and health dimensions have 2 indicators each (i.e., years of schooling and child school attendance for the education dimension; child mortality and nutrition for health dimension) and each indicator within each dimension is weighted equally (i.e., \( 1/6 \)). The standard of living dimension has

\(^5\)See Bourguignon and Chakravarty, 2003 equation (18) for multidimensional poverty index in the form of CES-like with additional \( \theta \) parameter which sets the level of substitutability between shortfalls; the higher the \( \theta \), the lower the degree of substitutability. Through the paper, we assume \( \theta = 1 \).

\(^6\)Throughout the paper, we use standardized weights where all dimensions’ weights sum to 1.
six indicators: electricity (deprived if household has no electricity), sanitation (deprived if the household’s sanitation is not improved or it is improved but shared with other households), water (deprived if the household does not have any access to clean water or clean water is more than 30 minutes walking from home), floor (deprived if household’s main floor has dirt, sand or dung), cooking fuel (deprived if household cooks with dung, wood or charcoal) and assets (deprived if household does not own more than one of: radio, TV, telephone, bike, or motorbike, and do not own a car or tractor) and each indicator within dimension is equally weighted (i.e., 1/18). Thus a household being deprived in only one dimension (i.e., deprived in all indicators located under any dimension), or being deprived in one indicator in either health or education dimension and three indicators in standard of living dimension, is considered as multidimensionally poor. Therefore, a household being deprived in two to six indicators may be considered multidimensionally poor. Obviously, not only choice of weights to each dimension and indicator (i.e., \(w\)) but also \(k\) plays an important role in identifying the multidimensionally poor.\footnote{See Alkire and Santos (2010) for further discussion of MPI or empirical application of the paper.}

The above mentioned FGT-type of measures have the properties of measuring poverty for sub-groups or dimension specific contribution to overall poverty (i.e., decomposition of poverty), however the main contribution of this paper is to analyze how the choice of \(l, k, w, \alpha\) parameters in \(H(l, k, w; F)\) and \(P_\alpha(l, k, w; F)\) affects the multidimensional poverty of a given population. Therefore, the next section we test the robustness of weighting schemes assigned to each dimension for the multidimensional poverty of a given population. Furthermore, a discussion on how a change in either \(l\) (dimension specific poverty lines) or \(k\) (across dimension cut-off) or \(\alpha\) (poverty aversion parameter) play a role on robust weighting scheme of multidimensional poverty measurement.

### 3 Poverty frontiers based on SDE

We consider a process \(Y = (Y_1, Y_2, ..., Y_d)\) that denotes a random population with joint distribution of achievements taking values in \(\mathbb{R}^d\). These data correspond to observed values of \(d\) attributes of poverty. Let us consider a set of the pre-determined weights assigned to the attributes of poverty, \(w \in \mathbb{L}\) where \(\mathbb{L} := \{w \in \mathbb{R}_+^d : e'w = 1\}\) with \(e\) for a vector made of ones. This means that each attribute of poverty has nonnegative weight, and that the weights sum to one. We denote by \(F(y)\), the continuous cdf of \(Y = (Y_1, ..., Y_d)'\) at point \(y = (y_1, ..., y_d)'\) of a given population. In this respect, let us denote an alternative possible weighting scheme assigned to dimensions which sums to one but different than the pre-determined set of weights,
Let us denote by $G(l, k, w; F)$ the cdf of the weighted sum of the deprived dimensions \( \mathbb{1}(Y \leq l) \) at a given point $k$ given by

$$G(l, k, w; F) := \int _\mathbb{E} \{w' u \leq k\} dF(u)^8.$$

We denote by $a$ the largest scalar so that $G(l, a, w; F) = 0$ for all sets $w \in \mathbb{L}$, and $b$ the smallest scalar so that $G(l, b, w; F) = 1$ for all sets $w \in \mathbb{L}$. Then the interval made of the upper bound $a$ and the lower bound $b$ forms the narrowest common support of all distributions, and the nature of $\mathbb{E}$ and $\mathbb{L}$ ensures that the interval $[a, b]$ is finite.

In this paper, testing for multidimensional poverty in a given population involves hypothesis testing with the robust choice of weights for each level of $k$ when $\alpha = 0$ (i.e., weighted sum of deprived dimensions) for a given $l$ (i.e., tests are based on difference between $G(l, k, w; F)$ and $G(l, k, w^a; F)$).

Moreover, the tests of multidimensional poverty involves with the robust choice of weights for each measurement of deprivation. For example, $z = \left( \frac{l_j - Y_j}{l_j} \right)^\alpha$ when $\alpha > 0$ (i.e., poverty aversion parameter) for a given $l$ and $k$ where in this case, tests will be based on difference between $G(P_\alpha(l, k, w; F))$ and $G(P_\alpha(l, k, w^a; F))$. This would be the difference between the cdf’s of the poverty gaps with the predetermined weights and an alternative weights assigned to each dimension for a given $z$ when $\alpha = 1$. One could also change the poverty lines, $l$, for various dimensions simultaneously and find a robust weighting for each respective case which will allow one to test the choice of poverty lines in different dimensions.

### 3.1 Stochastic Dominance Efficiency

SD is a term which refers to a set of relations that may hold between distributions. A natural application of the SD efficiency approach arises when there exists already a set of weight assignment to each poverty attribute, as in the case of the official HPI and MPI. A given set of weights for each poverty attribute or index (e.g., equal weights to each dimension in MPI and equal weights to each sub-index for the official HPI) offers a certain level of poverty. However, any valid choice of different weights assigned to each dimension (or index) may not only correspond to different levels of poverty for some households, but also may decrease (increase) the overall level of measured poverty. Choosing among all possible weighting schemes is the focus of our paper. For example, we take the official MPI weights to each dimension to obtain the weighted sum of deprived dimensions of each household, $k$, as the benchmark and we apply the SD efficiency approach to derive weights assigned to each sub-dimension (i.e., education, health and standard of living) that result in the lowest possible measured level of well-being (highest possible poverty) among all possible alternatives when compared with this benchmark.

---

8 The matrix $u$ is a $dxn$ matrix $\mathbb{1}(Y \leq l)$ of zeros or ones.
In this case that corresponds to the most pessimistic weighting scheme of the poverty dimensions. To do so, the SD efficiency approach maximizes the distributional distance between the given (equally-weighted) sum of deprived dimensions, \((w)\), and any possible alternative \((w^*)\). SD efficiency analysis allows us to derive the worst-case scenario weighting scheme \((w^*)\), where more individuals (households) achieve the lowest measured well-being levels (or the highest measured poverty levels) than any possible alternative for a given \(k\) (i.e., sum of weighted deprived dimensions). Thus, we test whether a given set of weights to each attribute, \(w\), i.e., equal weights given to each poverty attribute, is the worst-case scenario, in the sense that it gives the minimum level of well-being (maximum poverty) at a given \(k\), or whether we can construct another set of attributes \(w^*\) (alternative weighting scheme) that dominates \(w\) at the first- or second-order of SD.

Similarly, given a set of weights to each dimension, we apply the SD efficiency approach to derive weights assigned to each sub-dimension that result in the highest possible measured level of well-being (lowest possible poverty) among all possible alternatives when compared with the equally-weighted benchmark. Obtaining the most optimistic weighting scheme of the poverty dimensions by employing SD efficiency analysis allows us to derive the best-case scenario weighting scheme \((w^{**})\), where more individuals (or households) achieve the highest measured well-being levels (or the lowest measured poverty levels) than any possible alternative for a given \(k\) (i.e., sum of weighted deprived dimensions)\(^9\).

It is worth noting that as \(k\) represents a deprivation level (i.e., socially ‘bad’ condition), when poverty distribution with pre-determined weights, \(w\), dominates the poverty distribution with any alternative weighting scheme, \(w\) offers the weighting scheme which would give the highest measured multidimensional poverty for a given population. In this case, \(w\) dominates any alternative weighting scheme, there is always more proportion of population that has a poverty level that is above the deprivation level of \(k\). Similarly, when poverty distribution with pre-determined weights, \(w\), is dominated by the poverty distribution with all alternative weighting scheme, \(w\) offers the weighting scheme which would give the lowest measured multidimensional poverty (or highest multidimensional wellbeing) for a given population. Therefore, dominating set of weights, \(w^*\), offers the worst-case scenario (i.e., highest multidimensional poverty), and dominated set of weights, \(w^{**}\), gives the best-case scenario (i.e., lowest multidimensional poverty) for a given deprivation level when compared to the pre-determined weighting scheme, \(w\).

---

\(^9\)Even though both \(w^*\) and \(w^{**}\) may take any possible weighting for a given argument \(k\), they are different from each other since the former dominates the distribution with the pre-determined weights and the latter is dominated by the distribution with the pre-determined weights. In order to make the distinction, we labelled them differently, yet each weighting scheme is obtained through full diversification of weights.
The distribution of the set $w^*$ dominates the distribution of $w$ stochastically at first order (SD1) if, for any argument $k$, $G(l, k, w; F) > G(l, k, w^*; F)$. If $k$ denotes the level of weighted sum of deprived dimensions$^{10}$, then the inequality in the definition means that the proportion of individuals (households) in distribution $w^*$ with level of weighted sum of deprived dimensions smaller than $k$ is not larger than the proportion of such individuals in $w$ and $U$ is any decreasing monotonic function of $k$ — i.e., $U'(k) \leq 0$. In other words, there is at least as high a proportion of weighted sum of deprivation in $w^*$ as in $w$. If $w^*$ dominates $w$ at first order, then there is always more poverty (less well-being) in $w^*$ than in $w$. Figure 3.1 displays the dominance of $w^*$ over $w$ for different levels of $k$ where for a given level of $k$, there is always an alternative weighting $w^*$, which may be different for different levels of $k$, that dominates the pre-determined weights in the first-order sense.

Analogously, when the distance between the cdf of weighted sum of deprivation with the pre-determined weights $w$, and the cdf of the weighted sum of deprivation with alternative weighting scheme $w^{**}$ is maximized, then distribution of the set $w$ dominates the distribution of $w^{**}$ stochastically at first-order (SD1) for any argument $k$, if $G(l, k, w^{**}; F) > G(l, k, w; F)$. For each $k$, the proportion of individuals (households) in distribution $w$ with level of weighted sum of deprived dimensions smaller than $k$ is not larger than the proportion of such individuals in $w^{**}$ and therefore, there is at least as high a proportion of weighted sum of deprivation in $w$ as in $w^{**}$. In this case, for a given level of $k$, there is always an alternative weighting $w^{**}$, which may be different for different levels of $k$, that is dominated by the the pre-determined weights in the first-order sense$^{11}$.

However, the cdf of the weighted sum of deprivation with the pre-determined weights, $G(l, k, w; F)$, is not necessarily dominated by or dominates the cdf of the weighted sum of deprivation with alternative weighting schemes $G(l, k, w^*; F)$ and $G(l, k, w^{**}; F)$ respectively, for all arguments of $k$. If this is the case for some arguments of $k$, then we estimate the poverty frontier $k_1$. There are two possible cases that this can happen, either the cdf of the weighted sum of deprivation with the pre-determined weights, $G(l, k, w; F)$, is cut from below or above by the cdf of the weighted sum of deprivation with alternative weighting scheme. If $G(l, k, w; F)$ is cut from below by another cdf curve of weighted sum of deprivation with alternative weighting scheme, then $G(l, k, w; F)$ is dominated by another cdf up to this point. Therefore, up to this point, distribution of the set $w^*$ dominates the distribution of $w$, $G(l, k, w; F) > G(l, k, w^*; F)$.

$^{10}$ $k$ lies between 0 and 1 representing that the individual (household) is deprived in no dimensions and deprived in all dimensions respectively.

$^{11}$ The graphical presentation of this case is similar to Figure 3.1 but in this case cumulative distribution with $w^{**}$ lies above the cumulative distribution with $w$ for all deprivation levels.
Figure 3.1: First Order Stochastic Dominance of the set $w^*$ over $w$ for different levels of $k$.

If the inequality $G(l, k, w; F) > G(l, k, w^*; F)$ holds for all values of $k$ up to $k_1$, then we have restricted first-order SD up to $k_1$. This is equivalent to the statement that the proportion of individuals (households) below the weighted sum of deprivation level, $k_1$, is always greater with weights, $w$ than with the weights, $w^*$, for any weighted sum of deprivation level not exceeding $k_1$. In other words, there is a higher proportion of the population having lower levels of deprivation with $w$ weights than with weights $w^*$ up to $k_1$ deprivation level. On the other hand, if $G(l, k, w; F)$ is cut from above by another cdf curve of weighted sum of deprivation with an alternative weighting scheme, then $G(l, k, w; F)$ dominates $G(l, k, w^{**}; F)$ up to this point. Therefore, up to this point, the cdf of the set $w^{**}$ is dominated by the cdf of $w$, $G(l, k, w^{**}; F) > G(l, k, w; F)$. In this case, the proportion of individuals (households) below the weighted sum of deprivation level, $k_1$, is always greater with weights, $w^{**}$ than with the
weights, \( w \), for any weighted sum of deprivation level not exceeding \( k \)\(^{12}\).

Let us define for \( k \leq b \):

\[
J_1(l, k, w; F) = G(l, k, w; F),
\]
\[
J_2(l, k, w; F) = \int_a^k G(l, u, w; F) \, du = \int_a^k J_1(l, u, w; F) \, du,
\]
\[
J_3(l, k, w; F) = \int_a^k \int_a^u G(l, v, w; F) \, dv \, du = \int_a^k J_2(l, u, w; F) \, du,
\]

and generally\(^{13}\)

\[
J_{s+1}(l, k, w; F) = \int_a^k J_s(l, u, w; F) \, du
\]

From equation (2) of DD, we know that

\[
J_s(l, k, w; F) = \int_a^k \frac{1}{(s-1)!} (k - u)^{s-1} dG(l, u, w; F),
\]

which can be rewritten as

\[
J_s(l, k, w; F) = \int_{\mathbb{R}^d} \frac{1}{(s-1)!} (k - w'u)^{s-1} \mathbb{1}\{w'u \leq k\} dF(u)
\]

The general hypotheses for testing SD efficiency of order \( s \) of \( w \), hereafter \( SD_s \), whether the cdf of the pre-determined weights, \( w \), dominates the cdf of an alternative weighting scheme, \( w^* \), for all weighted sum of deprivation levels, \( k \) (i.e., null hypothesis), or there exists an alternative weighting scheme with a cdf that dominates the cdf of the pre-determined weights for some levels of \( k \), can be written compactly as:

\[
H_0^s : J_s(l, k, w; F) \leq J_s(l, k, w^*; F) \text{ for all } k \in \mathbb{R} \text{ and for all } w^* \in \mathbb{L}, \quad (3.1a)
\]
\[
H_1^s : J_s(l, k, w; F) > J_s(l, k, w^*; F) \text{ for some } k \in \mathbb{R} \text{ or for some } w^* \in \mathbb{L}. \quad (3.1b)
\]

Similarly, we can write the general hypotheses for testing SD efficiency of order \( s \) of \( w \),

\(^{12}\)It is worth noting that the argument of the cdf in our case refers to the deprivation level, a “bad” thing, whereas in other applications of SD analysis, the argument is typically refers to a “good” such as income.

\(^{13}\)These multiple integrals of the cdf are not well defined if \( a \) or \( b \) are infinite. This is the case for instance when the support of \( F \) is \( \mathbb{R}^d \) instead of \( E \). Note however that we may consider any finite interval containing \( \{a, b\} \) and define the integrals accordingly. Hence if we take such an interval sufficiently large, this should not affect the stochastic ordering in practice.
hereafter $SD_s$, where we test whether the cdf of the pre-determined weights, $\mathbf{w}$, is dominated by the cdf of some alternative weighting scheme, $\mathbf{w}^{**}$, for all weighted sum of deprivation levels, $k$ (i.e., null hypothesis) and the corresponding alternative hypothesis can be written compactly as:

\[ H_{s0}^s : J_s(l, k, \mathbf{w}^{**}; F) \leq J_s(l, k, \mathbf{w}; F) \text{ for all } k \in \mathbb{R} \text{ and for all } \mathbf{w}^{**} \in \mathbb{L}, \]  
\[ H_{s1}^s : J_s(l, k, \mathbf{w}^{**}; F) > J_s(l, k, \mathbf{w}; F) \text{ for some } k \in \mathbb{R} \text{ or for some } \mathbf{w}^{**} \in \mathbb{L}. \]  

(3.2a)

(3.2b)

Under the first (second) null hypothesis $H_{s0}^s$ there is no cdf with the set of alternative weighting scheme, $\mathbf{w}^* (\mathbf{w}^{**})$ constructed from the set of attributes that dominates (is dominated by) the cdf of $\mathbf{w}$ at order $s$. In this case, the function $J_s(l, k, \mathbf{w}; F)$ is always lower (higher) that the function $J_s(l, k, \mathbf{w}^*; F) \ (J_s(l, k, \mathbf{w}^{**}; F))$ for all possible $\mathbf{w}^* (\mathbf{w}^{**})$ for any argument $k$. On the other hand, under the alternative hypothesis $H_{s1}^s$, we can obtain a set of attributes $\mathbf{w}^* (\mathbf{w}^{**})$ for some arguments $k$, where the function $J_s(l, k, \mathbf{w}; F)$ is greater (lower) than the function $J_s(l, k, \mathbf{w}^*; F) \ (J_s(l, k, \mathbf{w}^{**}; F))$. Thus, $\mathbf{w}$ is robust as long as its cdf dominates (or is dominated by) the cdf of all alternative weighting schemes for all arguments $k$. In that case one can unambiguously state that the pre-determined weights of the given dimensions result in the highest (lowest) possible multidimensional poverty for all arguments of $k$. Of course, the opposite will be true under the alternative hypothesis\(^{14}\).

The empirical counterpart is simply obtained by integrating with respect to the empirical distribution $\hat{F}$ of $F$, which yields:

\[ J_s(l, k, \mathbf{w}; \hat{F}) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(s-1)!} (k - \mathbf{w}' \mathbb{1}(Y \leq l))^{s-1} \mathbb{1}\{\mathbf{w}' \mathbb{1}(Y \leq l) \leq k\}, \]

and can be rewritten more compactly for $s \geq 2$ as:

\[ J_s(l, k, \mathbf{w}; \hat{F}) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(s-1)!} (k - \mathbf{w}' \mathbb{1}(Y \leq l))^{s-1}. \]

In this paper, we consider a given population where an empirical distribution of multidimensional poverty with the predetermined weights, $\mathbf{w}$, is compared with the empirical distribution of multidimensional poverty with any alternative weighting scheme, $\mathbf{w}^a$\(^{15}\).

\(^{14}\)The analysis so far has been done using poverty headcounts, an analogously similar approach can be followed using normalized poverty gaps, but it is omitted to conserve space.

\(^{15}\)It could be either an alternative one that dominates the empirical distribution of the multidimen-
The empirical counterpart of the multidimensional poverty is given by \( \mathcal{J}_s(l, k, w; \hat{F}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(s-1)!} (k - w^T Y_i)^{s-1}_+ \) and the characterization of the limiting distributions of the test statistics under the null hypothesis such that \( \sqrt{n}(\hat{F} - F) \) tends weakly to a mean zero Gaussian process \( B \circ F \) in the space of continuous functions on \( \mathbb{R}^n \) in \( C([0, \overline{k}]) \) where the space is the continuous function on \( [0, \overline{k}] \) for \( k_2 > k_1 \) (see Rio (2000) for the multivariate functional central limit theorem for stationary strongly mixing sequences). Therefore, one can derive the limiting behavior of above empirical counterpart using the Continuous Mapping Theorem (see Lemma 1 of BD or Lemma 2.1 of ST).

**Lemma 3.1** \( \sqrt{n}[\mathcal{J}_s(\cdot; \hat{F}) - \mathcal{J}_s(\cdot; F)] \) tends weakly to a Gaussian process \( \mathcal{J}_s(\cdot; B \circ F) \) with mean zero and covariance function given by:

- for \( s = 1 \):

\[
\Omega_1(l, k_1, k_2, w^1, w^2) := E[G(l, k_1, w^1; B \circ F)G(l, k_2, w^2; B \circ F)] = \sum_{n \in \mathbb{Z}} E \left[ \mathbb{I}\{w^1_\alpha(Y \leq l) \leq k_1\} \mathbb{I}\{w^2_\alpha(Y \leq l) \leq k_2\} - G(l, k_1, w^1; F)G(l, k_2, w^2; F), \right]
\]

- for \( s \geq 2 \):

\[
\Omega_s(l, k_1, k_2, w^1, w^2) := E \left[ \mathcal{J}_s(l, k_1, w^1; B \circ F)\mathcal{J}_s(l, k_2, w^2; B \circ F) \right] = \sum_{n \in \mathbb{Z}} \frac{1}{((s-1))!} E \left[ \left( k_1 - w^1_\alpha(Y \leq l) \right)^{s-1}_+ \left( k_2 - w^2_\alpha(Y \leq l) \right)^{s-1}_+ \right] - \mathcal{J}_s(l, k_1, w^1; F)\mathcal{J}_s(l, k_2, w^2; F),
\]

with \((k_1, k_2)' \in \mathbb{R}^2\) and \((w^1_\alpha, w^2_\alpha)' \in \mathbb{L}^2\).

When the data is i.i.d., the covariance kernel reduces to

for \( s = 1 \):

\[
\Omega_1(l, k_1, k_2, w^1, w^2) = E \left[ \mathbb{I}\{w^1_\alpha(Y \leq l) \leq k_1\} \mathbb{I}\{w^2_\alpha(Y \leq l) \leq k_2\} - G(l, k_1, w^1; F)G(l, k_2, w^2; F), \right]
\]

and for \( s \geq 2 \):

\[
\Omega_s(l, k_1, k_2, w^1, w^2) = \frac{1}{((s-1))!} E \left[ \left( k_1 - w^1_\alpha(Y \leq l) \right)^{s-1}_+ \left( k_2 - w^2_\alpha(Y \leq l) \right)^{s-1}_+ \right] - \mathcal{J}_s(l, k_1, w^1; F)\mathcal{J}_s(l, k_2, w^2; F).
\]

sional poverty with pre-determined weights, i.e. \( w^* \), or dominated by it, i.e. \( w^{**} \).
In the next section, we derive the poverty efficiency lines based on tests of the null hypothesis that the pre-assigned weights to well-being dimensions offer the highest (lowest) multidimensional poverty against the alternative hypothesis that other weighting schemes given to each well-being dimension might offer highest (lowest) multidimensional poverty at a given weighted sum of deprivation level.

3.2 Estimation of poverty efficiency lines

We estimate SD efficiency lines (i.e., poverty frontiers) in which the cdf of the weighted sum of deprivation with predetermined weights is crossed with the cdf of a weighted sum of deprivation with alternative weights. At the first-order SDE, either we have robust first-order SDE ordering, namely $G(l, k, w^a; F) \leq G(l, k, w; F)$ or $G(l, k, w; F) \leq G(l, k, w^a; F)$ everywhere (i.e., for any argument $k$), and for all possible sets of weights, or else there exists at least one set of weights of attributes of poverty, $w^{**}$ (or $w^*$) for some level of $k$ that is dominated by (or dominates) $w$ so that there is a strict reversal in the inequality. In other words, we analyze whether pre-determined weights result in the highest or lowest possible headcount ratio for all possible weighted sums of deprivation, when compared with all possible alternative weighting scheme.

If the pre-determined weights are robust in one way or another, irrespective of the selection of $k$, the pre-determined weights would assign the highest (or lowest) possible headcount ratio for a given population. On the other hand, if there exists some weighted sum of deprivation level $k$, where an alternative weighting scheme other than $w$ dominates and/or is dominated by $w$, therefore offering a higher and lower headcount ratio for a given $k$ level respectively, then the pre-determined weights would not be robust. If there exist weighting schemes other than $w$ that dominate and are dominated by $w$ for all arguments of $k$ (see Figure 3.1), there exists an alternative weighting that is different than $w$ for each $k$ that dominates $w$ (i.e., cdf with $w^*$) and is dominated by $w$ (i.e., cdf with $w^{**}$). These conditions may not hold for all $k$ levels, and as such the empirical distributions may intersect at some $k$ level, which is the poverty frontier where either the inequality, $G(l, k, w^*; F) < G(l, k, w; F)$ or $G(l, k, w; F) < G(l, k, w^{**}; F)$, is reversed.

The worst-case scenario (most pessimistic) weighting scheme $w^*$ constructed from the set of components reaches the highest level of weighted sum of deprivation (highest poverty) for a given probability, implying that the number of observations (individuals or households) having a relative weighted deprivation level above a given argument, $\bar{k}$, is maximized (i.e., $k > \bar{k}$ is maximized). In other words, the number of observations (individuals or households) having a weighted deprivation level below a given argument is minimized, and as such there is a higher multidimensional headcount ratio with $w^*$ than with $w$. In this (minimization) problem,
therefore, we find the weighting scheme of dimensions that minimizes the number of observations that are below a given $k$. This minimization problem either can hold for all arguments $k$ or can hold for up to some argument $k$ or after some argument $k$. Natural estimators of $w^*$ (say $\hat{w}^*$) for a given $k$ can be obtained by substituting $\hat{F}$ for $F$.

$$
\sup_{k} \inf_{w^*} G(l, k, w^*; \hat{F})
$$

s.t.

$$
G(l, k, w; \hat{F}) > G(l, k, w^*; \hat{F}).
$$

(3.3a)

The best-case scenario (most optimistic) hybrid weighting scheme $w^{**}$ reaches the lowest level of weighted sum of deprivation (lowest poverty) for a given probability, implying that the number of observations (individuals or households) having a relative weighted deprivation level above a given argument, $\bar{k}$, is minimized (i.e., $k > \bar{k}$ is minimized). In other words, the number of observations (individuals or households) having a weighted deprivation level below a given argument is maximized and as a result there is a lower multidimensional headcount ratio with $w^{**}$ than with $w$. In this (maximization) problem we find the weighting scheme of dimensions that maximizes the number of observations that are below a given $k$. This maximization problem either can hold for all arguments $k$ or can hold for up to some argument $k$ or after some argument $k$. Natural estimators of $w^{**}$ (say $\hat{w}^{**}$) for a given $k$ can be obtained by substituting $\hat{F}$ for $F$.

$$
\sup_{k} \inf_{w^{**}} G(l, k, w^{**}; \hat{F})
$$

s.t.

$$
G(l, k, w; \hat{F}) < G(l, k, w^{**}; \hat{F}).
$$

(3.4)

As mentioned earlier, it may not be possible that the cdf of deprivation with pre-determined weights, $w$ dominates all other cdf’s or is dominated by all other cdf’s with alternative weighting schemes for all arguments of $k$. Multidimensional poverty is at its maximum (minimum) level up to or after a common efficiency (poverty) line, $k_1$, with $w^*$ ($w^{**}$) constructed from the set of attributes that dominates (is dominated by) $w$ at order $s$. In other words, for first-order SD, the cdf curves may intersect at some level(s) of $k$. Let us assume then that we can face at most one single crossing between $G(l, k, w; F)$ and $G(l, k, w^*; F)$, $w^* \neq w$ and at most one single
crossing between $G(l, k, w; F)$ and $G(l, k, w^{**}; F)$, $w^{**} \neq w$. Then we have two possibilities.

In the first case, $G(l, k, w; F)$ dominates all distributions with all possible alternative weighting, $w^{*}$ at low levels of weighted sums of deprivation, but there exist some higher deprivation levels where $G(l, k, w; F)$ is dominated at these higher levels. Let us denote by $\bar{k}_1$ as the upper efficiency deprivation level (i.e., poverty frontier). In this case, the cdf with pre-determined weight set $w$ dominates the cdf of all sets of $w^*_1$ up to $\bar{k}_1$, whereas the alternative weight set, $w^*_1$, dominates the pre-determined set $w$ after $\bar{k}_1$. Figure 3.2 suggests that up to $\bar{k}_1$, there is more multidimensional poverty in $w$ than in $w^*_1$. In other words, if one were to set a cross-dimensional cut-off value, $k$, less than $\bar{k}_1$, then $w$ would result in higher levels of multidimensional headcount ratio since there is a higher proportion of observations above this $k$ with $w$ than in $w^*$. On the other hand, if the cross-dimensional cut-off value, $k$, is higher than $\bar{k}_1$, then $w^*$ would confer higher levels of multidimensional headcount ratio for a given population, since there would more proportion of observations above this $k$ with $w^*$ than in $w$. Therefore, the deprivation level $\bar{k}_1$ is the critical poverty frontier where the reversal of multidimensional poverty with alternative weighting takes place.

Natural estimator $\hat{k}_1$ of $k_1$ is then obtained by substituting $\hat{F}$ for $F$. To this end assume that $G(l, k, w; \hat{F})$ is smaller than $G(l, k, w^*; \hat{F})$ for low levels of weighted sum of deprivation $k$ and for all sets $w^*$. If $G(l, k, w; \hat{F})$ is larger than $G(l, k, w^*; \hat{F})$ for high levels of weighted sum of deprivation $k$ for some sets $w^*$, then the estimator $\hat{k}_1$ is obtained where the dominance between pre-determined weights and any alternative weight is reversed at this deprivation level:

$$\sup_k \inf_{w^*} G(l, k, w^*; \hat{F})$$

s.t.

$$G(l, k, w; \hat{F}) < G(l, k, w^*; \hat{F}) \text{ for all } k < \bar{k}_1 \text{ and for all } w^*,$$  

(3.5a)

$$G(l, k, w; \hat{F}) > G(l, k, w^*; \hat{F}) \text{ for all } k > \bar{k}_1 \text{ and for some } w^*.$$  

(3.5b)

In the second case, there exist some sets which provide dominance over $G(l, k, w; F)$ at low levels of weighted sum of deprivation, but not at high levels. In that case we denote by $\underline{k}_1$ as the lower efficiency line (poverty frontier). In this case, the cdf with pre-determined weights $w$ is dominated by the cdf with some weights $w^{**}$ up to $\underline{k}_1$, whereas the pre-determined weighting set, $w$ would dominate any alternative weighting scheme $w^{**}$ after $\underline{k}_1$ level. Figure 3.3 suggests that up to $\underline{k}_1$, there is more multidimensional headcount ratio in $w^{**}$ than in $w$. In other words, if one were to set a cross-dimensional cut-off value, $k$, less than $\underline{k}_1$, then $w^{**}$ would result in higher levels of multidimensional headcount ratio since there is a higher proportion of
observations above this $k$ with $w^*$ than with $w$. On the other hand, if the cross-dimensional cut-off value, $k$, is higher than $\tilde{k}_1$, then $w$ would confer a higher levels of multidimensional headcount ratio than $w^*$. Therefore, $\tilde{k}_1$ is the critical poverty frontier where the reversal of multidimensional poverty with alternative weighting takes place.

Natural estimator $\hat{k}_1$ of $k_1$ is then obtained with the empirical counterpart by substituting $\hat{F}$ for $F$. To this end assume that $G(l,k,w;\hat{F})$ is larger than $G(l,k,w^*;\hat{F})$ for low levels of weighted sum of deprivation $k$ for some sets of $w^*$. If $G(l,k,w;\hat{F})$ is lower than $G(l,k,w^*;\hat{F})$ for high weighted sum of deprivation levels $k$ for all sets $w^*$, then the estimator $\hat{k}_1$ is obtained where the dominance between pre-determined weights and any alternative weight is reversed at this deprivation level:

Figure 3.2: First Order Stochastic Dominance of $w$ over $w^*$ at low poverty levels.
Figure 3.3: First Order Stochastic Dominance of $w$ over $w^{**}$ at high poverty levels.

\[
\sup_{k} \inf_{w^{**}} G(l, k, w^{**}; \hat{F})
\]

s.t.
\[
G(l, k, w^{**}; \hat{F}) < G(l, k, w; \hat{F}) \text{ for all } k < k_1 \text{ and for some } w^{**}, \quad (3.6a)
\]
\[
G(l, k, w^{**}; \hat{F}) > G(l, k, w; \hat{F}) \text{ for all } k > k_1 \text{ and for all } \ w^{**}. \quad (3.6b)
\]

At the first-order SDE, we will have robust first-order SDE ordering, if $G(l, k, w^{*}; F) < G(l, k, w; F)$ or $G(l, k, w; F) < G(l, k, w^{**}; F)$ everywhere (i.e., for any argument $k$). Furthermore, first-order SDE, $G(l, k, w^{*}; F) < G(l, k, w; F)$ or $G(l, k, w; F) < G(l, k, w^{**}; F)$ for all $k$, would suggest second-order dominance (see Figure 3.1). However, if two cdf curves intersect
and there is a reversal of inequalities at some \(k\) level (see Figures 3.2 and 3.3), we move to the second-order dominance.

Second-order SDE of \(w\) over \(w^{**}\) corresponds to \(\mathcal{J}_2(l, k, w; F) < \mathcal{J}_2(l, k, w^{**}; F)\) for all \(k\) and the weighted sum of deprivation in the population summarized by \(G(l, k, w; F)\) is at least as large as that in the \(G(l, k, w^{**}; F)\) population, for any utility function \(U\) that is monotonically decreasing and convex, that is \(U''(k) \leq 0\) and \(U''(k) \geq 0\). Second-order stochastic dominance is verified, not by comparing the \(cdf\)'s themselves, but comparing the integrals below them (i.e., the \(\mathcal{J}_2\) functions). We examine the area below the \(G(l, k, w; F)\) and \(G(l, k, w^{**}; F)\) curves. Given lower and upper boundary levels, we determine the area beneath the curves and, if the area beneath the \(G(l, k, w^{**}; F)\) distribution is larger than the one of \(G(l, k, w; F)\), then in this case \(w\) stochastically dominates \(w^{**}\) in the second-order sense. Since we look at the area under the distributions, second-order dominance implies simply a total sum of deprivation and not a point-wise dominance over all the points of the support of one distribution over another.

Following the same idea, we assume that \(\mathcal{J}_2(l, k, w; \hat{F})\) is smaller than \(\mathcal{J}_2(l, k, w^{*}; \hat{F})\) for low weighted sum of deprivation levels \(k\) and for all sets \(w^{*}\). If \(\mathcal{J}_2(l, k, w; \hat{F})\) is larger than \(\mathcal{J}_2(l, k, w^{*}; \hat{F})\) for high weighted sum of deprivation levels \(k\) and some sets \(w^{*}\), then the estimator \(\hat{k}_2\) of the upper poverty efficiency line at second-order is obtained through:

\[
\sup_k \inf_{w^*} \mathcal{J}_2(l, k, w^{*}; \hat{F})
\]

s.t.
\[
\begin{align*}
\mathcal{J}_2(l, k, w; \hat{F}) &< \mathcal{J}_2(l, k, w^{*}; \hat{F}) \quad \text{for all } k < \bar{k}_2 \text{ and for all } w^{*}, \\
\mathcal{J}_2(l, k, w; \hat{F}) &> \mathcal{J}_2(l, k, w^{*}; \hat{F}) \quad \text{for all } k > \bar{k}_2 \text{ and for some } w^{*}.
\end{align*}
\]

We may also estimate analogously the lower efficiency line (poverty line) at second-order. To this end assume that \(\mathcal{J}_2(l, k, w; \hat{F})\) is larger than \(\mathcal{J}_2(l, k, w^{**}; \hat{F})\) for low levels of weighted sum of deprivation \(k\) for some sets \(w^{**}\). If \(\mathcal{J}_2(l, k, w; \hat{F})\) is lower than \(\mathcal{J}_2(l, k, w^{**}; \hat{F})\) for high weighted sum of deprivation levels \(k\) for all sets \(w^{**}\), then the estimators \(\hat{k}_2\) of the lower poverty efficiency line at second-order can be obtained by:

\[
\sup_k \inf_{w^{**}} \mathcal{J}_2(l, k, w^{**}; \hat{F})
\]

s.t.
\[
\begin{align*}
\mathcal{J}_2(l, k, w; \hat{F}) &< \mathcal{J}_2(l, k, w^{**}; \hat{F}) \quad \text{for all } k < \bar{k}_2 \text{ and for all } w^{**}, \\
\mathcal{J}_2(l, k, w; \hat{F}) &> \mathcal{J}_2(l, k, w^{**}; \hat{F}) \quad \text{for all } k > \bar{k}_2 \text{ and for some } w^{**}.
\end{align*}
\]
\[
\sup_k \inf_{\mathbf{w}^{**}} J_2(l, k, \mathbf{w}^{**}; \hat{F}) \\
\text{s.t.}
\]
\[
J_2(l, k, \mathbf{w}^{**}; \hat{F}) < J_2(l, k, \mathbf{w}; \hat{F}) \quad \text{for all } k < k_2 \text{ and for some } \mathbf{w}^{**}, \quad (3.9a)
\]
\[
J_2(l, k, \mathbf{w}^{**}; \hat{F}) > J_2(l, k, \mathbf{w}; \hat{F}) \quad \text{for all } k > k_2 \text{ and for all } \mathbf{w}^{**}. \quad (3.9b)
\]

If there is clear second-order SDE, we will have robust second-order SDE ordering, if \(J_2(l, k, \mathbf{w}^{*}; F) < J_2(l, k, \mathbf{w}; F)\) or \(J_2(l, k, \mathbf{w}; F) < J_2(l, k, \mathbf{w}^{**}; F)\) everywhere (i.e., for any argument \(k\)), there is also higher-order dominance. However, if two integrals intersect, as in the case of poverty efficiency lines given above, there is a reversal of inequalities at some \(k\) level, we move to the higher-order dominance.

Generalising the above cases, there is a robust SDE ordering of order \(s\) if there is some alternative weighting scheme, \(\mathbf{w}^{*}\), which would dominate the pre-determined weights at all arguments \(k\) offering the worst-case scenario:

\[
\sup_k \inf_{\mathbf{w}^{*}} J_s(l, k, \mathbf{w}^{*}; \hat{F}) \\
\text{s.t.}
\]
\[
J_s(l, k, \mathbf{w}; \hat{F}) > J_s(l, k, \mathbf{w}^{*}; \hat{F}) \text{ for all } k \text{ and for some } \mathbf{w}^{*}, \quad (3.10a)
\]

and if pre-determined weights dominates, \(\mathbf{w}\), all alternative weighting schemes, \(\mathbf{w}^{**}\), at all arguments of \(k\):

\[
\sup_k \inf_{\mathbf{w}^{**}} J_s(l, k, \mathbf{w}^{**}; \hat{F}) \\
\text{s.t.}
\]
\[
J_s(l, k, \mathbf{w}; \hat{F}) < J_s(l, k, \mathbf{w}^{**}; \hat{F}) \text{ for all } k \text{ and for all } \mathbf{w}^{**}. \quad (3.11a)
\]

In the first case, there is an alternative weighting scheme, \(\mathbf{w}^{*}\), which offers a worst-case scenario at SDE order of \(s\). Whereas, in the second case, pre-determined weights offer the worst-case scenario at all deprivation levels and therefore it is the worst-case scenario at SDE order of \(s\).

Whereas, if there is no clear SDE orderings at order of \(s\), one can similarly find the upper
efficiency lines (i.e., poverty lines), $\hat{k}_s$, by:

$$\sup_k \inf_{w^*} J_s(l, k, w^*; \hat{F})$$

s.t.

$$J_s(l, k, w; \hat{F}) < J_s(l, k, w^*; \hat{F}) \quad \text{for all} \quad k < \hat{k}_s \quad \text{and for all} \quad w^*,$$

(3.12a)

$$J_s(l, k, w; \hat{F}) > J_s(l, k, w^*; \hat{F}) \quad \text{for all} \quad k > \hat{k}_s \quad \text{and for some} \quad w^*.$$

(3.12b)

Similarly, the lower efficiency lines at the SDE order of $s$, $\hat{k}_s$, can be obtained by

$$\sup_k \inf_{w^{**}} J_s(l, k, w^{**}; \hat{F})$$

s.t.

$$J_s(l, k, w^{**}; \hat{F}) < J_s(l, k, w^*; \hat{F}) \quad \text{for all} \quad k < \hat{k}_s \quad \text{and for some} \quad w^{**},$$

(3.13a)

$$J_s(l, k, w^{**}; \hat{F}) > J_s(l, k, w^*; \hat{F}) \quad \text{for all} \quad k > \hat{k}_s \quad \text{and for all} \quad w^{**}.$$

(3.13b)

### 3.3 Test statistics

For the cases above, we employ bootstrap techniques to obtain inferences for p-values for testing the first (second) null hypothesis $H^*_s$. In other words, we test whether there is no empirical distribution of deprivation with the set of alternative weighting scheme, $w^*$ ($w^{**}$) constructed from the set of attributes that dominates (is dominated by) the empirical distribution of the deprivation with the pre-assigned weights, $w$, at order $s$. We consider the weighted Kolmogorov-Smirnov type test

$$\hat{S}_s := \sqrt{N} \sup_{k, w^*} [J_s(l, k, w; F) - J_s(l, k, w^*; F)]$$

for $H^*_0$ in 3.1a,

$$\hat{S}_s := \sqrt{N} \sup_{k, w^{**}} [J_s(l, k, w^{**}; F) - J_s(l, k, w; F)]$$

for $H^*_0$ in 3.2a,

and a test based on the decision rule of the form

$$\text{reject } H^*_0 \text{ if } \hat{S}_s > c_s,$$

where $c_s$ is some critical value which will be discussed later.

The following result characterizes the properties of the tests respectively, where

---

16Assuming that $(\hat{k}_s, \hat{w}^{**})'$ exists and is unique, it can be shown that $(\hat{k}_s, \hat{w}^{**})'$ converges in probability to $(\hat{k}_s, \hat{w}^{**})'$. The same holds true for $\hat{k}$ and $\hat{w}^{**}$. The proof is available from the authors.
\[ \bar{S}_s := \sup_{k, w^*} [J_s(l, k, w; B \circ F) - J_s(l, k, w^*; B \circ F)], \]
\[ \bar{S} := \sup_{k, w^{**}} [J_s(l, k, w^{**}; B \circ F) - J_s(l, k, w; B \circ F)]. \]

**Proposition 3.2** Let \( c_s \) is a finite constant.

(i) if \( H^*_0 \) is true,
\[ \lim_{N \to \infty} P \left[ \text{reject } H^*_0 \right] \leq P \left[ \bar{S}_s > c_s \right] := \alpha(c_s), \]
with equality when \( G(l, k, w^*; F) = G(l, k, w; F) \) (respectively \( G(l, k, w^{**}; F) = G(l, k, w; F) \)) for all \( k \in \mathbb{R} \) and for some \( w^* \in \mathbb{L} \) (respectively \( w^{**} \in \mathbb{L} \));

(ii) if \( H^*_0 \) is false,
\[ \lim_{N \to \infty} P \left[ \text{reject } H^*_0 \right] = 1. \]

The result provides two random variables where one dominates (and the other one dominated by) the limiting random variable corresponding to the test statistic under the null hypothesis. The inequality yields a test that never rejects more often than \( \alpha(c_s) \) for any weighting scheme of poverty attributes \( w^* \) (respectively \( w^{**} \)) satisfying the null hypothesis. As noted in the result, the probability of rejection is asymptotically exactly \( \alpha(c_s) \) when \( G(l, k, w^*; F) = G(l, k, w; F) \) (respectively \( G(l, k, w^{**}; F) = G(l, k, w; F) \)) for all \( k \in \mathbb{R} \) and for some \( w^* \in \mathbb{L} \) (respectively \( w^{**} \in \mathbb{L} \)). The result in (i) implies that if we can find a \( c_s \) to set the \( \alpha(c_s) \) to some desired probability level (for example, conventional 0.05 or 0.01) then this will be the significance level for composite null hypotheses in the sense described by Lehmann (1986). The result in (ii) suggests that the test is capable of detecting any violation of the full set of restrictions of the null hypothesis\(^{17}\).

In order to make the result operational we need to find, for each null hypothesis, an appropriate critical value, \( c_s \), to satisfy \( P \left[ \bar{S}_s > c_s \right] = \alpha(c_s) \). Since the distribution of both test statistics depend on the underlying distribution (see e.g., McFadden, 1989), this is not an easy task, and we decide to rely on bootstrap method to simulate \( p \)-values. In the next section, we will discuss the bootstrap methods where one could also apply the multiplier method to simulate \( p \)-values (see section 3.1 of BD). One reason that the bootstrap method is chosen to simulate \( p \)-values is that although existence of a limiting distribution is usually needed, one might not be able to characterize it and therefore bootstrapping is more suitable in more complicated cases.

\(^{17}\)The proof of proposition 3.2 could be obtained similar to that of proof of Proposition 2.2 of ST by altering time dependence with iid case. Similarly, Proof of Theorem 1 of Horvath et al. (2006) could be extended to accomodate the full diversification.
4 Bootstrap methods to simulate p-values

In this section, we discuss the bootstrap method employed to simulate p-values. In this paper, we extend the nonparametric i.i.d. bootstrap where we allow for full diversification of weights. We use resampling techniques to obtain critical values for our Kolmogorov-Smirnov type of test statistic (see Abadie, 2002; Barrett and Donald, 2003 for resampling approaches used in i.i.d. bootstrap application for pairwise comparisons, and Linton et al., 2005 for serially dependent data) where i.i.d. case is extended for full diversification of weights.

The bootstrap method is based on Proposition 3.3 and we simulate the random variable corresponding to $S_s$. We define the sample as $Y = \{Y_1, Y_2, ..., Y_N\}$ (i.e., vector of achievements for each individual or household) and compute the distribution of random quantity for each case respectively

$$S^*_s := \sqrt{N} \sup_{k, w^*} \left[ J_s(l, k, w; \hat{F}^*) - J_s(l, k, w^*; \hat{F}^*) \right],$$

$$S^{**}_s := \sqrt{N} \sup_{k, w^{**}} \left[ J_s(l, k, w^{**}; \hat{F}^*) - J_s(l, k, w; \hat{F}^*) \right],$$

where

$$J_s(l, k, w; \hat{F}) = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{(s-1)!} (k - w^I(Y \leq l))^{s-1} \right]$$

for a sample of $Y^*_i$ is drawn randomly with replacement from $Y$.

Let us define $p^*_s := P[S^*_s > S_s]$ (similarly $p^{**}_s := P[S^{**}_s > S_s]$).

This approach is justified with the following proposition.

**Proposition 4.1** Assuming $\alpha < \frac{1}{2}$, a test for SDEs for pre-determined weights on well-being attributes based on the rule

"reject $H_0^s$ if $p^*_s < \alpha","$

satisfies the following:

$$\lim P[\text{reject } H_0^s] \leq \alpha \text{ if } H_0^s \text{ is true},$$

$$\lim P[\text{reject } H_0^s] = 1 \text{ if } H_0^s \text{ is false}.$$

Similar procedure is applied to obtain $p^{**}_s$ (see proof of Proposition 3.1 in ST or Proposition 3 of BD for the proof).
In order to approximate p-values, we need to use Monte Carlo in practice. The p-value for each case is simply approximated by $\tilde{p}_s = \frac{1}{R} \sum_{r=1}^{R} \mathbb{I}(\tilde{S}_{j,r} > \hat{S}_s)$, where the averaging is made on $R$ replications. The replication numbers can be chosen to make approximations as accurate as possible given the computer constraints.

5 Mathematical formulation for first-order SDE

As test statistics are obtained when all possible weighting schemes on multidimensional well-being attributes is considered, we use mixed integer programming to obtain the first-order test statistic. Testing for SDE$_1$ is based on the following test statistic, $\hat{S}_1$, is given below:

$$\max_{k, w^*} \hat{S}_1 = \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} (L_i - W_i)$$

(5.1a)

s.t. $M(L_i - 1) \leq k - w' \mathbb{I}(Y \leq l) \leq ML_i, \quad \forall i$ (5.1b)

$M(W_i - 1) \leq k - w^* \mathbb{I}(Y \leq l) \leq MW_i, \quad \forall i$ (5.1c)

$e'w^* = 1$, (5.1d)

$w^* \geq 0$, (5.1e)

$W_i \in \{0, 1\}, L_i \in \{0, 1\}, \quad \forall i$ (5.1f)

with $M$ being a large constant.

The model is a mixed integer program maximizing the distance between the sum over all scenarios of two binary variables, $\frac{1}{N} \sum_{i=1}^{N} L_i$ and $\frac{1}{N} \sum_{i=1}^{N} W_i$ which represent $G(l, k, w; \hat{F})$ and $G(l, k, w^*; \hat{F})$, respectively (empirical cdf of multidimensional poverty with the predetermined weights, $w$, and with alternative weighting scheme, $w^*$, at deprivation level of $k$). According to inequalities (5.1b), $L_i$ equals 1 for each scenario $i \in N$ for which $k \geq w' \mathbb{I}(Y \leq l)$, and 0 otherwise. In other words, binary variable counts the number of individuals (or households) that have deprivation levels that are lower than $k$ level. Analogously, inequalities (5.1c) ensure that $W_i$ equals 1 for each scenario for which $k \geq w^* \mathbb{I}(Y \leq l)$. Equation (5.1d) defines the sum of weights given to all multidimensional poverty attributes to be unity, while inequality (5.1e) suggests that attributes could either get zero or positive weights. This formulation allows for a test of the dominance of the multidimensional poverty with predetermined weights, $w$, over any potential linear combination $w^*$ of the attributes for different deprivation levels. As the number of individuals (or households) below a deprivation level is known, the problem could be
further reduced to a minimization problem where $W_i$ is minimized for each deprivation level $k$ and therefore offering the weighting scheme of multidimensional poverty attributes that offers the highest poverty levels for a given $k$ level.

Similarly, mixed integer programming could be written to find the weighting scheme that maximizes the number of individuals (or households) that have a deprivation level below a given $k$ level which would give the weighting scheme that offers the lowest possible poverty (see ST for the implication of the mixed integer programming for the first-order SDE implemented for time dependant data)$^{18}$.

6 Empirical Analysis of SDE poverty frontiers

6.1 Data and multidimensional poverty indicators

In this section, we will apply the above methodology to data from the northeastern region of Kenya which experiences the highest multidimensional poverty among all other regions in the country and the Canadian multidimensional poverty for two different urban populations. For the former, we analyze whether the whether the pre-determined weights of MPI is the robust to alternative weighting or there exists an alternative weighting schemes which would offer the highest (lowest) possible weighted sum of deprivation for above a given weighted sum of deprivation, $k$. For the latter one, we analyze given the poverty lines for each dimension, whether the pre-determined weights assigned to each dimension is robust, or there exists a weighting schemes which would offer the highest (lowest) possible normalized poverty gaps above a given level of normalized poverty gap, $z$. We also change the poverty lines for some indicators to analyze how a change in poverty line in one dimension affects the robust weighting schemes.

For Kenya, we use the same dimensions and indicators as in MPI and apply AF methodology to identify who is deprived or not at a given indicator. However, we will derive a weighting scheme for each dimension, where multidimensional headcount ratio is highest (lowest) for a given weighted sum of deprivation level, $k$. The data set we use comes from the Demographic and Health Survey of 2003 which was carried out by Central Bureau of Statistics (CBS) in partnership with the Ministry of Health (MOH) of Kenya and the National Council for Population and Development (CBS et al., 2004). We have three dimensions namely health, education and standard of living. Each dimension has different numbers of indicators and each indicator in

$^{18}$Mathematical formulation of the second-order SDE could be written for multidimensional poverty for the iid case following the section 4.2 of ST.
each dimension is assigned an equal weight.

The health dimension has two indicators: nutrition and mortality. An individual is deprived in nutrition if he/she lives in a household in which there is at least one member of the household who is malnourished. Adults are considered malnourished if their BMI is below 18.5. Children are considered malnourished if their z-score of weight-for-age is below minus two standard deviations from the median of the reference population. An individual is deprived in the mortality indicator if he/she lives in a household that experienced a child death in the family. The education dimension has two indicators: years of schooling and child enrolment. An individual is deprived in the years of schooling indicator if there is no household member who has completed five years of schooling, whereas he/she is deprived in the child enrolment indicator if a school aged child is not attending school in years 1 to 8.

Finally, the standard of living dimension has six indicators: electricity (deprived if household has no electricity), sanitation (deprived if the household’s sanitation is not improved or it is improved but shared with other households), water (deprived if the household do not have any access to clean water or clean water is more than 30 minutes walking from home), floor (deprived if household’s main floor has dirt, sand or dung), cooking fuel (deprived if household cooks with dung, wood or charcoal) and assets (deprived if household does not own more than one of: radio, TV, telephone, bike, or motorbike, and do not own a car or tractor)\(^{19}\).

Finally, each dimension is equally weighted (i.e., each dimension gets a weight of 1/3) and each indicator in each dimension is equally weighted (i.e., each indicator in the health and education dimension has a weight of 1/6 and the indicators in standard of living have a weight of 1/18 respectively in arriving at the overall final index). If an individual is deprived 30% of the weighted indicators, then that person is considered as multidimensionally poor for the calculation of MPI.

Whereas for the Canadian individual multidimensional poverty, we choose two cardinal well-being dimensions which allow us further to analyze the poverty gaps with the pre-determined poverty thresholds. The data for this application is obtained from Statistics Canada’s Survey of Labour and Income Dynamics (SLID) for 2006. In order to reduce the level of heterogeneity within sample that we analyze, we consider individuals that are residing in a similar urban area and having same amount of individuals in a given household. For the first one, we only consider the individuals residing in urban areas with populations above 500,000 and having two persons in household that are above 19 years or older (4531 adults). Whereas the second one consist individuals that are living in urban areas with the populations below 29,999 and

\(^{19}\)see Alkire and Santos (2010) p.17. Also see Alkire and Santos (2010) for details for treatment of missing information and treatment of households of the non-applicable population.
having two persons in household that are above 19 years or older (4418 adults). Therefore, the only difference between population sizes where individuals resides and a direct comparisons of the robust weighting schemes for different levels of normalized poverty gaps may give different policy directions. For both populations, we consider two well-being attributes, after-tax income of the household and highest education level of the individual. To measure poverty in income dimension, we use the poverty line of Statistics Canada’s so-called low-income cutoff (LICO) for the households that are living in urban areas with populations below 29,999 and above 500,000 with household size of two, which are $16,008 and $21,381 respectively. We consider an individual to be poor in the education dimension if she/he has not completed elementary or secondary school for two exercises for two populations\textsuperscript{20}.

6.2 Multidimensional poverty in Kenya

In this subsection, we derive a robust weighting scheme for each dimension obtaining the highest (lowest) number of individuals who are multidimensionally poor for different levels of weighted deprivation. The health and education dimensions have two indicators and each individual is deprived 0%, 50% or 100% in the respective dimension (i.e., having values 0, 1/2 or 1 respectively). On the other hand, the standard of living dimension has six indicators and an individual is deprived 0%, 16.7%, 33.3%, 50%, 67.7%, 83.3% or 100% (i.e., having values 0, 1/6, 2/6, 3/6, 4/6, 5/6 or 1 respectively).

Given that each dimension gets the same weight, the number of people who are deprived above a given weighted deprivation level is known. We can find a weighting scheme to each dimension where we can reach the highest (lowest) number of people who are deprived above the given weighted deprivation level. The panel (a) of Table 1 presents the weighting scheme for each dimension for a given set of weighted deprivation level (i.e., deprivation levels between 0.1 and 0.9 with 0.1 increments), which offers the highest number of people who are deprived above that deprivation level (i.e., pessimistic case). For each given deprivation level $k$, there always exists an alternative weighting scheme which offers a higher number of individual deprived above a given $k$ level with alternative weighting than the pre-determined equal weights to each dimension. For example, in MPI that is published by UNDP and OPHI, an individual is considered multidimensionally poor if that individual is deprived 30% or above of the weighted indicators (i.e., $k \geq 0.3$). In this case, we find that if the health dimension has a weight of

\textsuperscript{20}Even though years of schooling vary across Canada for completing elementary and/or secondary school, in order to analyze poverty gaps, we take 8 and 12 years of schooling as poverty lines for elementary and secondary school. Therefore, our analysis are mainly for illustrative rather than a deep multidimensional poverty for a given population.
The education dimension has a weight of 0.25 and the standard of living dimension has a weight of 0.7, then the number of people who are deprived over 30% of the weighted indicators is the highest. Whereas, when the health, education and standard of living dimensions get weights of 0.1, 0.3 and 0.6 respectively, the number of people who are deprived over 20% of the weighted indicators is the highest. Overall, standard of living dimension gets the highest weight for all different levels of deprivation levels $k$ in the most pessimistic case, therefore, no matter the across dimensional cut-off (i.e., $k$ level), the standard of living dimension is the main contributor to the multidimensional poverty to the northeast region of Kenya.

The panel (b) of Table 1 presents the weighting scheme for each dimension at given deprivation levels (i.e., similarly for deprivations levels between 0.1 and 0.9 with 0.1 increments) which yields the lowest number of people who are deprived over that deprivation level (i.e., optimistic case). When the health dimension gets a weight of 0.8 and the other two dimensions get a weight of 0.1, the number of people who are deprived over 20% of the weighted indicators is the lowest. The panel (b) of Table 1 also presents the weighting schemes for each dimension at given weighted deprivation levels which offer the lowest number of people deprived over that deprivation levels. In this case, the health dimension gets the highest weight for all different levels of deprivation levels $k$ in the most optimistic case, therefore, except the case when $k = 0.9$, no matter the across dimensional cut-off (i.e., $k$ level), the health dimension is the least contributor to the multidimensional poverty to the northeast region of Kenya.

In the panel (c) of Table 1, to make our findings more operational, we present the number of people who are deprived equal and above a given deprivation level when each dimension is equally weighted as in MPI. We also present the number of people who are deprived over a given deprivation level with the weights offered in the panel (a) and (b) of Table 1 for every given deprivation level (i.e., the pessimistic and optimistic cases respectively). There are 2353 individuals who are deprived at least some level with the equally weighted. For example, with equal weights given to each dimension, 2280 individuals (i.e., 96.90% of the people) in the northeastern region of Kenya is deprived equal or above 30% of the weighted indicators. On the other hand, with the pessimistic case, if the health, education and standard of living dimensions take weights of 5%, 25% and 70% respectively, 2340 individuals (99.45% of the people) in northeast region of Kenya is deprived over 30% of the weighted indicators. With the optimistic case, on the other hand, if the health, education and standard of living dimensions get weights of 70%, 15% and 15% respectively, then there are 1553 individuals (66% of the people) are deprived over 30% of the weighted indicators. The panel (c) of Table 1 also presents the number of people who are deprived over given weighted deprivation levels when the dimensions are weighted equally, and when weights from the panel (a) and (b) of Table 1 are used.
Another interesting finding in Table 1 is the number of people who are deprived over a weighted deprivation level is the same at different weighted deprivation levels with different weighting schemes on each dimension. For example, for the pessimistic case, when the health, education and standard of living dimensions take weights of 10%, 30%, and 60% respectively, 2340 individuals (99.45% of the population) are deprived over 20% of the weighted deprivation level. On the other hand, when the health, education and standard of living dimensions get weights of 5%, 25%, and 70% respectively, 2340 individuals (99.45% of the population) are deprived over 30% of the weighted deprivation level. In other words, changing the weights of each dimension slightly (i.e., decreasing the weight of health and education dimensions by 5% and increasing the weight of standard of living dimension by 10%) leads to higher multidimensional poverty for the same proportion of the population analysed. Similarly, one can see that different weights on each dimension offer the same number of people deprived over 10%, 20% and 30% of the weighted deprivation levels for the most optimistic case.

With our methodology applied to data from the northeastern region of Kenya, we show that different weighting schemes on each dimension result in different numbers of people who are deprived over a given weighted deprivation level. In other words the percentage of people being multidimensionally poor is different with different weighting schemes. Moreover, we show that different weighting schemes on each dimension yield the same percentage of people deprived over different deprivation levels. Hence, different weighting schemes on poverty attributes result in different levels of multidimensional poverty. In this paper, we derive extreme bounds where one can obtain the weighting scheme applied to each dimension which offers highest and lowest number of people who are deprived over a given weighted deprivation level.

Furthermore, if one were to follow AF methodology and eliminate the deprivation levels of individuals (households) if these were below $k$ (cross dimension cut-off value) prior to aggregation (i.e., to censor the deprivation levels of individuals who are not multidimensionally poor to zero) may distort the poverty analysis and one may find different weighting scheme for a given deprivation level which offer the most pessimistic (optimistic) level of poverty. In other words, the empirical distribution of the weighted sum of deprivations is being distorted with the elimination stage prior to aggregation. We further conducted our analysis to disregard some of individuals from the analysis as in AF methodology. For example, suggest we propose $k$ being 0.3 as in AF methodology, 0.5 and 0.7 respectively. In these cases, the individuals which were deprived below 0.3, 0.5 and 0.7 are eliminated from the analysis and similarly, we applied our methodology as before to find the most optimistic and pessimistic case for different

\footnote{We conducted analysis for all levels of $k$, 0.1 to 0.9, and the similar pattern is in place no matter the $k$ level but given the space limitations, we only reported the results for $k = 0.3, 0.5$ and $0.7$ respectively. Detailed findings are available upon request from the authors.}
weighed deprivation levels that are above each respective level. Specifically, we find the most optimistic and pessimistic weighting schemes for the weighted deprivation levels that are above the censoring level. It is possible that we may find the same weighting schemes for the most pessimistic and optimistic cases as in panels (a) and (b) of the Table 1 or we may find different weighting scheme that offers the worst- and best-case scenario. In other words, after dropping some individuals from the analysis due to the censoring process may alter the results found when all individuals were part of the analysis.

When we apply our analysis after censoring the people deprived less than 0.3, 0.5, and 0.7 respectively, we find that the weights offered in panel (a) and (b) of Table 1 for each deprivation level are still offering the most optimistic and pessimistic case for the majority of the cases even after censoring. Specifically, for the most optimistic case, the weights offered in the panel (b) of Table 1 are always the ones that offer the best-case scenario for a given deprivation level and therefore, even after censoring, the most optimistic weights given in the panel (b) of Table 1 for different $k$ levels are still the most optimistic ones. In other words, before and after censoring, similar weights are found for the most optimistic case for all different levels of $k$. However, we also find that there are some $k$ levels where weighting scheme for the most pessimistic case were different than the ones found before censoring. In the panels (a) of Table 2 and 3, we only offer the cases where different weighting schemes found for the worst-case scenario before and after censoring when censoring is done for $k < 0.3$ and $k < 0.5$ respectively. In panel (a) of Table 2, we find that if the health, education and standard of living dimensions get weights of 2%, 26% and 72% respectively, there are more individuals deprived above 60% deprivation after censoring. Whereas, the most pessimistic weights for the health, education and standard living are 4%, 4% and 92% respectively before censoring (see panel (a) of Table 1). To make the result operational, we propose the number of individuals who are deprived above 60% with the equal weights, the most pessimistic weights after censoring (i.e., weights from panel (a) of Table 2), and before censoring (i.e., weights from panel (a) of Table 1). There are 2206 and 2005 individuals who are deprived above 60% of the weighted deprivation with the weights after and before censoring respectively. Similarly, the panel (a) of the Table 3 offers the weighting scheme of the dimensions for the most pessimistic case after censoring the deprived individuals below the weighted sum of deprivation 0.5. If the health and standard of living dimensions get weights of 25% and 75% respectively, and education dimension with no weight, there are more individuals deprived above 60% deprivation after censoring. The panel (b) of the Table 3 gives the number of individuals which were deprived above 60% of the weighted deprivation with the weights after and before censoring (i.e., 2049 and 2037 individuals deprived above 60% of
weighted deprivation levels respectively)\textsuperscript{22}.

Overall, we find that the similar weighting schemes for the most optimistic and pessimistic case may be found before and after censoring since the dropped individuals may have been taking lower deprivation levels. However, the optimistic and pessimistic case weights for a given deprivation level may change after censoring the data. In other words, some individuals are now being kept outside the analysis after censoring may have been deprived above a given deprivation level with an alternative weighting. Therefore, a censoring step that is applied in AF methodology may potentially distort the results and our approach should also be applied after the censoring.

6.3 Multidimensional poverty in Canada

In this subsection, we further analyze multidimensional poverty when all dimensions are cardinal variables which allows us to assess the shortfalls of deprivations (i.e., normalized poverty gaps) when each dimension is weighted equally (i.e., given equal importance). We have two wellbeing attributes, after-tax income of the household and highest attained education level of the individual for the Canadian multidimensional poverty analysis. To measure poverty in each dimension, we use the poverty line of Statistics Canada’s so-called low-income cutoff (LICO) for the individuals that is living in urban areas with populations below 29,999 and above 500,000 with household size of two, which is $16,008 and $21,381 respectively. We consider an individual to be poor in the education dimension if she/he has not completed elementary or secondary school for two exercises for each population. Therefore, individual is considered to be income-poor if living in a household that has income less than $16,008 and $21,381 for each respective population and/or education-poor if she/he has not completed elementary or secondary school for each respective application. The panels (a) and (b) of Table 4 offer the number and percentage of deprived individuals in each dimension and the percentage of headcount ratio, poverty gap and the severity of poverty (i.e., when $\alpha = 0, 1, \text{and } 2$ respectively) if the union approach were taken to define the poor when income and education dimensions are weighted equally and the poverty lines are $l_1=(16008, 8\text{years})$ and $l_2=(16008, 12\text{years})$ for the individuals residing urban areas below 29,000. Whereas, the panels (a) and (b) of Table 5 offers the same information for the individuals living urban areas above 500,000 when the poverty lines are $l_1=(21381, 8\text{years})$ and $l_2=(21381, 12\text{years})$\textsuperscript{23}.

\textsuperscript{22}Note that the number of individuals who are deprived above 0.6 sum of deprivation level is different in panels (b) of Table 2 and 3 than panel (c) of Table 1 as some of the individuals are dropped from the analysis due to censoring.

\textsuperscript{23}The intersection approach may also be chosen for our application to define the multidimensionally poor, however, we only rely on the union approach in this paper since the intersection approach leaves
When one takes the union approach, there is higher multidimensional headcount ratio in urban areas with population above 500,000 than the one in urban areas with population below 29,999 where headcount ratios are 15.40% and 12.22% respectively when poverty lines are LICO levels and 8 years of schooling for income and education respectively. Whereas, if the poverty line for income for each population is kept the same but the education poverty line is increased to 12 years of schooling, there is higher multidimensional headcount ratio in urban areas with population below 29,999 than the one in urban areas with population above 500,000 where the headcount ratios are 41.78% and 33.59% respectively. However, the main application in this section is the derivation of robust weights for each dimensions which offers the highest (lowest) possible normalized poverty gaps (i.e., shortfall of individual achievement from the poverty line). In this case, we analyze when $\alpha = 1$ from FGT type of poverty measurement. We derive a robust weighting scheme for each dimension obtaining the highest (lowest) number of individuals who are above a given level of normalized poverty gap, $z$, for each respective exercise.

Initially, we obtain the normalized poverty gaps for each poor individual, given the dimension specific poverty lines in each respective application. Each dimension is assigned a pre-determined weight, $w=(1/2, 1/2)$. Given the $k = 0.5$ (i.e., individual is being considered poor if deprived in at least one dimension, i.e., union approach), the normalized poverty gaps (i.e., $\alpha = 1$) are obtained for each poor individual by $w_1z_1 + w_2z_2$. In the traditional FGT type of application, a population has one representative number for average normalized gaps and severity of poverty. However, these numbers only rely on the first moment, which are obtained by the average of the normalized poverty gaps and severity of poverty. Whereas, in our application, rather than relying on the average of normalized poverty gaps and severity of poverty, we analyze the empirical distribution of normalized poverty gaps, $G_{\alpha=1}(l, k, z; w; F)$ which assesses the importance of all the moments of normalized poverty gaps$^{24}$.

The panel (a) of Table 6 offers the robust weighting scheme where the number of deprived people above a given normalized poverty gap is maximized and minimized respectively (i.e., the most pessimistic and optimistic case respectively) for the deprived individuals residing urban areas below 29,999 when $l_1=(16008, 8\text{years})$ and $k = 0.5$. For the most pessimistic case scenario, there is always an alternative weighting for different levels of normalized poverty gap, $z$, that offer more deprived people above that level when compared the benchmark (i.e., only few number of individuals who deprived in both dimensions. For example, there are only 151 individuals being deprived both dimensions when poverty lines are $21,381$ and 12 years of schooling for income and education dimensions respectively.

$^{24}$In this case, we do not analyze the empirical distribution of the squared normalized poverty gaps since the empirical distribution of normalized poverty already give emphasis to all moments.
equal weights to each dimension). On the other hand, for the most optimistic scenario, there is always alternative weighting which offers least number of deprived people above a given z level for all levels of z except when $z = 0.7, 0.8$ and $z = 0.9$ in which case, there is no deprived people above those normalized poverty gaps with equally weighted case and obviously no alternative weighting can offer a better case\textsuperscript{25}. For the most pessimistic scenario, weighting education more for the lower z levels would assign higher number of deprived people above 0.1, 0.2 and 0.3; however, weighting income relatively more when $z = 0.4$ and consecutive cases, always offer more people who are deprived above these levels. Findings in the most pessimistic scenario highlights the distribution of shortfalls in two dimensions, if one were to consider lower deprivation levels, there are more number of deprived people when education is weighted relatively more and the poverty levels can be decreased by concentrating education levels and poverty levels can be relatively easily decreased. However, the most deprived people (i.e., the poor of the poor) are mainly driven by the shortfalls in the income dimension and if a policy should concentrate the most deprived, more needs to be done in income dimension. Whereas the most optimistic case of deprivation will mostly be measured weighting education more and for the higher levels of shortfalls, the equally weighted case is actually the best-case. However, the most interesting finding of this application is that for the lower normalized gap levels, slightly different weighting in education actually offers extremely different results. For example, when $z = 0.2$, weighting education 84.6% provides 191 individuals being deprived above that level, whereas, weighting education 77.87% (i.e., almost 7% less than former case), would result the best-case where in this case there would be 134 individuals have shortfall above 0.2 (see panel (b) of Table 6). This example, clearly shows the importance of the weighting scheme and how a small difference in weighting may result absolutely contradicting results.

The panel (b) of Table 6 offers the number of people deprived above a given normalized gap with the equally weighted, the most pessimistic and optimistic cases for the population residing in urban areas with population below 29,999 when the poverty lines are $l_1=($16008, 8\textit{years})$. For example, when $k = 0.5$ and $l_1=($16008, 8\textit{years})$, there are 66 individuals who has a normalized poverty gaps equal to and above 0.4 with the equal weights given to each dimension. However, with the most pessimistic and optimistic weights, there are 98 and 27 who have normalized gaps over 0.4 respectively\textsuperscript{26}.

\textsuperscript{25}In the best-case scenario, there are weighting schemes which offer no individual deprived above a given $z$ level for 0.7, 0.8, and 0.9 but these cases are equally well with the equally-weighted case.

\textsuperscript{26}We also apply our methodology when $l_1=($16008, 12\textit{years})$ and $k = 0.5$ for the individuals residing urban areas below 29,999. We find that the weighting scheme pattern for the most pessimistic and optimistic case have been more consistent. In other words, for all levels of normalized poverty gaps, $z$, education gets close to full weight in the most pessimistic case and income dimension gets relatively more weight in the most optimistic case. The similar type of patterns but different weighting schemes
Overall, we find that not only the selection of dimension specific poverty lines, \( l \), and across dimension cut-off level, \( k \), matters for the multidimensional poverty analysis but also the choice of weights after the assignment of \( l \) and \( k \). As proposed, the policy targets may be different for each dimension in each country and therefore the dimension specific poverty lines. Furthermore, given the progress in a country, dimension specific poverty lines are subject to change. Therefore, one can consider a further step-wise application of our methodology through changing the dimension specific poverty lines respectively one at a time. One can choose various poverty lines for each dimension, \( l \), and analyse whether which dimension is more deprived with respect to other ones given \( w \) and \( k \) and therefore can obtain a robust comparison between well-being dimensions\(^{27}\).

7 Conclusion

In this paper, we offer multidimensional poverty analysis that are robust to different weighting schemes assigned to each indicator prior to aggregation where we allow for full diversification of weights to each well-being dimension rather than relying on fixed set of arbitrarily chosen points. We provide robust weighting scheme of well-being dimensions for different levels of weighted sum of deprivation and normalized poverty levels of a given population. This paper is an extension of the stochastic dominance efficiency in which we provide robust weighting schemes for a given deprivation level by using the Kolmogorov-Simirnov type of test statistic.

We provide two empirical applications which analyse the robustness of the weighting scheme in two identification steps. Identification of poor in multidimensional case requires two steps, dimension specific poverty lines, \( l \), and across dimension cut-off, \( k \). We derive a robust weighting scheme for each dimension obtaining the highest (lowest) number of individuals who are multidimensionally poor for different levels of weighted deprivation. In other words, given that the equal weights are allocated to standard of living, health and education dimension in MPI, we provide weighting scheme of these three dimensions which offers the highest (lowest) possible number of individuals (households) who are multidimensionally poor above a given \( k \) level. Whereas, if all dimensions of well-being are cardinal, one can further analyse the distribution for a given level of normalized gap, \( z \), are found for the case of multidimensional poverty analysis for the individuals that are living in urban areas that have a population above 500,000 people when \( l_1 = ($21,381, 8\text{years} ) \) and \( l_2 = ($21,381, 12\text{years} ) \). Therefore a different set of policies may be followed for two different urban populations to decrease multidimensional poverty. However, we do not report the results in detail to converse space but the results may be requested from authors.

\(^{27}\)For example, possible different thresholds could be chosen for Canadian poverty for after-tax income and years of education, such as between 20,000 to 36,000 after-tax income levels with a given level of income increments and between 8 and 12 years of schooling.
of normalized poverty gaps rather than an average normalized gaps for a given population. In this case, we provide robust weighting schemes for different levels of normalized poverty gap for a given population which offer the highest number of individuals (households) being above a given normalized poverty gap for the Canadian data. In this application, we allow for different dimension specific poverty lines, $l$, where one can obtain robust weights for each dimension which provides a highest (lowest) possible normalized poverty gaps above a given level. Furthermore, changing the dimension specific poverty lines, $l$, step-wise, one can obtain dimension specific poverty lines, where one dimension is clearly contributing most (least) to the multidimensional poverty since that dimension receives full weight (no weight) for all levels of normalized poverty gaps.

Our approach sheds light for policy makers. Firstly, two population may give same amount of multidimensional headcount ratio above a given weighted deprivation level, $k$, however, the pre-determined weights may underscore (overscore) the multidimensional poverty for that level and each population’s poverty for a given deprivation level may be driven with different set of well-being dimensions, and therefore, our approach provides the combination of dimensions which would offer the highest (lowest) multidimensional headcount ratio above a given $k$ level. Therefore a policy maker can concentrate certain dimensions of two different populations even though the proportion of population poor is the same level. In other words, empirical applications will give a direction to a policy maker how to allocate received aid or government budget to specific dimensions if the target were to reduce the multidimensional poverty. For example, if a policy maker targets two dimensions for simplicity since it could be even more and sets a dimension-specific cut-off values to be reached in each dimension, and would like to allocate its resources efficiently to reduce multidimensional poverty at a given region. Rather than pre-determined set of importance and allocation of budget according to this pre-determined importance, robust weighting scheme offered in this paper could give a direction to a policy maker on how to allocate its resources if one were to concentrate on most deprived populations.

In this paper, we provided consistent SD tests to obtain robust weighting schemes for different wellbeing dimensions which offer the lowest (highest) multidimensional poverty at a given time period (i.e., snapshot poverty). However, one can further extend the SD tests to obtain robust weighting scheme of time periods when one considers the lifetime poverty. In this case, lifetime poverty of a household (or individual) is measured as a weighted sum of the snapshot poverty levels in different time periods. Similar to the case of snapshot poverty, now different set of weights are allocated for the snapshot poverty levels at different time periods to obtain the lifetime poverty of a household. Existing literature suggests different weight allocation to the periods such as giving higher weights to the consecutive spells of poverty than that of spells
that are isolated (see, e.g., Hoy and Zheng, 2011; Bossert et al. 2012). However, allocation of weights to different time periods consist some set of rules which avoids full comparison of different set of weight assignments. Therefore, one could extend SD tests to the lifetime poverty context.

Another future research is to extend the multidimensional poverty analysis to multidimensional inequality analysis by eliminating the dimension specific poverty lines and analyze the full distribution of well-being dimensions. In this case, one can obtain a robust combination of well-being dimensions which offers the highest (lowest) possible inequality for a given population when the full distribution is considered. Moreover, this analysis might only consider the inequality level of a population after given levels of different dimensions such as the inequality level of a given population above a given percentile of different dimensions where one could obtain the contribution of each well-being dimension to the inequality above that percentile.

8 References


<table>
<thead>
<tr>
<th>Deprivation level</th>
<th>Robust weighting scheme</th>
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<th>Education</th>
<th>Standard of living</th>
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<td>0.15</td>
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<td>0.04</td>
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<table>
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<th>Robust weighting scheme</th>
<th>Health</th>
<th>Education</th>
<th>Standard of living</th>
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<th>Equally-weighted case</th>
<th>Most pessimistic case</th>
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<td>210</td>
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Table 2 (a)  
Weighting scheme for the most pessimistic case for Kenya northeast region MPI for $k \geq 0.3$

<table>
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<tr>
<th>Deprivation level</th>
<th>Robust weighting scheme</th>
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<td></td>
<td>Health</td>
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<td>$k = 0.6$</td>
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Table 2 (b)  
The number of individuals who are deprived with the equally-weighed, and the most pessimistic weighting schemes after and before censoring

<table>
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<tr>
<th>Weighted deprivation level</th>
<th>Equally-weighted case $\geq$</th>
<th>Most pessimistic case (with weights from Table 2a) $&gt;$</th>
<th>Most pessimistic case (with weights from Table 1a) $&gt;$</th>
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<tbody>
<tr>
<td>$k \geq 0.6$</td>
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<td>2206</td>
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Table 3 (a)  
Weighting scheme for the most pessimistic case for Kenya northeast region MPI for $k \geq 0.5$

<table>
<thead>
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Table 3 (b)  
The number of individuals who are deprived with the equally-weighed, and the most pessimistic weighting schemes after and before censoring

<table>
<thead>
<tr>
<th>Weighted deprivation level</th>
<th>Equally-weighted case $\geq$</th>
<th>Most pessimistic case (with weights from Table 3a) $&gt;$</th>
<th>Most pessimistic case (with weights from Table 1a) $&gt;$</th>
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</thead>
<tbody>
<tr>
<td>$k \geq 0.6$</td>
<td>1655</td>
<td>2049</td>
<td>2037</td>
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Table 4
Multidimensional poverty measures for individuals living urban areas that have population below 29,999: cardinal variables

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of deprived individuals</th>
<th>Percentage</th>
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<tr>
<td>Income</td>
<td>232</td>
<td>5.25</td>
</tr>
<tr>
<td>Education</td>
<td>324</td>
<td>7.33</td>
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</tbody>
</table>

Aversion for poverty measures, income weight=50% and education weight=50%

| Alpha=0 (headcount) | 540 | 12.22 |
| Alpha=1 (poverty gap) |   | 2.19  |
| Alpha=2 (severity of poverty) | | 1.23  |

b) l=(16008, 12 years)

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of deprived individuals</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Income</td>
<td>232</td>
<td>5.25</td>
</tr>
<tr>
<td>Education</td>
<td>1700</td>
<td>38.48</td>
</tr>
</tbody>
</table>

Aversion for poverty measures, income weight=50% and education weight=50%

| Alpha=0 (headcount) | 1846 | 41.78 |
| Alpha=1 (poverty gap) | | 5.92  |
| Alpha=2 (severity of poverty) | | 2.51  |

Table 5
Multidimensional poverty measures for individuals living urban areas that have population above 500,000: cardinal variables.

a) l=(21381, 8 years)

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of deprived individuals</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Income</td>
<td>466</td>
<td>10.28</td>
</tr>
<tr>
<td>Education</td>
<td>262</td>
<td>5.78</td>
</tr>
</tbody>
</table>

Aversion for poverty measures, income weight=50% and education weight=50%

| Alpha=0 (headcount) | 698 | 15.40 |
| Alpha=1 (poverty gap) | | 2.98  |
| Alpha=2 (severity of poverty) | | 1.66  |

b) l=(21381, 12 years)

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number of deprived individuals</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Income</td>
<td>466</td>
<td>10.28</td>
</tr>
<tr>
<td>Education</td>
<td>1207</td>
<td>26.64</td>
</tr>
</tbody>
</table>

Aversion for poverty measures, income weight=50% and education weight=50%

| Alpha=0 (headcount) | 1522 | 33.59 |
| Alpha=1 (poverty gap) | | 5.44  |
| Alpha=2 (severity of poverty) | | 2.56  |
Table 6 (a)
Weighting scheme for the most pessimistic and optimistic cases for individuals living in urban areas that have population below 29,999 when \( l=(16008, 8 \text{ years}) \) and \( k=0.5 \)

<table>
<thead>
<tr>
<th>Level of normalized poverty gap</th>
<th>Most pessimistic weighting scheme</th>
<th>Most pessimistic weighting scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Education</td>
<td>Income</td>
</tr>
<tr>
<td>( z = 0.1 )</td>
<td>80.13%</td>
<td>19.87%</td>
</tr>
<tr>
<td>( z = 0.2 )</td>
<td>84.60%</td>
<td>15.40%</td>
</tr>
<tr>
<td>( z = 0.3 )</td>
<td>60.9%</td>
<td>39.1%</td>
</tr>
<tr>
<td>( z = 0.4 )</td>
<td>0.46%</td>
<td>99.54%</td>
</tr>
<tr>
<td>( z = 0.5 )</td>
<td>0.66%</td>
<td>99.34%</td>
</tr>
<tr>
<td>( z = 0.6 )</td>
<td>0.66%</td>
<td>99.34%</td>
</tr>
<tr>
<td>( z = 0.7 )</td>
<td>1.06%</td>
<td>98.94%</td>
</tr>
<tr>
<td>( z = 0.8 )</td>
<td>3.47%</td>
<td>96.53%</td>
</tr>
<tr>
<td>( z = 0.9 )</td>
<td>2.54%</td>
<td>97.46%</td>
</tr>
</tbody>
</table>

Table 6 (b)
The number of individuals who are deprived with the equally-weighted, the most pessimistic and optimistic weighting schemes of the dimensions

<table>
<thead>
<tr>
<th>Sum of normalized poverty gap, ( z )</th>
<th>Number of individuals with equal weights ( \geq )</th>
<th>Number of individuals with the most pessimistic case ( (&gt;) )</th>
<th>Number of individuals with the most optimistic case ( (&gt;) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z \geq 0.1 )</td>
<td>345</td>
<td>395</td>
<td>202</td>
</tr>
<tr>
<td>( z \geq 0.2 )</td>
<td>161</td>
<td>191</td>
<td>134</td>
</tr>
<tr>
<td>( z \geq 0.3 )</td>
<td>111</td>
<td>115</td>
<td>68</td>
</tr>
<tr>
<td>( z \geq 0.4 )</td>
<td>66</td>
<td>98</td>
<td>27</td>
</tr>
<tr>
<td>( z \geq 0.5 )</td>
<td>26</td>
<td>83</td>
<td>13</td>
</tr>
<tr>
<td>( z \geq 0.6 )</td>
<td>3</td>
<td>74</td>
<td>1</td>
</tr>
<tr>
<td>( z \geq 0.7 )</td>
<td>0</td>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>( z \geq 0.8 )</td>
<td>0</td>
<td>44</td>
<td>0</td>
</tr>
<tr>
<td>( z \geq 0.9 )</td>
<td>0</td>
<td>36</td>
<td>0</td>
</tr>
</tbody>
</table>