Abstract

Prompt Corrective Action (PCA) is a banking regulation system of predetermined capital/asset ratios that trigger supervisory actions by regulator. Our paper addresses the optimality of this regulation system by adapting the dynamic model of entrepreneurial finance to banking regulation. In a dynamic moral hazard setting, we first derive the optimal contract between the banker and the regulator and then implement it by a menu of regulatory tools. Our main findings are the following: first, the insurance premium is risk-based premium where the risk is measured by the capital level; second, our model-implied capital regulation share several similarities with the US PCA. According to our capital regulation, regulatory supervision should be realized in the spirit of gradual intervention and the book-value of capital is used as information to trigger intervention. Banks with high capital are not subject to any restrictions. Dividend distribution is prohibited in banks with intermediate level of capital. When banks have low capital level, a plan of recapitalization is required and in the worse case, banks are placed in liquidation.

Key words: Prompt Corrective Action, Capital Regulation, Dynamic contracting, Recapitalization.

JEL Codes: D82, G21, G28

1 Introduction

Following the implementation of the first Basel Accord (1988), academic research has spend a lot of effort in assessing the effects of minimum capital requirement on excessive risk taking incentives. A conclusion derived from these works is that imposing minimum regulatory capital requirements itself does not constitute an adequate solution for reducing excessive risk taking, particularly in today’s world, financial innovation has produced new markets and instruments that make it easy for banks and their employees to make huge
bets easily and quickly. This thinking drives the Basel Committee to incorporate in Basel Accord II the pillar 2 - supervisory review - and the pillar 3 - market discipline - as complementary to the pillar 1 - minimum capital requirement. The Basel Committee states that the goal of the pillar 2 is to enable early supervisory intervention if the capital does not provide a sufficient buffer against risk. However, it remains silent on the way to implement this principle in practice, or in other words, it remains silent on the threshold and forms of intervention.

In the US, a system of predetermined capital/asset ratios that trigger structured actions by supervisor, which is called as Prompt Corrective Action (PCA), was introduced in the 1991 Federal Deposit Insurance Corporation Improvement Act (FDICIA). PCA classifies banks in five categories depending on capital ratios: well capitalized, adequately capitalized, undercapitalized, significantly undercapitalized and critically undercapitalized. Imposition of regulatory restraints on banks becomes more and more severe the lower their capital ratios. For instance, well capitalized and adequately capitalized banks face no restrictions. Undercapitalized banks don’t have right to capital distribution (dividend or stock repurchase). Significantly undercapitalized banks must submit a recapitalization plan. Critically undercapitalized banks have to be placed in receivership within 90 days. Some positive observed effects of FDICIA in creating the appropriate incentives for banks, the deposit insurer and the prudential supervisor result in the increasing number of recommendations to introduce PCA - type provisions in other countries. Over the past years, Japan, Korea and Mexico have adopted a similar system of the US PCA. Recently, the European Shadow Financial Regulatory Committee (ESFRC) made a proposal aimed at dealing with problem banks. One of the recommendations in their proposal is to implement a PCA regime in each individual Member State. In such circumstances, a rigorous study of the optimality of PCA-type regulation seems timely and relevant.

Our paper will address this issue by adapting the dynamic model of entrepreneurial finance to banking regulation. We consider an infinitely repeated relationship between the banker and the Deposit Insurance Corporation (DIC) which is subject to moral hazard problem. The banker runs a bank whose profitability depends on her effort. High effort improves the expected cash-flows of the bank. However, exerting high effort generates a loss of private benefits to the banker. The DIC offers the deposit insurance services and supervises the banker on behalf of the depositors. To provide the banker with appropriate incentives, the DIC can control her compensations, require her to inject more capital to the bank or force her to liquidate it. In this framework, we first characterize the optimal contract between the banker and the DIC. The method we use to solve for the optimal contract is the dynamic programming technique. Specifically, we use the banker’s expected discounted utility as state variable and the optimal contract will be contingent on it. After the characterization, we construct a regulatory menu that can implement the optimal contract. Our menu includes three instruments: bank chartering, capital regulation and deposit insurance premium. Bank chartering determines the condition to set up a bank. Deposit insurance premium defines the payments paid to the DIC at every period. Capital regulation is characterized by the regulatory restrictions on dividend distribution, recapitalization plan and liquidation. Our main findings are the
following: first, the insurance premium is risk-based premium where the risk is measured by the amount of capital; second, our model-implied capital regulation shares several similarities with the US PCA. According to our capital regulation, regulatory supervision should be realized in the spirit of gradual intervention and the book-value of capital is used as information to trigger intervention. Banks with high capital are not subject to any restrictions. Dividend distribution is prohibited in banks with intermediate level of capital. When banks have low capital level, a plan of recapitalization is required and in the worse case, banks are placed in liquidation.

Recently, there is a growing literature analyzing dynamic moral hazard. It typically consists of DeMarzo and Fishman (2004); DeMarzo and Sannikov (2006); Sannikov (2006); Biais, Mariotti, Plantin and Rochet (2006) (BMPR (2006)); Biais, Mariotti, Rochet and Villeneuve (2007) (BMRV(2007)) and DeMarzo and Sannikov (2007). In general, these papers study the optimal dynamic financial contract in a setting in which a risk neutral entrepreneur seeks funding from risk neutral investors to finance a project that pays stochastic cash-flows over many periods. Their contracting relationship is subject to moral hazard problem coming either from the unobservability of cash-flows or from the hidden effort. The entrepreneur is liable for payments to the investors only to the extent of current revenues. In addition to some variations in modelling, these papers propose different methods to implement the optimal contract and so, generate different insights. For example, to get interesting implications for an optimal capital structure, DeMarzo and Sannikov (2006) consider to implement the optimal contract by a combination of equity, long-term debt and credit line. Having the same objective but BMPR (2006) study an implementation realized via debt, equity and cash reserves. By implementing the optimal contract through the firm’s payout policy, DeMarzo and Sannikov (2007) provides an explanation for the smoothness of corporate dividends relative to earnings and cash-flows.

Our paper is also based on a dynamic moral hazard model. However, compared to the above papers, we relax the limited liability of the agent (the banker) and so, allow the principal (the DIC) to require the banker to inject money during their relationship. Moreover, in this paper, we consider the implementation through a menu of regulatory instruments available in practice of banking regulation. Therefore, we are able to discuss the issue of optimality of PCA-type regulation.

The literature on PCA is mainly empirical. Since the introduction of the US PCA, there have been several attempts to assess its functioning. Some papers recognized significant impacts of PCA in terms of raising capital ratios and reducing risk for banks. Nevertheless, Barth et al. (2004) in a study of bank regulation and supervision in 107 countries raise doubts about government policies that rely excessively on direct government regulation and supervision of banks. For the theoretical analysis of PCA, we can cite Shim (2006), Freixas and Parigi (2007). The most relevant for our work is Shim (2006). Our paper takes the same approach as its, i.e. applying the dynamic moral hazard model of entrepreneurial finance to banking regulation. However, this paper uses the discrete - time model and does not account for the possibility of recapitalization by the banker. Our paper can be seen as its extension since we take into consideration the costly recapitalization option. Due to the advantage in tractability of a continuous - time model, we succeed
to fully characterize the optimal contract in the extended setting.

The paper proceeds as follows. In section 2, we briefly describe the current system of Prompt Corrective Action applied in the US. In section 3, we present the model in a continuous-time setting. Section 4 is devoted to the characterization of the optimal contract. Section 5 shows how this optimal contract is implemented by regulatory instruments. Finally, section 6 concludes.

2 Description of the US PCA

Prompt Corrective Action (PCA) is part of a package of measures adopted in the Federal Deposit Insurance Corporation Improvement Act (FDICIA) which was enacted in 1991 after the US banking and thrift breakdowns of the 1980s. It is a version of the Benston and Kaufman’s (1988) proposal for structured early intervention and resolution (SEIR). PCA creates five categories for banks: well capitalized, adequately capitalized, undercapitalized, significantly undercapitalized and critically undercapitalized. Bank classification into these categories depends on three different capital ratios: (1) the total risk-based capital ratio; (2) the Tier 1 risk-based capital ratio and (3) the Tier 1 leverage ratio, as shown in the following table.

<table>
<thead>
<tr>
<th>Category</th>
<th>Total risk-based capital (%)</th>
<th>Tier 1 risk-based ratio (%)</th>
<th>Tier 1 leverage ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well capitalized</td>
<td>≥10</td>
<td>≥6</td>
<td>≥5</td>
</tr>
<tr>
<td>Adequately capitalized</td>
<td>≥8</td>
<td>≥4</td>
<td>≥4</td>
</tr>
<tr>
<td>Undercapitalized</td>
<td>&lt;8</td>
<td>&lt;4</td>
<td>&lt;4</td>
</tr>
<tr>
<td>Significantly undercapitalized</td>
<td>&lt;6</td>
<td>&lt;3</td>
<td>&lt;3</td>
</tr>
<tr>
<td>Critically undercapitalized</td>
<td></td>
<td>Tangible equity ≥2</td>
<td></td>
</tr>
</tbody>
</table>

Some mix of mandatory and discretionary restrictions is prescribed for banks in each category. Mandatory restrictions become increasingly severe as the bank’s capital ratios decrease. For example, no bank may make capital distribution if it belongs to any of the three undercapitalized categories. Significantly undercapitalized banks are subject to a multitude of constraints such as required capital restoration; restrictions on transactions with affiliates and affiliated banks, on asset growth... For critically undercapitalized banks, they face not only more stringent restrictions on activities but also the appointment of a conservator (receiver) within 90 days.

The FDICIA requires each appropriate federal banking agency to take prompt corrective actions to resolve the problems of insured depository institutions at the least possible

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1Source: Table 1 in Aggarwal and Jacques (2001)
2Total capital is the sum of Tier 1 and Tier 2 capital. Tier 1 mainly comprises permanent shareholders’ equity, i.e. common stock and disclosed reserves or retained earnings. Tier 2 comprises loan loss reserves, subordinated debts, asset revaluation reserves, hybrid capital instruments, etc. Total risk-based capital ratio is the ratio of total capital to risk-weighted assets.
3Tier 1 risk-based ratio is the ratio of Tier 1 capital to risk-weighted assets.
4Tier 1 leverage ratio is the ratio of Tier 1 capital to total assets.
5The tangible equity ratio equals the total of Tier 1 capital plus cumulative preferred stock and related surplus less intangibles except qualifying purchased mortgage servicing rights divided by the total of bank assets less intangible assets except qualifying purchased mortgage servicing rights.
long-term loss to the deposit insurance fund. To increase the accountability of the regulators in carrying out their delegated responsibilities, the Office of the Inspector General at the relevant agencies is required to file audit reports in cases that generate material losses to the deposit insurance fund. These reports review the timeliness and cost effectiveness of corrective actions taken.

As noted by Benston and Kaufman (1997), the system of predetermined capital/asset ratios that trigger actions by the regulatory authorities serves two purposes. One is to give banks an incentive to strive for high capital levels. The second purpose is to place limits on the discretion of regulators.

3 Model

We consider here a repeated relationship between a risk-neutral banker who wants to operate a bank and the risk-neutral Deposit Insurance Corporation (DIC) who is in charge of insuring the deposits and supervising the bank.

More specifically, at the initial time, the banker has some endowment of cash. If she transfers $E_0$ to the DIC, she can set up a bank, collects $D$ units of deposits and invests them in a long-term risky loan portfolio. The cumulated cashflows of this portfolio evolve according to the following diffusion process

$$dR_t = \mu A_t dt + \sigma dZ^A_t$$

where $\mu > 0$ and $\sigma$ are constants; $A_t$ denotes the effort level of the banker at time $t$ and $Z^A = \{ Z^A_t, \mathcal{F}_t, 0 \leq t < \infty \}$ is a standard Brownian motion defined on the measurable space $(\Omega, \mathcal{F})$ equipped with a probability measure $P^A$ induced by the effort process $A = \{ A_t, 0 \leq t < \infty \}$. For simplicity, we assume that the set of feasible effort levels contains two elements $\{0, 1\}$. Effort is costly for the banker in the sense that she enjoys a private benefit $B$ if exerting low effort ($A_t = 0$). Denote by $v(A_t)$ the banker’s private benefits associated with effort level $A_t$. Hence, $v(0) = B$ and $v(1) = 0$. We assume that $B < \mu$, i.e. exerting high effort is efficient.

The relationship between the banker and the DIC is subject to a moral hazard problem which comes from the unobservability of the banker’s effort. That means, whereas the cashflows process $R = \{ R_t, 0 \leq t < \infty \}$ is publicly observable by both the DIC and the banker, the effort level $A_t$ is private information of the latter. A contract between the banker and the DIC specifies, based on the entire history of cashflow realizations, a liquidation time $\tau (R_s, 0 \leq s < \tau)$ and the payments for the banker. Denote by $C = \{ C_t (R_s, 0 \leq s \leq t), 0 \leq t < \tau \}$ the process describing the cumulative payments to the banker. At any time, the bank can also be closed if the banker decides not to run the bank any more and switches to her second-best business which gives her an expected util-

\footnote{In our model, the choice of effort level affects the expected value of the cashflow but not its volatility. Moreover, the effort at time $t$ affects only the distribution of the cashflows at this time.}
ity $\tilde{W} \in \left[0, \frac{W}{\rho}\right]$ where $\rho$ is the discount rate of the banker. We assume that the value of the loan portfolio at the time of termination is zero.

Regarding the payments to the banker, we consider two alternatives. First, we assume that the banker can not inject capital to the bank during its operation and so, the payments to the banker must be non negative, i.e. $dC_t \geq 0 \ \forall t < \tau$. Next, we will relax this limited liability constraint and assume that the DIC possesses an option of requiring the banker to contribute capital. We interpret this option as the recapitalization possibility and model it by allowing the lower bound of the banker’s payments to be a negative number. More concretely, we assume that for all $t < \tau$, $dC_t$ must be greater or equal to $-K dt$ where $K > 0$. Furthermore, in our paper, the capital injection is costly to the banker who bears a cost $\alpha$ for each unit of contributed capital.

If the banker discounts the future at the rate $\rho$ and the DIC at the riskless interest rate $r < \rho$, then, given a contract $(\tau, C)$ and an effort strategy $A$, the total expected utility for the banker as of time 0, if she never quits, is given by

$$E^A \left[ \int_0^\tau e^{-\rho t} \left( (1 + \alpha 1_{dC_t < 0}) dC_t + v(A_t) dt \right) + e^{-\rho \tau} \tilde{W} \right]$$

and for the DIC by

$$E^A \left[ \int_0^\tau e^{-rt} dR_t - \int_0^\tau e^{-rt} dC_t \right] = E^A \left[ \int_0^\tau e^{-rt} (\mu A_t dt - dC_t) \right]$$

where $E^A$ denotes the expectation under the probability measure $P^A$.

An effort strategy is defined as incentive compatible with respect to the contract $(\tau, C)$ if it maximizes the total expected utility of the banker given $(\tau, C)$. Here we focus on the contracts that induce the banker to choose the effort strategy $A^* = \{A_t = 1 \ \forall 0 \leq t < \tau\}$ (i.e. the banker exerts high effort every time) and if facing these contracts, the banker will never choose to quit. We label such a class of contracts as incentive compatible one. The DIC’s problem is to find, among this class, the contract which provides him with highest payoff.

We denote by $P$ the probability measure generated by the effort process $A^*$ and by $E$ the expectation under $P$. Then, the DIC’s problem can be formulated as follows

$$\max E \left[ \int_0^\tau e^{-rt} (\mu A_t dt - dC_t) \right]$$

(1)

Since $E \left[ \int_0^{+\infty} e^{-\rho t} dR_t \right] = \int_0^{+\infty} e^{-\rho t} \mu dt = \frac{\mu}{\rho}$, if $\tilde{W} > \frac{\mu}{\rho}$ then, for the banker, running a bank is worse than outside options.
subject to the following constraints

\[ A^* = \{ A_t = 1 \ \forall 0 \leq t < \tau \} \] is incentive compatible w.r.t \((\tau, C)\)

\[ W_0 = E \left[ \int_{0}^{\tau} e^{-\rho t} (1 + \alpha 1_{\{dC_t < 0\}}) \, dC_t + e^{-\rho \tau} W \right] \]

\[ E \left[ \left( \int_{t}^{\tau} e^{-\rho(s-t)} (1 + \alpha 1_{\{dC_s < 0\}}) \, dC_s + e^{-\rho(\tau-t)} W \right) \bigg| F_t \right] \geq W \quad \forall 0 \leq t < \tau \]

\[ \forall 0 \leq t < \tau : \ dC_t \in \Gamma - \text{ an exogenous interval} \]

The formulation of the constraint (3) is in line with DeMarzo and Sannikov (2006). By varying \(W_0\), we can use this solution to consider different divisions of bargaining power between the banker and the DIC. For example, if the DIC charters a banker from a competitive pool, then \(W_0\) is chosen such that the DIC’s expected payoff as of time 0 is maximal subject to the constraint that the banker receives at least \(\hat{W}\) (i.e. \(W_0 \geq \hat{W}\)).

4 Optimal contract

In this section, we present the derivation of the optimal contract. It will proceed in three steps. First, we state a result that relates the incentive compatibility condition of the effort process \(A^*\) to the dynamic evolution of the banker’s continuation value. Next, we prove that the DIC’s payoff function can be determined as solution to an ordinary differential equation. Finally, by solving this equation, we find the optimal contract.

4.1 Incentive compatibility condition

Here, we derive the incentive compatibility constraint for the banker relying on the martingale techniques introduced by Sannikov (2006).

Given an allocation \((\tau, C)\), for each \(t < \tau\), denote by \(W_t^A\) the banker’s continuation utility corresponding to an effort strategy \(A = \{ A_t, 0 \leq t < \tau \}\). It is the total expected utility the banker receives from the transfers from time \(t\) on if she follows the strategy \(A\).

\[ W_t^A = E^A \left[ \left( \int_{t}^{\tau} e^{-\rho(s-t)} \left[ (1 + \alpha 1_{\{dC_s < 0\}}) \, dC_s + v(A_s) \, ds \right] + e^{-\rho(\tau-t)} W \right) \bigg| F_t \right] \]

The following lemma provides a useful representation of \(W_t^A\)

**Lemma 1** There exists a stochastic process \(G^A = \{ G_t^A, 0 \leq t < \infty \}\) that represents the sensitivity of the banker’s continuation value to the cashflows, i.e.

\[ dW_t^A = (\rho W_t^A - v(A_t)) \, dt - (1 + \alpha 1_{\{dC_t < 0\}}) \, dC_t + \frac{G_t^A}{\sigma} (dR_t - \mu A_t \, dt) \]
Proof. Define by $V_t^A$ the lifetime utility of the banker under the allocation $(\tau, C)$ and an effort strategy $A = \{A_t, 0 \leq t < \tau\}$ conditionally on the information available at time $t < \tau$, then

$$V_t^A = E^A \left[ \left( \int_0^\tau e^{-\rho s} \left[ (1 + \alpha 1_{(dC_s < 0)}) dC_s + v(A_s) ds \right] + e^{-\rho t} \tilde{W} \right) \bigg| \mathcal{F}_t \right]$$

So, we can rewrite $V_t^A$ as follows

$$V_t^A = \int_0^t e^{-\rho s} \left[ (1 + \alpha 1_{(dC_s < 0)}) dC_s + v(A_s) ds \right] + e^{-\rho t} W_t^A$$

that implies

$$dV_t^A = e^{-\rho t} \left( (1 + \alpha 1_{(dC_t < 0)}) dC_t + v(A_t) dt - \rho W_t^A dt \right) + e^{-\rho t} dW_t^A$$

(7)

On the other hand, by construction, we have that $V_t^A$ is $\mathcal{F}_t$-measurable and that for all $s \leq t < \tau$, $E^A [V_t^A \mid \mathcal{F}_s] = V_s^A$. So, $V_t^A$ is a $\mathcal{F}_t$-martingale. By the martingale representation theorem, there exists a progressively measurable stochastic process $G^A = \{G_t^A, \mathcal{F}_t, 0 \leq t < \infty\}$ defined on the probability space $(\Omega, \mathcal{F}, P^A)$ and satisfying

$$E^A \left[ \int_0^t (e^{-\rho s} G_s^A)^2 ds \right] < \infty$$

for all $0 \leq t < \infty$ such that

$$V_t^A = V_0^A + \int_0^t e^{-\rho s} G_s^A dZ_s^A$$

(8)

therefore,

$$dV_t^A = e^{-\rho t} G_t^A dZ_t^A$$

(9)

(6) is automatically derived from (7) and (9). Q.E.D. ■

The lemma 1 provides a representation of the banker’s continuation utility as a diffusion process. This representation is valid for any effort strategy $A$. The question arised now is to determine under what conditions the banker will optimally choose to follow the effort strategy $A^\ast$.

Let $W_t$ and $Z_t$ be correspondently defined for the effort process $A^\ast$. Applying the lemma 1 to this effort process, we have

$$dW_t = \rho W_t dt - (1 + \alpha 1_{(dC_t < 0)}) dC_t + G_t dZ_t$$

(10)

where $G = \{G_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is defined on the probability space $(\Omega, \mathcal{F}, P)$. Obviously, at each time $t$, to decide what level of effort should be taken, the banker will rely on how
this decision affects her continuation utility. Exerting high effort at time \( t (A_t = 1) \) immediately causes a loss of private benefit \( B \) to the banker but it improves her continuation value in expected term by \( \frac{G_t}{\sigma} \mu \). Hence, intuitively, choosing high effort is profitable to the banker as long as \( \frac{G_t}{\sigma} \mu \geq B \). A formal statement of this result is the following:

**Proposition 1** The strategy of exerting high effort at any time is optimal for the banker if and only if the volatility of her continuation utility \( G_t \) is at least equal to \( \frac{B}{\mu} \sigma \) for all \( t \in [0, \tau) \).

**Proof.** See appendix 1. ■

The proposition 1 means that for the incentive provision purpose, the banker has to bear some minimum risk, which is materialized by requiring that her continuation utility must be sensitive enough to the cashflows.

### 4.2 DIC’s continuation payoff function

The incentive compatibility condition being defined, it is the time for characterizing the DIC’s payoff function. Denote it by \( F(W_t)^8 \). It stands for the maximal continuation payoff the DIC can earn from all incentive compatible allocations if a continuation utility \( W_t \) is promised to the banker at time \( t \).

Consider an infinitesimal time interval \([t, t+dt)\), given a promised utility \( W_t \) for the banker, if a transfer \( dC_t \) is paid to her during this period, the DIC immediately receives \( dR_t - dC_t \) and his continuation payoff is equal to \( F(W_t + dW_t) \). In other words, given the payment \( dC_t \) to the banker, the actual change in the DIC’s payoff is measured by the sum of \( dR_t - dC_t \) and \( F(W_t + dW_t) - F(W_t) \). Since the DIC discounts the future at the rate \( r \), the expected change of his payoff during the considered time interval is represented by the term \( r F(W_t) dt \). Therefore, by construction, the following equality should be satisfied

\[
r F(W_t) dt = \max_{dC_t} \{E[dR_t - dC_t + F(W_t + dW_t) - F(W_t)]\}
\]

equivalently,

\[
r F(W_t) dt = \max_{dC_t} \{\mu dt - dC_t + E[F(W_t)]\}
\]

To compute the second term on the right-hand side of the equation (11), we need to find the dynamic evolution of the DIC’s payoff. Using the Ito’s formula and based on the dynamic evolution of the banker’s continuation utility given by the equation (10), we have

\[
dF(W_t) = F'(W_t) \rho W_t dt + \frac{1}{2} F''(W_t) G_t^2 dt - F'(W_t)(1 + \alpha 1_{dC_t < 0}) dC_t + F'(W_t) G_t dZ_t
\]

Replacing \( E[F(W_t)] = F'(W_t) \rho W_t dt + \frac{1}{2} F''(W_t) G_t^2 dt - F'(W_t)(1 + \alpha 1_{dC_t < 0}) dC_t \) obtained from the equation (12) into the equation (11), we find that the DIC’s payoff function

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8Note that by the stationnary property of the allocation, the DIC’s payoff function is common to all dates. Hence, we write it without any time label on the function symbol.
is solution to the following Hamilton-Jacobi-Bellman (HJB) equation

\[ rF(W_t)dt = \max \left\{ \mu dt - dC_t + F'(W_t)\rho W_t dt + \frac{1}{2} F''(W_t)G_t^2 dt - F'(W_t)(1 + \alpha 1_{dC_t<0})dC_t \right\} \]

(13)

On the left-hand side of the equation (13), we have the expected change in the DIC’s payoff. On the right-hand side, we have the sum of the expected cashflows accruing to him and of the expected change in his continuation value. The maximization means that the current choice is managed optimally, bearing in mind not only the immediate payments but also the consequences for future payoffs.

We are now in position to consider the features of the optimal controls in (13). In the following, we will assume that the function \( F(.) \) is concave, which will be checked later. This concavity implies that it is optimal to set \( G_t \) at its minimal possible value \( B \). Hence, because the expected discounted payoff of the DIC is concave with respect to the banker’s utility, reducing her exposure to risk is desirable for the DIC. The equation (13) is then rewritten as follows

\[ rF(W_t)dt = F'(W_t)\rho W_t dt + \frac{1}{2} F''(W_t) B^2 \frac{\sigma^2}{\mu^2} dt + \mu dt + \max \left\{ -dC_t - F'(W_t)(1 + \alpha 1_{dC_t<0})dC_t \right\} \]

(14)

Concerning the compensation policy, we distinguish between two alternatives: with or without recapitalization possibility

4.2.1 Optimal contract without recapitalization possibility

In case the banker cannot inject capital to the bank during its operation, the maximization in (14) is realized over the interval \([0, +\infty)\). The question arises here is to determine when a positive transfer should be paid to the banker. To provide the banker with the utility \( W_t \), the DIC has the option to pay a lump-sum transfer of \( dC_t \) and switching to the allocation with promised utility \( W_t - dC_t \). The DIC’s payoff corresponding to this compensation structure is equal to \( F(W_t - dC_t) - dC_t \). So, giving a positive compensation to the banker is optimal for the DIC if and only if

\[ F(W_t) \leq F(W_t - dC_t) - dC_t \]

In other words, paying a positive transfer is optimal over the range of \( W_t \) where the function \( F(W_t) + W_t \) is nonincreasing. Define \( W \) by

\[ W = \inf \left\{ W : F'(W) = -1 \right\} \]

Owing to the concavity of the function \( F \), we obtain

\[ dC_t > 0 \text{ if and only if } W_t > W \]

Since the banker has the possibility to quit and take outside option with reservation utility \( \bar{W} \), on the equilibrium path, \( W_t \) can not fall below \( \bar{W} \) and the DIC must close the bank once \( W_t \) reaches this boundary value, which implies \( F(\bar{W}) = 0 \).
In summary, over the interval \([\bar{W}, W]\), the function \(F\) is determined by the following ordinary differential equation

\[
rF(W_t) = F'(W_t)\rho W_t + \frac{1}{2}F''(W_t)\frac{B^2}{\mu^2}\sigma^2 + \mu
\]

(15)

There are already two boundary conditions associated with (15). The first one is \(F(\bar{W}) = 0\). The second is the usual smooth-pasting condition \(F'(W) = -1\) which ensures that \(W\) is dividend threshold. However, since the value \(W\) itself is unknown, we need a supplementary third condition. That is the condition \(F''(W) = 0\), which determines the optimality of \(W\).

**Proposition 2** The optimal contract is characterized through the continuation utility \(W\) of the banker whose the dynamic evolution is governed by the following stochastic differential equation

\[
dW_t = \rho W_t dt + \frac{B}{\mu}\sigma dZ_t - dC_t
\]

The DIC’s payoff is determined by the function \(F\) - solution to

\[
\begin{cases}
  rF(W) = F'(W)\rho W + \frac{1}{2}F''(W)\frac{B^2}{\mu^2}\sigma^2 + \mu & \text{for } W \in [\bar{W}, \underline{W}]
  \\
  F(W) = F(W^*) - (W - W^*) & \text{for } W \geq \underline{W}
\end{cases}
\]

with three boundary conditions: \(F(\bar{W}) = 0, F'(\underline{W}) = -1\) and \(F''(\underline{W}) = 0\). The banker receives no compensations as long as \(W_t \in [\bar{W}, \underline{W}]\). The bank will be closed in the first time \(W\) reaches \(\bar{W}\)

\[
\tau = \inf \left\{ t : W_t = \bar{W} \right\}
\]

**Proof.** See appendix 2 \(\blacksquare\)

4.2.2 Optimal contract with costly recapitalization possibility

In this subsection, we extend the previous one by assuming that during the operation of the bank, the DIC has possibility to ask the banker to contribute capital. So, the maximization in (14) is realized over the interval \([-K dt, \infty)\).

With this extension, we have to find two thresholds: one for dividend region and the other for recapitalization region. To be able to do that, we should compare the DIC’s payoff between different payment structures. The arguments for finding dividend threshold is similar with the preceding subsection, the banker will get a positive transfer if and only if the function \(F(W_t) + W_t\) is nonincreasing. Denote such threshold by \(W^*\).

What about the recapitalization threshold? Given the utility \(W_t\) promised to the banker, if the DIC requires the banker to contributes \(-dC_t > 0\), because of the cost of capital contribution, he has to move to the allocation with promised utility \(W_t - (1 + \alpha)dC_t\) and gets a payoff \(F(W_t - (1 + \alpha)dC_t) - dC_t\). Therefore, demanding a recapitalization is optimal if and only if

\[
F(W_t) \leq F(W_t - (1 + \alpha)dC_t) - dC_t
\]
or

\[ F(W_t) + \frac{W_t}{1 + \alpha} \leq F(W_t - (1 + \alpha)dC_t) + \frac{W_t - (1 + \alpha)dC_t}{1 + \alpha} \]

A recapitalization should happen over the range of \( W_t \) where the function \( F(W_t) + \frac{W_t}{1 + \alpha} \) is nondecreasing. Define \( \bar{W} \) by

\[ \bar{W} = \inf \left\{ W : F'(W) = -\frac{1}{1 + \alpha} \right\} \]

Since the function \( F \) is concave, it is obvious that \( \bar{W} \) is less than \( W^* \). Hence, the banker collects a positive compensation once \( W_t > W^* \). When \( W_t \) is in between \( \bar{W} \) and \( W^* \), no transfer between the DIC and the banker occurs. Finally, if \( W_t \) falls into the region \([\bar{W}, W] \), the banker must inject capital to the bank. Over this last interval, the equation (14) becomes

\[ rF(W_t) = F'(W_t)\rho W_t + \frac{1}{2} F''(W_t) \frac{B^2}{\mu^2} \sigma^2 + \mu + \max \left[ -\frac{dC_t}{dt} \left( 1 + (1 + \alpha)F'(W_t) \right) \right] \] (16)

Because \( F'(W_t) > -\frac{1}{1 + \alpha} \), the last term of the equation (16) is decreasing in \( dC_t \) and so, at the optimum, \( dC_t = -K dt \).

Lastly, the DIC’s payoff function will be composed of two functions. Over the interval \([\bar{W}, W] \), it coincides with the function \( F_1 \) which satisfies the following differential equation

\[ rF_1(W_t) = F'_1(W_t)(\rho W_t + (1 + \alpha)K) + \frac{1}{2} F''_1(W_t) \frac{B^2}{\mu^2} \sigma^2 + \mu + K \]

Over the interval \((\bar{W}, W^*) \), it corresponds to the function \( F_2 \) - solution to the differential equation as follows

\[ rF_2(W) = F'_2(W)\rho W + \frac{1}{2} F''_2(W) \frac{B^2}{\mu^2} \sigma^2 + \mu \]

Since these two functions are determined from second-order differential equations with two free boundaries, we need to specify 6 boundary conditions. The first condition, namely \( F_1(\bar{W}) = 0 \), ensures that on the equilibrium path, \( W_t \) will not fall below \( \bar{W} \). Three other conditions, namely \( F'_1(\bar{W}) = -\frac{1}{1 + \alpha}, F''_2(\bar{W}) = -\frac{1}{1 + \alpha}, F'_2(W^*) = -1 \), come from the fact that \( \bar{W} \) and \( W^* \) are respectively recapitalization and dividend threshold. The fifth condition \( F''_2(W^*) = 0 \) guarantee the optimality of \( W^* \). Finally, in order to insure that the DIC’s payoff function is continuous, we have to impose that the second-order derivatives of two functions \( F_1 \) and \( F_2 \) match at the boundary \( \bar{W} \), i.e. \( F'_1(\bar{W}) = F''_2(\bar{W}) \).

**Proposition 3** The optimal allocation is characterized through the continuation utility \( W \) of the banker whose the dynamic evolution is governed by the following stochastic differential equation

\[ dW_t = \rho W_t dt + \frac{B}{\mu} \sigma dZ_t - (1 + \alpha 1(dC_t < 0)) dC_t \]
The DIC’s payoff is determined by the function $F$ as follows

$$F(W) = \begin{cases} 
F_1(W) & \text{for } W \in \left[ \hat{W}, \bar{W} \right) \\
F_2(W) & \text{for } W \in \left[ \bar{W}, W^* \right) \\
F_2(W^*) - (W - W^*) & \text{for } W \geq W^*
\end{cases}$$

where $F_1$ and $F_2$ are specified by

$$\begin{align*}
rF_1(W) &= F_1'(W) (\rho W + (1 + \alpha)K) + \frac{1}{2} F_1''(W) \frac{B^2}{\nu^2} \sigma^2 + \mu + K \\
rF_2(W) &= F_2'(W) \rho W + \frac{1}{2} F_2''(W) \frac{B^2}{\nu^2} \sigma^2 + \mu
\end{align*}$$

with six conditions: $F_1(\hat{W}) = 0$, $F_1'(\hat{W}) = F_2'(\hat{W}) = -\frac{1}{1+\alpha}$, $F_1''(\hat{W}) = F_2''(\hat{W}) = F_2''(W^*) = -1$ and $F_2''(W^*) = 0$. The banker has to inject more capital to the bank when $W_t \in \left[ \hat{W}, \bar{W} \right)$. The bank will be closed in the first time $W$ reaches $\bar{W}$

$$\tau = \inf \left\{ t : W_t = \bar{W} \right\}$$

Proof. See appendix 3.

4.3 Determination of initial rent for the banker

Proposition 2 and 3 describe the optimal contract for a given initial promised utility $W_0$ for the banker. We now study how this value is determined. In the context of banking regulation, it is reasonable to think of the situation where the DIC has the right to charter a banker from a competitive pool. Hence, the DIC retains all bargaining power. The initial rent for the banker is determined by

$$W_0^* = \arg \max_{W_0 \geq \hat{W}} F(W_0)$$

5 Implementation of the optimal contract with recapitalization possibility

5.1 Implementation results

So far, we characterized the optimal contract that induces the banker to exert high effort every time. Now, we consider to implement this optimal contract through a regulatory menu which is designed by the DIC and informed to all potential bankers from the initial time. The DIC acting as a supervisory authority commits to pursue it. Our regulatory menu consists of three tools: bank chartering, capital regulation and deposit insurance premium.

Bank chartering. It defines the initial amount of capital $E_0$ the banker must contribute to open the bank. Once the banker obtains the charter to set up the bank, she will collect $D$ units of deposits. This amount of deposits is invested in the risky loan portfolio whereas
the initial capital is kept as cash to meet possible future liquidity needs. The amount of cash grows at the risk-free rate $r$.

*Deposit insurance premium.* It is characterized by a sequence of payment $dP_t$ from the banker to the DIC during each time interval $[t, t + dt]^9$.  

*Capital regulation.* It determines the restrictions regarding the policies of dividend, recapitalization and liquidation.

Note that all provisions of the regulatory menu will be made contingent on the amount of banks’ capital $E_t$. In other words, in our implementation, the level of banks’ capital plays the role of a record-keeping device, as $W_t$ does in the abstract characterization of optimal contract. For implementation purpose, we define two thresholds $E^*, \bar{E}$ which respectively correspond to $W^* - \bar{W}, \bar{W} - \bar{W}$.

**Proposition 4** The following regulatory menu will implement the optimal allocation

In order to get a licence of opening a bank, the banker must contribute an amount of capital $E_0 = W_0 - \bar{W}$. During each infinitesimal time interval $[t, t + dt)$, the banker has to pay to the DIC

$$dP_t = \left\{ \begin{array}{ll}
[B - \rho \bar{W} - (\rho - r)E_t] dt + \left(1 - \frac{\rho}{\mu}\right) dR_t & \text{for } E_t > \bar{E} \\
[B - \rho \bar{W} - (\rho - r)E_t] dt + \left(1 - \frac{\rho}{\mu}\right) dR_t - \alpha K dt & \text{for } E_t \leq \bar{E}
\end{array} \right.$$

The bank is prevented to distribute dividends as long as the amount of capital is not greater than $E^*$. When its level of capital is larger than this threshold, all excess capital is distributed as dividends. The DIC orders a recapitalization from the bank if its capital level falls below $\bar{E}$. The bank is placed into the liquidation procedure if its amount of capital is zero.

In the above implementation, the bank’s capital level $E_t$ at each time $t$ before the liquidation time is related to the banker’s continuation utility $W_t$ by the functional relationship $E_t = W_t - \bar{W}$ for all $t$. The initial amount of cash is financed by equity which is capital contributed by the banker. To ensure the voluntary participation of the banker, $E_0$ could not exceed $W_0 - \bar{W}$. Since we assume that the DIC charts a banker from a competitive pool, it is optimal for the former to set $E_0 = W_0 - \bar{W}$. This prescription regarding the minimum starting capital level for opening a bank is present in law of a number of countries.

The periodic payment to the DIC (i.e. the deposit insurance premium) is determined to coordinate the evolution of the bank’s capital level with the motion of the banker’s continuation utility characterized in the proposition 3. We observe that this insurance premium is decreasing with the amount of bank’s capital at the beginning of each period. Hence, our insurance premium can be interpreted as risk-based premium where the risk is measured by bank’s capital level$^{10}$.

$^9$Our insurance premium is paid after the realization of the bank’s cash-flows.

$^{10}$Notice that differently with Shim (2006), our insurance premium is increasing with the bank’s cash-flows. The reason for this difference is that in Shim (2006), the bank’s cash-flows are unobservable and so, negative relationship between the bank’s return and the payment to the DIC is necessary to truthfully reporting incentive. In our model, we don’t have asymmetric information concerning the bank’s returns.
Relatively to the capital regulation, we see that our model-implied regulation and the US PCA have several similarities. Indeed, our model-implied regulation specifies that restrictions on banks become more stringent the less capitalized banks are. According to our regulation, banks can be classified in four categories based on their capital level. Banks with high level of capital (more than $E^*$) would be subject to minimum prudential intervention, they can distribute dividends to shareholders. In banks with intermediate capital level (i.e. their capital level falls into the interval $[E^*, \bar{E}]$), dividend distributions are suspended but banks are still allowed to continue in normal operation, no capital restoration is required. If the banks’ capital level still falls lower, the regulator will order banks to recapitalise promptly and in the worst situation, the banks’ authorities resolve banks through liquidation. A remark is that when the bank has to proceed a recapitalization, the DIC will reduce the premium by $\alpha K$ which exactly corresponds to the total cost of recapitalization. This premium reduction can be seen as a subsidy from the DIC to the undercapitalized banks in replenishing capital. Therefore, our optimal regulatory menu exhibits the combination of private and public recapitalization when banks are undercapitalized.

5.2 Discussion

Banking supervisors’ discretion: A key component of any regulatory arrangement is the nature, timing and form of intervention. The novelty of the US PCA is that it recommends a reduction of supervisory discretion by requiring the supervisors to take some prespecified intervention actions at some predetermined thresholds of banks’ capital. Our approach to design the banking regulation is to implement the ex-ante optimal allocation without the possibility of renegotiation. Therefore, as is done in the US PCA, in our model-implied capital regulation, all actions of the regulator is specified ex-ante by the law, which means that the regulators’ discretion is limited. Another aspect introduced in the US PCA in favor of limiting regulatory forbearance is the provisions calling for timely resolution. According to these provisions, banks should be closed before the economic value of their capital becomes negative. Our liquidation policy exhibits the same property in the sense that it claims to liquidate the banks as soon as their capital is wiped out. Insolvent banks with negative capital should not be allowed to continue in operation.

Book - value vs. Market - value: One of the major issues about the effectiveness of the USPCA is related to its intervention triggering device. The triggers for prompt corrective actions in FDICIA are based on historical - cost accounting measures (i.e. on the book-value of capital), which raises the concerns about the adequacy of such indicators. Some studies (e.g. Peek and Rosengren (1996, 1997) and Jones and King (1992, 1995)) have noted that the capital ratio thresholds used in PCA are lagging indicators of a bank’s financial status. In our model-implied regulation, the regulatory restrictions are contingent on the book-value of capital. However, this result should not be seen as a support for the use of book - value against the use of market - value measure. Indeed, what matters for the discrepancy between book - value and market - value is possible changes in the interest rates and default probability of loans compared to the initial situations when liabilities
and assets are acquired\textsuperscript{11}. In our model, both variables \((r\text{ and }\sigma)\) are assumed to be fixed over time. So, there is no interest to distinguish these two measures. A rigorous analysis of the choice between two measures requires a richer setting than ours.

6 Conclusion

In this paper, we apply the approach of designing prudential regulation of banks as a mechanism to implement the socially optimal allocation proposed by Shim (2006) to study the optimality of the current US Prompt Corrective Action. In a dynamic setting where the regulator (the DIC) can not observe the effort chosen by the banker and can require the banker to inject capital at each period, we first derive the optimal allocation specifying the payments to the banker and the liquidation policy, using the banker’s expected discounted utility as state variable. Then, we show that this allocation can be implemented by a combination of capital regulation and risk-based deposit insurance premium. From the implementation results, we observe that the PCA version applied in US closely mimics properties of an optimal regulation.

A Appendix 1: Proof of proposition 1

Define by \(V_t\) the total utility the banker expects to get from the allocation \((\tau, C)\) if she chooses the effort strategy \(A^*\) conditionally on the information available at time \(t \leq \tau\)

\[
V_t = E\left[ \left( \int_0^\tau e^{-\rho s} (1 + \alpha 1_{(dC_s < 0)}) dC_s + e^{-\rho \tau} \tilde{W} \right) \right] \]  

and by \(\tilde{V}_t\) the one the banker receives if she follows an effort strategy \(A\) up to time \(t \leq \tau\) and then, switches to the strategy \(A^*\)

\[
\tilde{V}_t = \int_0^t e^{-\rho s} \left[ (1 + \alpha 1_{(dC_s < 0)}) dC_s + v(A_s)ds \right] + E\left[ \left( \int_0^\tau e^{-\rho(s-t)} (1 + \alpha 1_{(dC_s < 0)}) dC_s + e^{-\rho(\tau-t)} \tilde{W} \right) \right] \]  

So,

\[
\tilde{V}_t = V_t + \int_0^t e^{-\rho s} v(A_s)ds
\]

\textsuperscript{11}Book value is the amount paid for an asset or acquired upon issuance of a liability in the past, net of some accounting adjustments, such as reserving against expected losses from default. But changing interest rates affect the present value of fixed-interest obligations, and changing economic conditions affect the probability that loans will not be repaid as was expected when they were made. Consequently, the book value of equity often is discrepant from its market value.
Similarly to (8), we can represent $V_t$ as $V_t = V_0 + \int_0^t e^{-\rho s} G_s dZ_s$. Hence, the dynamic evolution of $\tilde{V}_t$ under the probability measure $P$ is the following

$$d\tilde{V}_t = e^{-\rho t} v(A_t) dt + e^{-\rho t} G_t dZ_t$$

Since $Z_t$ and $Z_t^A$ are related by the equality $dZ_t = dZ_t^A + \frac{1}{\sigma} (\mu A_t - \mu) dt$, under the probability measure $P^A$, $\tilde{V}_t$ evolves according to

$$d\tilde{V}_t = e^{-\rho t} \left( v(A_t) + \frac{G_t}{\sigma} \mu A_t - \frac{G_t}{\sigma} \mu \right) dt + e^{-\rho t} G_t dZ_t^A$$

**Conclusion 1** If $\tilde{V}_t$ is $P^A - \text{submartingale}$, then the effort strategy $A^*$ is suboptimal for the banker

**Proof.** Indeed, the fact that $\tilde{V}_t$ is $P^A - \text{submartingale}$ means for all $s \leq t$,

$$E^A \left( \tilde{V}_t \bigg| F_s \right) \geq \tilde{V}_s$$

Therefore, for all $t \geq 0$,

$$V_0 = \tilde{V}_0 \leq E^A \left( \tilde{V}_t \right)$$

(17)

Note that $E^A \left( \tilde{V}_t \right)$ represents the total utility the banker expects to get at date 0 if she follows a strategy $A$ until the time $t$ and then, follows the strategy $A^*$. Obviously, (17) implies that the strategy $A^*$ is suboptimal compared to $A$. ■

**Conclusion 2** If $\tilde{V}_t$ is $P^A - \text{supermartingale}$, then the effort strategy $A^*$ is at least as good as the strategy $A$ for the banker

**Proof.** Since $\tilde{V}_t$ is $P^A - \text{supermartingale}$, we have

$$E^A \left( \tilde{V}_t \bigg| F_s \right) \leq \tilde{V}_s$$

for all $s \leq t$. Applying the optional sampling theorem, we get

$$E^A \left( \tilde{V}_t \right) \leq \tilde{V}_0 = V_0$$

(18)

$E^A \left( \tilde{V}_t \right)$ accounts for the total utility the banker expects to get at date 0 if she always follows the strategy $A$ and so, (18) concludes the proof. ■

From two conclusions above, we obtain the necessary and sufficient condition for the optimality of the strategy $A^*$. That is, the drift coefficient of $\tilde{V}_t$ under $P^A$ is non positive:

$$v(A_t) + \frac{G_t}{\sigma} \mu A_t - \frac{G_t}{\sigma} \mu \leq 0 \text{ for all } A_t \in \{0, 1\}$$

It is equivalent to $G_t \geq \frac{\sigma B}{\mu}$. Q.E.D
Appendix 2: Proof of proposition 2

For the formal proof, we have to establish the following conclusions:

1) the contract characterized in this proposition is incentive compatible
2) It is optimal among the class of incentive compatible contracts

Because of the proposition 1, the incentive compatibility of the characterized contract is derived directly from the specification of the dynamic evolution of banker’s continuation utility. The proof for the optimality proceeds as follows:

B.1 Upper bound of the DIC’s expected payoff

Here, we will prove that the function $F$ - solution, if exists, to the HJB equation (13) with boundary condition $F(\tilde{W}) = 0$ constitutes an upper bound for the expected payoff the DIC can earn from any incentive compatible contract that delivers the banker an initial expected discounted utility $W_0$.

Consider any incentive compatible contract $(\tau, C)$, the expected payoff of the DIC is evaluated by

$$E \left[ \int_0^\tau e^{-rt} dR_t - \int_0^\tau e^{-rt} dC_t \right] = E \left[ \int_0^\tau e^{-rt} (\mu dt - dC_t) \right]$$

Define a stochastic process $M = \{M_t\}$ by

$$M_t = \int_0^t e^{-rs} dR_s - \int_0^t e^{-rs} dC_t + e^{-rt} F(W_t)$$

where $W_t$ is defined by (10) with $G_t \geq \frac{B}{\rho} \sigma$. We have

$$dM_t = e^{-rt} (dR_t - dC_t - rF(W_t)dt + dF(W_t))$$

Using the dynamic of $F(W_t)$ given by the equation (12), we get

$$e^{rt} dM_t = \left( \mu dt - dC_t + F'(W_t)\rho W_t dt + \frac{1}{2} F''(W_t)G_t^2 dt - F'(W_t) (1 + \alpha 1_{\{dC_t < 0\}}) dC_t - rF(W_t) dt \right)_{N_t}$$

$$+ \left( \sigma + F'(W_t)G_t \right) dZ_t$$

Since $F(W_t)$ is solution to (13), then $N_t \leq 0 \ \forall t$, which implies that $M = \{M_t\}$ is super-martingale. Therefore,

$$E \left[ \int_0^\tau e^{-rt} (\mu dt - dC_t) \right] = E [M_\tau] \leq M_0 = F(W_0)$$

In (20), the first equality stems from $F(W_\tau) = F(\tilde{W}) = 0$; the inequality is due to the optional sampling theorem. Q.E.D.
B.2 DIC’s expected payoff from the optimal contract

Now, we show that the contract characterized in this proposition provide the DIC with expected payoff exactly equal to \( F(W_0) \). Notice that if \( M = \{M_t\} \) is martingale, then

\[
E \left[ \int_0^\tau e^{-rt} (\mu dt - dC_t) \right] = E [M_\tau] = M_0 = F(W_0)
\]

Therefore, we should prove that under the contract characterized in the proposition 2, the process \( M = \{M_t\} \) defined by (19) is a martingale.

To begin with, we establish the existence and uniqueness of the solution to the ordinary differential equation

\[
rF(W) = F'(W)\rho W + \frac{1}{2} F''(W) \frac{B^2}{\mu^2} \sigma^2 + \mu
\]

over the interval \([\bar{W}, \underline{W}]\) with boundary conditions: \( F(\bar{W}) = 0, F'(\bar{W}) = -1, F''(\bar{W}) = 0 \).

Indeed, in the proof, we will work with the social surplus function \( S(W) \) defined by

\[
S(W) = F(W) + W
\]

This function \( S(W) \) will satisfy

\[
rS(W) = \mu - (\rho - r) W + S'(W)\rho W + \frac{1}{2} S''(W) \frac{B^2}{\mu^2} \sigma^2 \tag{21}
\]

and the corresponding boundary conditions are the following: \( S(\bar{W}) = \bar{W}, S'(\bar{W}) = 0, S''(\bar{W}) = 0 \). The homogeneous differential equation associated with (21) is written as follows

\[
rS(W) = S'(W)\rho W + \frac{1}{2} S''(W) \frac{B^2}{\mu^2} \sigma^2 \tag{22}
\]

It is easy to see that \( \frac{\mu}{r} + W \) constitutes a particular solution to (21). Assuming that \( H_0(W) \) and \( H_1(W) \) are two particular solutions to the homogeneous differential equation (22) such that \( H_0(\bar{W}) = 1, H_0'(\bar{W}) = 0 \) and \( H_1(\bar{W}) = 0, H_1'(\bar{W}) = 1 \). Denote by \( \mathcal{L}(W) \) the Wronskian associated with \( H_0(W) \) and \( H_1(W) \). Since \( \mathcal{L}(\bar{W}) \) is non-zero, these two functions are linearly independent. Therefore, general solution to (21) will be written as follows

\[
S(W) = \frac{\mu}{r} + W + a_0 H_0(W) + a_1 H_1(W)
\]

In order to define \( a_0 \) and \( a_1 \), we rely on two conditions \( S(\bar{W}) = \bar{W}, S'(\bar{W}) = 0 \) with some fixed value \( \underline{W} \). We find that

\[
S(W) = \frac{\mu}{r} + W - \frac{\mu}{r} H_0(W) + \frac{\mu H_0'(W)}{H_1(W)} - \frac{1}{H_1(W)} H_1(W)
\]

Hence, what we have to show now is that there exists unique \( \underline{W} > \bar{W} \) such that \( S''(\underline{W}) = 0 \). By (21), we obtain

\[
\frac{1}{2} \frac{B^2}{\mu^2} \sigma^2 S''(\underline{W}) = \frac{\rho \underline{W}^2 H_1'(\underline{W}) - r H_1(\underline{W}) - \mu \mathcal{L}(\underline{W})}{H_1^2(\underline{W})}
\]
Define a function \( \Psi(W) = \frac{\rho W H_1(W) - r H_1(W)}{\mathcal{L}(W)} \). So \( \Psi(\bar{W}) = \rho \bar{W} \) and
\[
\frac{1}{2} B^2 \frac{\partial^2}{\partial W^2} \Psi(W) = \frac{[\Psi(W) - \rho \mathcal{L}(W)]}{H_1(W)}
\]
Using Abel’s identity, we get the following expression of the Wronskian: \( \mathcal{L}(W) = \exp\left(\frac{\mu(W^2 - W^2)}{B^2 \mu^2} \right) \).
So,
\[
\Psi'(W) = \frac{\rho - r}{\mathcal{L}(W)} H_1'(W)
\]
and
\[
\Psi''(W) = \frac{\rho - r}{\mathcal{L}(W)} \frac{2r}{B^2 \mu^2} H_1(W)
\]
Therefore, the characteristics of the function \( \Psi(W) \) depend on the properties of the function \( H_1(W) \). Here, we will show that \( H_1'(W) > 0 \) for all \( W > \bar{W} \). By contradiction, assuming that there exists a \( W > \bar{W} \) such that \( H_1'(W) \leq 0 \). Define \( W = \inf \left\{ W > \bar{W} : H_1'(W) \leq 0 \right\} \).
Hence
\[
H_1'(W) > 0 \quad \forall W \in [\bar{W}, \bar{W}]
\]
which implies that \( H_1(\bar{W}) > H_1(\bar{W}) = 0 \). By (22) and \( H_1'(\bar{W}) \leq 0 \), we have \( H_1''(\bar{W}) > 0 \).
In the neighbourhood \( [W - \varepsilon, \bar{W}] \), \( H_1'(W) \) is increasing function. So, over \([W - \varepsilon, \bar{W}]\), \( H_1'(W) < H_1'(\bar{W}) \leq 0 \), which contradicts (23). In summary, \( H_1'(W) > 0 \forall W > \bar{W} \), then, the function \( \Psi(W) \) is strictly increasing and convex, which implies that there exists unique \( \bar{W} \) such that \( \Psi(\bar{W}) = \mu \). Moreover, because \( \mu > \rho \bar{W} = \Psi(\bar{W}) \), such a \( \bar{W} \) must be greater than \( \bar{W} \).

Next, we show that solution to (21) is concave function. Differentiating the equation (21), we get
\[
\frac{1}{2} S'''(W) \frac{B^2}{\mu^2} \sigma^2 = (\rho - r) \left(1 - S'(W) \right) - S''(W) \rho W
\]
Hence, \( S'''(W) > 0 \), which implies that in the neighbourhood \( (W - \varepsilon, \bar{W}) \) of \( \bar{W} \), \( S''(W) < 0 \) and \( S'(W) > 0 \). We will prove that \( S'(W) > 0 \) for all \( W \in [\bar{W}, \bar{W} - \varepsilon] \). Suppose that \( S'(W) \leq 0 \) for some \( W < \bar{W} - \varepsilon \). Let \( \hat{W} = \sup \left\{ W < \bar{W} - \varepsilon : S'(W) \leq 0 \right\} \). So, over the interval \([\hat{W}, \bar{W}]\), \( S'(W) > 0 \) and \( rS(W) < rS(\bar{W}) = \mu - (\rho - r) \bar{W} < \mu - (\rho - r) W \). By (21), over this interval \( S'''(W) < 0 \). Thus, \( S'(W) = -\int^{W}_{\hat{W}} S'''(W)dW > 0 \), contradiction.

Hence, \( S'(W) > 0 \) for all \( W \in [\bar{W}, \bar{W}] \). By (21), for all \( W \in [\bar{W}, \bar{W}] \),
\[
\frac{1}{2} S'''(W) \frac{B^2}{\mu^2} \sigma^2 \leq rS(W) - \mu + (\rho - r) W < rS(\bar{W}) - \mu + (\rho - r) W = 0 \quad Q.E.D
\]

C Appendix 3: Proof of proposition 3

To prove this proposition, we must show the existence, uniqueness and concavity of the function \( F \). All other parts are similar to the proof of the proposition 2.
C.1 Existence, uniqueness and concavity of the function $F_2$

We fix some value $\bar{W}$. Define a function $S_2(W)$ by $S_2(W) = F_2(W) + W$. Hence, $S_2(W)$ satisfies the following differential equation

$$rS_2(W) = \mu - (\rho - r)W + S'_2(W)\rho W + \frac{1}{2}S''_2(W)\frac{B^2}{\mu^2}\sigma^2$$  \hspace{1cm} (24)

on the region $[\bar{W}, W^*)$ with boundary conditions $S'_2(\bar{W}) = \frac{a}{1+\alpha}$, $S'_2(W^*) = 0$ and $S''_2(W^*) = 0$. Let $P_0(W)$ and $P_1(W)$ be two particular solutions of the homogeneous differential equation associated with (24) such that $P_0(\bar{W}) = 0$, $P_0'(\bar{W}) = 1$ and $P_1(\bar{W}) = -1$, $P_1'(\bar{W}) = 0$. Since the Wronskian $L_{P_0,P_1}(W)$ associated with $P_0(W)$ and $P_1(W)$ has non-zero value at the point $\bar{W}$, the two function $P_0(W)$ and $P_1(W)$ are linearly independent. The general solution to the equation (24) can then be written as

$$S_2(W) = \frac{\mu}{r} + W + d_0P_0(W) + d_1P_1(W)$$

Based on two conditions $S'_2(\bar{W}) = \frac{a}{1+\alpha}$ and $S'_2(W^*) = 0$ we obtain

$$d_0 = -\frac{1}{1+\alpha}$$
$$d_1 = \frac{P'_0(W^*) - 1}{P'_1(W^*)}$$

Thus, the general solution to the equation (24) becomes

$$S_2(W) = \frac{\mu}{r} + W - \frac{1}{1+\alpha}P_0(W) + \frac{P'_0(W^*) - 1}{P'_1(W^*)}P_1(W)$$

Because of (24), we get

$$\frac{1}{2}B^2\frac{1}{\mu^2}\sigma^2S''_2(W^*) = \frac{\rho W^*P'_1(W^*) - rP_1(W^*) - \frac{r}{1+\alpha}L_{P_0,P_1}(W^*)}{P'_1(W^*)}$$ \hspace{1cm} (25)

Define a function $\Phi(W) = \frac{\rho W P'_1(W) - rP_1(W)}{L_{P_0,P_1}(W)}$, the equality (25) becomes

$$\frac{1}{2}B^2\frac{1}{\mu^2}\sigma^2S''_2(W^*) = \frac{\left(\Phi(W^*) - \frac{r}{1+\alpha}\right)L_{P_0,P_1}(W^*)}{P'_1(W^*)}$$ \hspace{1cm} (26)

Now we will prove that the function $\Phi(W)$ is strictly decreasing and concave. Indeed, from the following expression of the Wronskian: $L_{P_0,P_1}(W) = \exp\left(\frac{\rho(W^2 - W^2)}{\frac{B^2}{\mu^2}\sigma^2}\right)$ obtained
by applying the Abel’s identity, we get\(^\text{12}\)

\[
\Phi'(W) = \frac{(\rho - r) P'_1(W)}{L_{P_0 P_1}(W)}
\]
\[
\Phi''(W) = \frac{2r(\rho - r) P'_1(W)}{\mu^2 \sigma^2} \frac{B^2}{L_{P_0 P_1}(W)}
\]

Therefore, showing that the function \(\Phi(W)\) is strictly decreasing and concave is equivalent to prove that \(P'_1(W) < 0\) for all \(W \geq \bar{W}\). Assuming by contradiction that there exists \(W \in (\bar{W}, +\infty)\) such that \(P'_1(W) > 0\). Define \(\bar{W} = \inf \{W > \bar{W} : P'_1(W) > 0\}\), then

\[
\forall W \in (\bar{W}, \bar{W}) : P'_1(W) < 0
\]

which implies that \(P_1(\bar{W}) < P_1(\bar{W}) = -1 < 0\). Since the function \(P_1(W)\) satisfies

\[
\frac{1}{2} \frac{B^2}{\mu^2} \sigma^2 P''_1(W) + \rho WP'_1(W) - rP_1(W) = 0
\]

we get \(P''_1(\bar{W}) < 0\). Hence, in the neighbourhood \([\bar{W} - \varepsilon, \bar{W}]\), the function \(P'_1(W)\) is decreasing which means that for \(W \in [\bar{W} - \varepsilon, \bar{W}]\), \(P'_1(W) > P'_1(\bar{W}) > 0\), contradiction with (27). In summary, we have \(P'_1(W) < 0\) for all \(W > \bar{W}\) and so, \(\Phi(W)\) is strictly decreasing and concave function. From (26), we see that the equation \(S'_2(W^*) = 0\) equivalent to \(\Phi(W^*) = \frac{r}{1 + \alpha}\) will have unique solution \(W^*\). Moreover, because \(\Phi(\bar{W}) = r > \frac{r}{1 + \alpha}\), such a solution \(W^*\) will be greater than \(\bar{W}\).

**Concavity:** The proof for the concavity of the function \(S_2(W)\) is similar as in the appendix 2.

### C.2 Existence, uniqueness and concavity of the function \(F_1\)

Considering a function \(S_1(W)\) defined by \(S_1(W) = F_1(W) + W\). \(S_1(W)\) satisfies the following differential equation

\[
rS_1(W) = (\mu - \alpha K) - (\rho - r) W + S'_1(W) (\rho W + (1 + \alpha)K) + \frac{1}{2} S''_1(W) \frac{B^2}{\mu^2} \sigma^2
\]

over the interval \([\bar{W}, \bar{W}]\) with boundary conditions: \(S_1(\bar{W}) = \bar{W}, S'_1(\bar{W}) = \frac{\alpha}{1 + \alpha}\). As in the previous part, the general solution to the equation (28) can be written as

\[
S_1(W) = \frac{\mu + K}{r} + W + b_0 N_0(W) + b_1 N_1(W)
\]

where \(N_0(W)\) and \(N_1(W)\) are two particular solutions to the corresponding homogeneous differential equation such that \(N_0(\bar{W}) = 1, N'_0(\bar{W}) = 0\) and \(N_1(\bar{W}) = 0, N'_1(\bar{W}) = 1, b_0\) and \(b_1\) are calculated from two boundary conditions \(S_1(\bar{W}) = \bar{W}, S'_1(\bar{W}) = \frac{\alpha}{1 + \alpha}\) with some

\(^{12}\)Note that \(P_1(W)\) is solution to the homogeneous differential equation associated with the equation (24).
fixed value $\bar{W}$ as follows

$$b_0 = -\frac{\mu + K}{r}$$

$$b_1 = \frac{\mu + K}{r} \frac{N'_1(\bar{W}) - \frac{1}{1 + \alpha}}{N'_1(\bar{W})}$$

Hence

$$\frac{1}{2} \frac{B^2}{\mu^2} \sigma^2 S''_1(\bar{W}) = \left( K + \frac{\rho \bar{W}}{1 + \alpha} \right) N'_1(\bar{W}) - \frac{r}{1 + \alpha} N_1(\bar{W}) - \frac{\mu + K}{1 + \alpha} \mathcal{L}_{N_0N_1}(\bar{W})$$

where $\mathcal{L}_{N_0N_1}(W)$ is the Wronskian associated with $N_0$ and $N_1$. Define a function $\Pi(W)$ by

$$\Pi(W) = \left( K + \frac{\rho \bar{W}}{1 + \alpha} \right) N'_1(W) - \frac{r}{1 + \alpha} N_1(W)$$

then,

$$\frac{1}{2} \frac{B^2}{\mu^2} \sigma^2 S''_1(\bar{W}) = \left( \Pi(\bar{W}) - \frac{\mu + K}{1 + \alpha} \right) \mathcal{L}_{N_0N_1}(\bar{W})$$

Similarly to the appendix 2, we can prove that $N'_1(W) > 0$ for all $W > \bar{W}$ and that the function $\Pi(W)$ is strictly increasing and convex. Moreover $\Pi(\bar{W}) = K + \frac{\rho \bar{W}}{1 + \alpha} < K + \mu$. Therefore, there exists $\bar{W} > \bar{W}$ such that $\Pi(\bar{W}) < 0$. To determine the threshold $\bar{W}$, we will rely on the condition $F'_1(\bar{W}) = F''_1(\bar{W}) \leq 0$.

**Concavity:** We here prove that as long as $S'_1(\bar{W}) < 0$, then the solution to the equation (28) is concave function. The proof proceeds very similarly as in the precedent parts. Differentiating the equation (28), we get

$$\frac{1}{2} S''_1(W) \frac{B^2}{\mu^2} \sigma^2 = (\rho - r) \left( 1 - S'_1(W) \right) - S'_1(W) (\rho W + (1 + \alpha)K)$$

Hence, $S''_1(\bar{W}) > 0$, which implies that in the neighbourhood $(\bar{W} - \varepsilon, \bar{W})$ of $\bar{W}$, $S''_1(W) < 0$ and $S'_1(W) > \frac{\alpha}{1 + \alpha}$. We will prove that $S'_1(W) > \frac{\alpha}{1 + \alpha}$ for all $W \in [\bar{W}, \bar{W} - \varepsilon]$. Suppose that $S'_1(W) \leq \frac{\alpha}{1 + \alpha}$ for some $W < \bar{W} - \varepsilon$. Let $\tilde{W} = \sup \left\{ W < \bar{W} - \varepsilon : S'_1(W) \leq \frac{\alpha}{1 + \alpha} \right\}$. So, over the interval $(\tilde{W}, \bar{W})$, $S'_1(W) > \frac{\alpha}{1 + \alpha}$ and then for all $W \in (\tilde{W}, \bar{W})$:

$$r S_1(W) - \frac{r \alpha}{1 + \alpha} W < r S_1(\bar{W}) - \frac{r \alpha}{1 + \alpha} \bar{W} \leq \mu - \frac{\rho - r}{1 + \alpha} \bar{W} < \mu - \frac{\rho - r}{1 + \alpha} W$$

(29)

The first inequality comes from the fact that the function $S_1$ is strictly increasing on the interval $(\tilde{W}, \bar{W})$. The second inequality is obtained by replacing $S'_1(\bar{W}) = \frac{\alpha}{1 + \alpha}$ and $S''_1(\bar{W}) \leq 0$ into the equation (28). (29) implies that $r S_1(W) < \mu + \frac{\alpha}{1 + \alpha} \rho W - (\rho - r) W$.

By (28), we have $S''_1(W) < 0$ over the interval $(\bar{W}, \bar{W})$. Thus, $0 > \int_{\bar{W}}^{\bar{W}} S'_1(W) dW = \frac{\alpha}{1 + \alpha} - S'_1(\bar{W})$ and so, $S'_1(\bar{W}) > \frac{\alpha}{1 + \alpha}$ contradiction. Hence, $S'_1(W) > \frac{\alpha}{1 + \alpha}$ for all $W \in [\tilde{W}, \bar{W}]$.

By (28), for all $W \in [\tilde{W}, \bar{W})$, $\frac{1}{2} S''_1(W) \frac{B^2}{\mu^2} \sigma^2 \leq r S_1(W) - \mu - \frac{\alpha}{1 + \alpha} \rho W + (\rho - r) W <
$$rS_1(\tilde{W}) - \mu - \frac{\alpha}{1+\alpha} \rho \tilde{W} + (\rho - r) \tilde{W} \leq 0. Q.E.D.$$


