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Luc Bauwens, Arnaud Dufays
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Center for Operations Research
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Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium
http://www.uclouvain.be/core

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Estimating and forecasting structural breaks in financial time series

Luc BAUWENS [1], Arnaud DUFAYS[2]
and Bruno DE BACKER [3]

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Abstract

We present an algorithm, based on a differential evolution MCMC method, for Bayesian inference in AR-GARCH models subject to an unknown number of structural breaks at unknown dates. Break dates are directly treated as parameters and the number of breaks is determined by the marginal likelihood criterion. We prove the convergence of the algorithm and we show how to compute marginal likelihoods. We allow for both pure change-point and recurrent regime specifications and we show how to forecast structural breaks. We illustrate the efficiency of the algorithm through simulations and we apply it to eight financial time series of daily returns over the period 1987-2011. We find at least three breaks in all series.

Keywords: Bayesian inference, structural breaks, differential evolution, change-point, recurrent states, break forecasting, marginal likelihood.

JEL Classification: C11, C15, C22, C58

[1] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium. E-mail: luc.bauwens@uclouvain.be. This author is also member of ECORE, the association between CORE and ECARES.
[2] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium. E-mail: arnaud.dufays@uclouvain.be.
[3] Bruno de Backer started working on this project during his studies at the LSE.

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1 Introduction

The class of Generalized AutoRegressive Conditional Heteroskedastic (GARCH) models (see Engle (1982), and Bollerslev (1986)) has been widely applied to different kinds of time series since it is a simple and powerful way to model volatility. Nowadays, GARCH volatility models with fixed parameters are known to be too restrictive for long time series due to breaks in the volatility process.

Ignoring structural breaks generally produces poor forecasts and often gives the spurious impression of a nearly integrated behaviour of the time series (see, e.g., Diebold (1986), Lamoureux and Lastrapes (1990), and Hillebrand (2005)). Moreover, GARCH processes imply that long-run forecasts of conditional volatility tend to the unconditional variance of the error term in the mean equation. In order to tackle these issues more flexible models have been designed. They can be split into three categories: Component and smooth transition models (e.g. Amado and Terasvirta (2008)), Markov-switching models (e.g. Gray (1996), Haas, Mittnik, and Paolella (2004), Bauwens, Preminger, and Rombouts (2010), and He and Maheu (2010)), and models with time-varying unconditional variance (e.g. Engle and Rangel (2008)). In this paper we focus on change-point and recurrent state models. These models assume that structural changes can happen in an economy and affect the parameters of the considered econometric model but do not change the nature of the model itself.

Dealing with structural breaks is usually done by applying the forward-backward algorithm introduced by Chib (1996). However, the method cannot be applied to models with path dependence (see Fruhwirth-Schnatter (2004)) since the number of computations to be performed by the computer grows exponentially with the length of the time series. Indeed, path dependence is the fact that any value of the series depends on all the previous ones. For instance, ARMA and GARCH processes can be path dependent in the mean (cfr. ARMA) and conditional variance (cfr. GARCH).

Four recent papers tackle the issue of path dependence, all of them focusing on GARCH models. He and Maheu (2010) use a Sequential Monte Carlo *on-line* method in order to tell whether a break occurs when a new observation becomes available. Hence, their method is more useful to forecasting rather than detecting purposes. On the contrary, Liao (2008), Bauwens, Preminger, and Rombouts (2010), and Bauwens, Dufays, and Rombouts (2011) develop off-line MCMC methods. Similarly to the present work, Liao (2008) considers the break dates as parameters to be estimated and not anymore as hidden states underlying the data generating process. Liao (2008) separates the problem into two parts: drawing the model parameters and drawing the break dates. She uses a standard Gibbs sampler for the model parameters and chooses to draw the break dates via the Griddy-Gibbs method (see Bauwens, Lubrano, and Richard (1999)). She also proposes a sequential technique to determine the number of breaks in a series which is very different from what is done in this work. Still considering MCMC methods, the approach in Bauwens, Preminger, and Rombouts (2010) does not mix well because the sampling of the variables suffers from huge correlations across iterations of the algorithm. This problem is circumvented in Bauwens, Dufays, and Rombouts (2011) by embedding a Sequential Monte Carlo technique inside an MCMC algorithm. This approach works well and allows to compute marginal likelihoods for model selection but is computationally demanding.

The technical articles that influence the present paper are ter Braak and Vrugt (2008) and Vrugt, ter Braak, Diks, Robinson, Hyman, and Higdon (2009). ter Braak and Vrugt (2008) propose a Differential Evolution Markov Chain (DEMC) algorithm in which the proposal of the Metropolis algorithm is automatically generated. An extension of this algorithm is the DiffeRential Adaptative Evolution Metropolis (DREAM) algorithm of Vrugt, ter Braak, Diks, Robinson, Hyman, and Higdon (2009). In this paper we derive a Discrete-DREAM algorithm in order to sample the break dates in a Metropolis-type of block-at-a-time algorithm. We prove the convergence of our MCMC algorithm to the stationary distribution.

We thus introduce a novel MCMC method which deals with change-points in time series models and computes the resulting marginal likelihood. The proposed algorithm has the advantage to accommodate path dependence problems and to be fast to converge and easy to implement. Our specification allows for recurrent states which is a class of structural break model somewhere between Change-Point and Markov-Switching models. Moreover we also introduce a hierarchical structure to forecast structural break in the spirit of what Pesaran, Pettenuzzo, and Timmermann (2006) did for autoregressive models.

The paper is organized as follows. In section 2, we present the DREAM algorithm that we adapt to the present context of structural breaks. In section 3, we explain how to deal with the unknown number of breaks using the marginal likelihood criterion. In section 4, we propose two improvements, namely how to allow for recurrent regimes and how to forecast future breaks. In section 5, we test the efficiency of the new algorithm by means of simulations. In section 6, we apply the algorithm on daily data for the S&P500 index and give summary statistics for seven other financial series. Section 7 concludes.

2 D-DREAM Algorithm

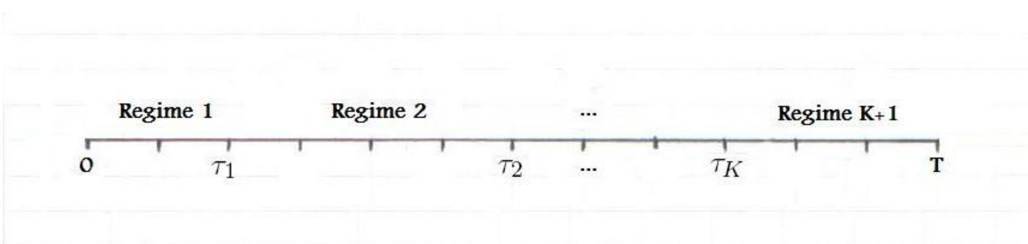
For the sake of simplicity, the present discussion focuses on an AR(1)-GARCH(1,1) specification though the algorithm could easily be adapted to more general models such as ARMA(p,q)-GARCH(r,s) and many other time series models.

We denote by $Y_T = (y_1, ..., y_T)'$ a time series of T observations. Instead of considering a whole vector of states as in Chib (1996), we focus on the break dates $\Gamma = (\tau_1, ..., \tau_K)'$, with the number of break dates K considered known for the moment. A Change-Point-AR(1)-GARCH(1,1) model is thus characterized by the following set of equations:

$$
\begin{aligned}
y_t &= \mu_k + \phi_k y_{t-1} + \epsilon_t, \\
\epsilon_t &= \sigma_t \eta_t, \ \textit{with} \ \eta_t \sim i.i.d. \, N(0,1), \\
\sigma_t^2 &= c_k + \alpha_k \epsilon_{t-1}^2 + \beta_k \sigma_{t-1}^2,
\end{aligned}
$$

if $t \in]\tau_{k-1}, \tau_k]$ with $\tau_0 = 0$ and $\tau_{K+1} = T$, with $k = \{1, 2, ..., K+1\}$ denoting the regime, μ_k and $\phi_k \in \Re$, $c_k \in \Re^+$, and α_k and $\beta_k \in [0, 1[$. We further impose, $\forall k$, $\alpha_k + \beta_k < 1$ as a stationarity condition.[1] The break date parameters are contained in a discrete vector which denotes the beginning and the end of each regime. See the figure 1 for an example.

Figure 1: Example of Break Date Configuration



[1] However, it should be noted that the Bayesian framework allows us to skip this stationarity condition. We choose to keep it as all series we use in this paper seem to be stationary, their volatility dynamics as well, and because we also save some computational time.

Let $\Theta = (\theta_1, \theta_2, ..., \theta_{K+1})'$ and $\theta_i = (\mu_i, \phi_i, c_i, \alpha_i, \beta_i)$, i.e. the parameters of the regime i. Our MCMC scheme simply consists in drawing first from the posterior of the model parameters Θ and, then, from the posterior of the break dates Γ as follows,

1. $f(\Theta|Y_T, \Gamma)$
2. $f(\Gamma|Y_T, \Theta)$

The difficulty is to sample from these two conditional densities since they have no closed form. We detail now our Metropolis-type of procedure.

2.1 Sampling Θ

The full conditional distribution of Θ does not have any closed form but is proportional to the likelihood times the prior:

$$f(\Theta|Y_T, \Gamma) \quad \propto \quad f(Y_T|\Theta, \Gamma) f(\Theta)$$

Sampling from this distribution can be done by the Metropolis-Hastings (M-H) algorithm. However the M-H algorithm is extremely versatile and depends on the proposal distribution. ter Braak and Vrugt (2008) propose a Differential Evolution Markov Chain (DEMC) algorithm in which the proposal of the Metropolis algorithm is automatically generated, that is, the size and the direction of the jump are endogenously determined by the Markov Chains. The idea is still to make a proposal from the previous draw to which a random term is added, however, since they run multiple chains in parallel they can make these chains interact between each other. An extension of this algorithm is the DiffeRential Adaptative Evolution Metropolis (DREAM) algorithm of Vrugt, ter Braak, Diks, Robinson, Hyman, and Higdon (2009). They show that the DREAM algorithm is more efficient than the DEMC algorithm in the sense that it mixes better and thus converges faster. We use the DREAM algorithm to sample from the full conditional distribution of Θ.

Following the DREAM algorithm, N Markov chains are launched and interact between each other. The proposal of a new location for a given chain is indeed constructed from the current locations of other chains. Focusing on the i^{th} chain ($i = 1, ..., N$) at the $t+1^{th}$ MCMC iteration, the procedure can be described in 3 steps.

1. Propose a new Θ_i^{t+1}, denoted by Z_i hereafter, according to the following rule (omitting superscripts referring to iterations for simplicity):

$$Z_i = \Theta_i + \gamma(\delta, d')(\sum_{g=1}^{\delta} \Theta_{r_1(g)} - \sum_{h=1}^{\delta} \Theta_{r_2(h)}) + \epsilon, \tag{1}$$

with $\forall g, h = 1, ..., \delta$, $i \neq r_1(g)$, $r_2(h)$; and $r_1(g) \neq r_2(h)$ if $g = h$. The fixed number of pairs of chains δ is smaller than the number of chains used in the differential evolution algorithm, $r_1(.)$ and $r_2(.)$ stand for random integers uniformly distributed between $[1, N]_{-i}$, $\epsilon \sim N(0, \eta_\Theta^2 I)$ where η_Θ^2 is small compared to the width of the target distribution, and $\gamma(\delta, d')$ is a multiplier of the size of the jump.

2. Replace each component of Z_i one by one by their old value in Θ_i according to a cross-over probability CR, d' being the number of "cross-overs". In proposal (1), $\gamma(\delta, d')$ is then defined as $\frac{2.38}{\sqrt{\delta d'}}$, following ter Braak and Vrugt (2008).[2]

3. Accept the proposed Z_i according to the acceptance probability ratio of the Metropolis algorithm $\alpha(\Theta_i, Z_i) = min\{\frac{f(Z_i|\Gamma, Y_T)}{f(\Theta_i|\Gamma, Y_T)}, 1\}$.

The second step of the algorithm can be overlooked by setting the cross-over probability CR equal to 1. However, in large dimensional spaces, it might not be optimal to resample the entire vector of parameters as a whole. Rather, the probability CR would be around .5 or .3.[3]

Since the support of the proposal is not constrained, we map the constrained space of the model parameters into the real space before applying the DREAM algorithm.[4]

2.2 Sampling Γ

The sampling procedure of the break date vector Γ is similar to the sampling of Θ and can be described in the same three steps as above. The only difference is that we need to round the elements of the proposed vector of break dates Z_i so as to have integers. Equation 1 then

[2]ter Braak and Vrugt show that this choice for $\gamma(\delta, d')$ is optimal under certain conditions.

[3]The DREAM algorithm in Vrugt, ter Braak, Diks, Robinson, Hyman, and Higdon (2009) also introduces some improvements to select the probability CR so as to maximize the jump length.

[4]If $x \in \Re^+$, then $\ln(x) \in \Re$. If $x \in [0, 1[$, $\ln(\frac{x}{1-x}) \in \Re$.

becomes:

$$Z_i = \Gamma_i + round[\gamma(\delta, d')(\sum_{g=1}^{\delta} \Gamma_{r1(g)} - \sum_{h=1}^{\delta} \Gamma_{r2(h)}) + \epsilon], \tag{2}$$

with $round[.]$ meaning that we take the nearest integer and $\epsilon \sim N(0, \eta_\Gamma^2 I)$.

The convergence and ergodicity of the chains in the Discrete-DREAM (D-DREAM hereafter) algorithm is insured by Theorem 1.

Theorem 1. For all $i = 1, ..., N$, let $X_i = (\Theta_i, \Gamma_i)'$. The D-DREAM algorithm yields a Markov chain that is ergodic with unique stationary distribution:

$$(X_1, ..., X_N) \sim \tilde{\pi}(X_1, ..., X_N | Y_T) = \pi(X_1|Y_T) \times ... \times \pi(X_N|Y_T).$$

Proof. See appendix.

2.3 Priors, Starting Point, and Burn-in

As is shown in table 1, we use standard prior distributions for this type of model. For the AR and GARCH parameters we choose respectively independent normal and uniform distributions. For the break dates we choose a prior that complies with the modelling constraints.[5]

Table 1: Prior Distributions

AR parameters			GARCH parameters			Break Date Parameters		
μ	$\sim$	Normal(0,1)	c	$\sim$	Uniform[0,5]	τ_1	$\sim$	Uniform[2,T-K+1]
ϕ	$\sim$	Normal(0,1)	α	$\sim$	Uniform[0,1]	$\forall k \in [2, K]$, τ_k	$\sim$	Uniform[$\tau_{k-1} + 1$,T-K+k]
			β	$\sim$	Uniform[0,1]			

The starting point of an MCMC algorithm is also a relevant practical issue since choosing a highly likely point with respect to the posterior distribution ensures faster convergence. Typically, MCMC algorithms start at the ML estimates but these cannot be computed in the case of structural breaks. In order to find good starting values, we randomly generate many break date vectors Γ and we maximize the parameters $(\mu_k, \phi_k, c_k, \alpha_k, \beta_k)$ for all regimes with respect to the likelihood by drawing a certain number of these parameters for each generated

[5]Namely, the chronological order of break dates should always be respected and no break date can fall outside of the interval [2,T].

break date vector. The initial points of the N chains are the N randomly generated vectors of break dates and associated maximized parameters with the highest global likelihoods.

We determine the length of the burn-in period using the criterion in Gelman and Rubin (1992). During this phase we replace outlier chains by the chain with the highest likelihood to ease convergence. These chains are defined similarly as in Vrugt, ter Braak, Diks, Robinson, Hyman, and Higdon (2009) as the ones with a data log-likelihood smaller than the third quartile minus 2 times the inter-quartile range (Q3-Q1) of the log-likelihoods of all chains at a given iteration.

3 Model Selection

From Section 2, assuming that a model is taken as given, the algorithm still requires the user to fix the number of breaks, K, which is often unknown. In this section, we remedy to this problem by selecting the best model after having run the algorithm with different values of K. Bayes factors are usually used for model selection, which requires the computation of the marginal likelihood. For complex models it may be difficult to compute and, consequently, several algorithms were developed. This paper focuses on the Bridge sampling method of Meng and Wong (1996) and on Chib and Jeliazkov (2001)'s formula.

3.1 Bridge Sampling

For simplicity, we will use the integral symbol to integrate out Γ (as if the variable was continuous). The marginal likelihood can then be written as

$$f(Y_T) = \int\int f(Y_T|\Theta,\Gamma)p(\Theta,\Gamma)d\Theta d\Gamma. \tag{3}$$

Taking a function $t(\Theta,\Gamma)$ and a proposal distribution $q(\Theta,\Gamma)$, we can define the following two quantities:

$$A = \frac{1}{G_1}\sum_{g_1=1}^{G_1} t(\Theta^{g_1},\Gamma^{g_1})q(\Theta^{g_1},\Gamma^{g_1}),$$
$$A_1 = \frac{1}{G_2}\sum_{g_2=1}^{G_2} t(\Theta^{g_2},\Gamma^{g_2})f(Y_T|\Theta^{g_2},\Gamma^{g_2})p(\Theta^{g_2},\Gamma^{g_2}),$$

where G_1 and G_2 are large. The sequence $\{\Theta^{g_1}, \Gamma^{g_1}\}_{g_1=1}^{G_1}$ is sampled from $f(\Theta, \Gamma|Y_T)$, which is done by MCMC, and $\{\Theta^{g_2}, \Gamma^{g_2}\}_{g_2=1}^{G_2}$ is sampled from $q(\Theta, \Gamma)$.

Meng and Wong (1996) show that the marginal likelihood can be computed using these two quantities as $P(Y_T) \approx \frac{A_1}{A}$.

The choice of $t(\Theta, \Gamma)$ can improve the accuracy of the estimate. If $t(\Theta, \Gamma) = \frac{1}{q(\Theta,\Gamma)}$, the method is equivalent to a simple importance sampling technique. In this paper, we use the asymptotically optimal choice of $t(\Theta, \Gamma)$ derived in Meng and Wong (1996) that minimizes the expected relative error of the estimator for i.i.d. draws from $f(\Theta, \Gamma|Y_T)$ and $q(\Theta, \Gamma)$:

$$t(\Theta, \Gamma) = \frac{1}{f(\Theta, \Gamma|Y_T) + q(\Theta, \Gamma)}$$

The proposal distribution $q(\Theta, \Gamma)$ is given in appendix.

3.2 Chib and Jeliazkov (2001)

We also compute the marginal likelihood from the local counterpart of formula (3) which is given by

$$f(Y_T) = \frac{f(Y_T|\Theta^*, \Gamma^*)p(\Theta^*, \Gamma^*)}{f(\Theta^*, \Gamma^*|Y_T)}, \quad (4)$$

where $\{\Theta^*, \Gamma^*\}$ is theoretically any point of the posterior distribution.

The goal is to compute each of the three terms : $f(Y_T|\Theta^*, \Gamma^*)$ is the likelihood, $p(\Theta^*, \Gamma^*)$ is the prior and, $f(\Theta^*, \Gamma^*|Y_T)$ is computed following the method of Chib and Jeliazkov (2001).

This method can be seen as a bridge sampling method with a specific choice for the function $t(\Theta, \Gamma)$ (see Meng and Schilling (2002), and Mira and Nicholls (2004)). Thus Chib and Jeliazkov (2001) formula should be less accurate than the bridge sampling method using the optimal choice for $t(\Theta, \Gamma)$, assuming that posterior draws are independently and identically distributed.

4 Improvements

It will be shown in Section 5 that the D-DREAM algorithm works well in practice. Nevertheless, one can argue that its scope is limited since it can only deal with change-point specifications. This section shows how to enhance the algorithm by allowing for recurrent regimes and by forecasting structural breaks.

4.1 Recurrent Regimes

The D-DREAM algorithm as it is up to now can only deal with change-point specifications. Hence, a natural extension is to allow for recurrent regimes, like in a Markov-switching framework. Therefore, we include a permutation variable λ which is a K+1 by 1 vector indicating which regime is active between two break dates. More precisely, at each MCMC iteration all possible combinations in λ are considered and the algorithm keeps the most likely permutation. As an example, assume that the number of regimes is four, all possible λ's, generally denoted by $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, are as follows: (1,2,3,4), (1,2,1,4), (1,2,3,1), (1,2,3,2), (1,2,1,2). All other combination possibilities would imply label switching problems. For example, the combination $(1, 3, 1, 3)$ is by no way different from labeling the regimes $(1, 2, 1, 2)$. The first possibility (1,2,3,4) is a pure change-point specification whereas the others are recurrent regime specifications. Take the case of the combination $(1, 2, 1, 4)$, the first regime is called recurrent as it is active before the first break date and between the second and third break dates. The parameter λ is easily inserted in the MCMC scheme. With all values of λ a priori equally probable, the computation of the full conditional density is straightforward since

$$f(\lambda|Y_T, \Theta, \Gamma) \propto f(Y_T|\Theta, \Gamma, \lambda)p(\lambda),$$

However it somewhat complicates the sampling of the parameter Θ since when a permutation parameter implies at least one recurrent regime we have at least one irrelevant regime, which does not alter the likelihood anymore. Any new proposal parameter for this regime will thus often be accepted although its value might be unrealistic. The algorithm must preclude such cases. We therefore condition the prior of Θ on the permutation parameter $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{K+1})$ as follows,

$$p(\Theta, \lambda) = \prod_{i=1}^{K+1} p(\theta_i|\lambda)p(\lambda)$$

where

$$p(\theta|\lambda) = p(\theta)\mathbb{1}_{\lambda_i=i} + \mathbb{1}_{\theta=\theta_i}\mathbb{1}_{\lambda_i\neq i}.$$

By introducing this new variable λ we hope to provide a formulation that encompasses both Change-Point and Markov-switching models.

4.2 Forecasting Breaks

Another possible improvement of the D-DREAM algorithm is to impose a structure on the break dates in order to forecast future breaks. Our approach is in the spirit of the seminal paper of Pesaran, Pettenuzzo, and Timmermann (2006) but is also more general as we can consider any interaction between regimes. Assuming a hierarchical distribution on the break dates, a forecast is drawn according to the following identity:

$$f(\tau_{K+1}|Y_T) = \int\int\int f(\tau_{K+1}|Y_T, \Theta, \Gamma, \nu) f(\Theta, \Gamma, \nu|Y_T) d\Theta d\Gamma d\nu,$$

where ν is a vector containing the parameters of the hierarchical distribution of the break dates. Obviously, τ_{K+1} is not anymore necessarily equal to the sample size T.

Like in Pesaran, Pettenuzzo, and Timmermann (2006), we could model the duration of regimes by a probability parameter of remaining in the same regime at the next observation. In this case, choosing a geometric distribution as prior, the expected value of the future duration is equal to the mean of all previous durations. Considering that the last duration should have more weight and that weights should decrease to zero as we move away to older durations, one could consider an AR(1) model for the duration process.

Let us define a duration vector $\kappa = [\kappa_1, ..., \kappa_K]$ where, $\forall i \in [2, K]$, $\kappa_i = \tau_i - \tau_{i-1}$ and $\kappa_1 = \tau_1$. Although the duration is a discrete variable we assume an AR(1) model with normally distributed innovations, as follows:

$$\kappa_i = \text{round}[\nu_1 + \nu_2 \kappa_{i-1} + \epsilon_i], \epsilon_i \sim N(0, \bar{\sigma}^2),$$

where $\bar{\sigma}^2$ is fixed since usually we do not have a large sample of durations. We choose uniform priors U[2,T-K] for the intercept and U[-5,5] for the autoregressive parameter. The MCMC scheme is now updated with a new vector of parameters ν and with a modified full conditional distribution for the break dates as follows:

$$f(\Gamma|Y_T, \Theta, \nu) \propto f(Y_T|\Theta, \Gamma) p(\Gamma|\nu).$$

The forecast of structural breaks could also be done using a Poisson regression since κ is a count vector.

5 Simulations

In order to illustrate the efficiency of the D-DREAM algorithm we generate a series of 5000 observations following an AR(1)-GARCH(1,1) model with four break dates. The parameters, displayed in Table 2, are chosen such as to generate changes in the mean equation (first and second breaks), the volatility dynamics (third break), and the unconditional variance (fourth break).

Table 2: Parameters of Simulated Series

Parameters	Regime 1	Regime 2	Regime 3	Regime 4	Regime 5
μ	0.10	0.90	0.90	0.90	0.90
ϕ	0.20	0.20	0.70	0.70	0.70
c	0.20	0.20	0.20	0.20	0.80
α	0.25	0.25	0.25	0.05	0.05
β	0.70	0.70	0.70	0.90	0.90
Break Dates	1000	1750	2750	4250	—

The D-DREAM algorithm is run successively with the number of breaks equal to at least 1 and at most 6 and the number of AR lags equal to 0, 1, and 2, while we keep the GARCH(1,1) specification for all models. In order to estimate each model, we let the algorithm run 1250 iterations per chain after convergence. The number N of chains run is fixed to 10, the cross-over probability CR is .7, the number of pairs of chains δ is 3, the standard deviations in the proposals η_Γ and η_Θ are equal to 1 and .0001 respectively, and the standard deviation in the AR(1) model for the regime durations $\bar{\sigma}$ is equal to the size of the series T.[6] Also, the Marginal Log-Likelihood (MLL) is calculated for each model following the two methods of section 5. Table 3 summarizes the MLL estimation results. Clearly, the criterion computed via the two methods selects the right model, that is, an AR(1)-GARCH(1,1) specification with 5 regimes. Figure 2 displays the series with the break dates of the selected (true) model evaluated at their posterior means.

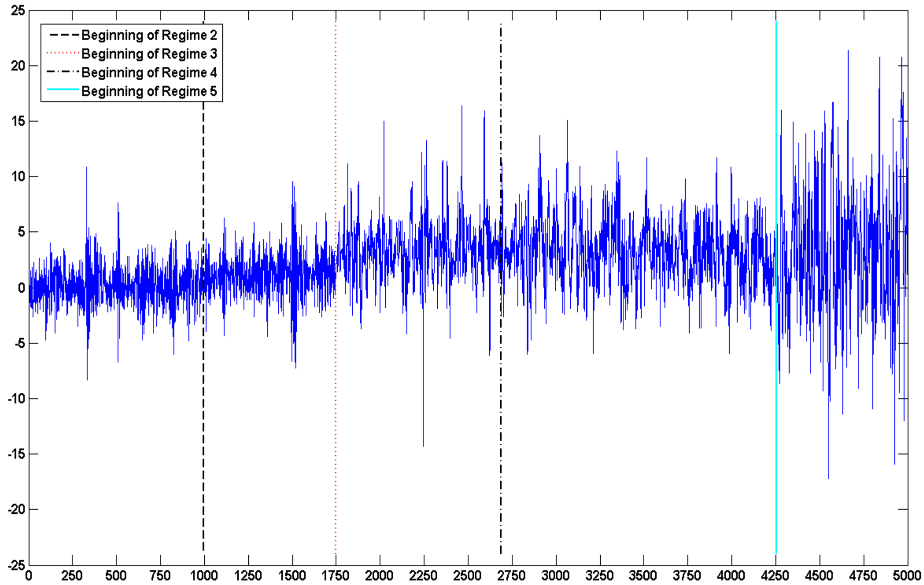
[6] One could also decrease CR as the number of regimes increases.

Table 3: MLL Estimations

#Regimes	1	2	3	4	5	6
			MLL via Chib and Jeliazkov (2001)			
AR(0)-GARCH(1,1)	-12559.10	-11838.40	-11742.90	-11695.90	-11686.00	-11693.40
AR(1)-GARCH(1,1)	-11068.80	-10816.00	-10770.20	-10760.80	**-10749.80**	-10755.70
AR(2)-GARCH(1,1)	-11042.90	-10819.90	-10812.40	-10771.10	-10770.10	-10783.30
			MLL via Bridge Sampling			
AR(0)-GARCH(1,1)	-12559.00	-11837.50	-11744.20	-11696.80	-11678.40	-11687.70
AR(1)-GARCH(1,1)	-11068.80	-10815.60	-10770.00	-10759.00	**-10749.10**	-10754.00
AR(2)-GARCH(1,1)	-11042.80	-10819.00	-10808.40	-10770.30	-10764.50	-10775.30

Table 3. The table reports the MLL of various AR(p)-GARCH(r,s) models with 1, 2, 3, 4, 5, and 6 regimes estimated on the simulated series. Values in bold indicate the highest values for each computation method.

Figure 2: Simulated Series with Break Dates (Posterior Means)



In order to test further our algorithm, we simulate the same DGP one hundred times and run the D-DREAM algorithm on each simulated series with the true specification. Table 4 reports the average values of the posterior means of each parameter. On average, the estimated values are close to the true ones and the standard deviations are small. In addition to summary statistics Table 4 also reports the break detection successes. These are defined as the number of times a posterior mean of a break date falls between a symmetric interval of two hundred observations around the true break date. It shows that a change in the mean is easier to detect than a change in volatility.

Table 4: Average of Posterior Means of 100 simulations

Parameters	Regime 1	Regime 2	Regime 3	Regime 4	Regime 5
μ	0.1039	0.8973	0.9203	0.9168	0.9157
	(0.0553)	(0.1113)	(0.1323)	(0.0903)	(0.1923)
ϕ	0.1947	0.2033	0.6896	0.6900	0.6973
	(0.0387)	(0.0654)	(0.0649)	(0.0300)	(0.0379)
c	0.2220	0.2290	0.2156	0.2917	1.0570
	(0.0631)	(0.0705)	(0.1000)	(0.2795)	(0.8074)
α	0.2568	0.2532	0.2450	0.0538	0.0447
	(0.0416)	(0.0438)	(0.0563)	(0.0319)	(0.0232)
β	0.6862	0.6845	0.7006	0.8720	0.8890
	(0.0483)	(0.0482)	(0.0552)	(0.0802)	(0.0590)
Break Dates	1000.12	1751.07	2737.01	4176.78	—
	(19.00)	(12.50)	(32.92)	(19.15)	—
	[97]	[99]	[89]	[93]	—

Table 4. The table reports the average of the posterior means of the parameters of the true model for the 100 simulated series. The true model is an AR(1)-GARCH(1,1) model with 4 breaks. Standard deviations are reported in parentheses. The break detection successes in percentage are reported in brackets

For a series of 5000 observations, the algorithm explores the posterior distribution quite quickly compared with other existing algorithms. It generates 12,500 draws in 25 minutes whereas the change-point SMC algorithm of He and Maheu (2010) takes more or less 4 hours (3 seconds per observation) and the particle MCMC of Bauwens, Dufays, and Rombouts (2011) takes about 5 hours.[7]

6 Empirical Study

First we study the Standard and Poor's 500 (S&P 500) index. Then, we apply the D-DREAM algorithm to other financial series and give some summary statistics. For each series, we run 1250 MCMC iterations for each chain after convergence and keep the same settings as in the simulation exercise.

[7] All codes in C++ on an Intel Core 2 Duo 3 Ghz with 3.48 Gb RAM.

6.1 S&P 500

The S&P 500 dataset was downloaded from the Center for Research in Security Prices. The dataset consists of daily observations from which we derive continuously compounded returns. The dataset covers the period from April 21, 1988 to April 25, 2011 (5800 returns).

Table 5 reports the MLL for three different specifications: AR(0)-GARCH(1,1), AR(1)-GARCH(1,1), AR(2)-GARCH(1,1). For each specification, the MLL selects a model with at least four regimes. Clearly, the best model which is an AR(0)-GARCH(1,1) with five regimes, is much preferred to the same specification without breaks.[8]

Table 5: MLL Estimations for S&P 500

#Regimes	1	2	3	4	5	6
			MLL via Chib and Jeliazkov (2001)			
AR(0)-GARCH(1,1)	-7869.80	-7841.67	-7830.73	-7828.07	**-7822.61**	-7825.25
AR(1)-GARCH(1,1)	-7872.48	-7846.48	-7838.95	-7832.87	-7828.53	-7832.95
AR(2)-GARCH(1,1)	-7874.28	-7852.10	-7847.35	-7845.62	-7848.37	-7852.05
			MLL via Bridge Sampling			
AR(0)-GARCH(1,1)	-7869.76	-7839.75	-7829.13	-7824.92	**-7817.81**	-7819.52
AR(1)-GARCH(1,1)	-7872.48	-7846.41	-7837.03	-7829.25	-7825.00	-7829.64
AR(2)-GARCH(1,1)	-7874.16	-7851.45	-7845.31	-7840.21	-7843.10	-7846.98

Table 5. The table reports the MLL of different models estimated with 1, 2, 3, 4, 5 and 6 regimes for the S&P 500 series. The left-hand side column indicates the model. The MLL criterion selects the model with 4 breaks, as indicated by the bold values.

The summary statistics for the selected model are displayed in Table 6. The algorithm finds structural breaks in December 30, 1991, November 27, 1996, March 14, 2003, and January 17, 2007. The unconditional variance and the persistence in particular change across regimes. The unconditional variance seems to exhibit a cycle as less volatile periods are followed by more agitated ones. It is indeed relatively small in regimes two and four. For forecasting purposes it is interesting to compare our results with a standard GARCH(1,1) specification without break. As it is shown in table 6, the posterior means of the parameters of the fifth regime are sensitively different from the posterior means of the parameters of the model without break ("No Break"). Hence, in this case but also in general the forecasting performance of models taking structural breaks into account can be expected to be quite

[8]Acceptance rates for the selected model are as follows: 43.01% for the AR parameters, 14.25% for the GARCH parameters, and 26.82% for the break dates.

higher compared with the performance of classical models not accounting for breaks.

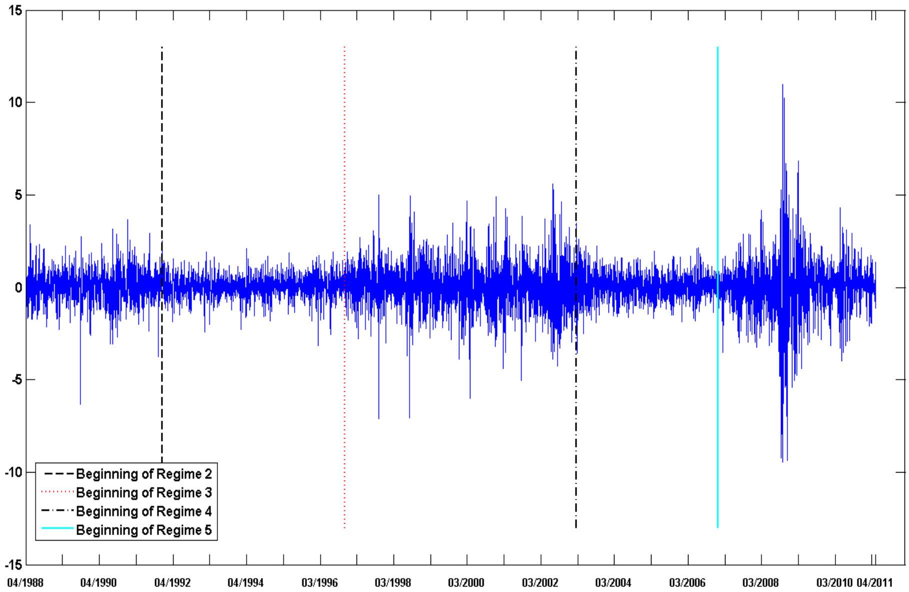
Table 6: S&P 500 Summary Statistics of Selected Model

Parameters	Regime 1	Regime 2	Regime 3	Regime 4	Regime 5	No Break
μ	0.0532	0.0547	0.0448	0.0530	0.0542	0.0506
	(0.0267)	(0.0120)	(0.0267)	(0.0147)	(0.0298)	(0.0109)
c	0.0712	0.0108	0.0889	0.0121	0.0296	0.0089
	(0.0424)	(0.0028)	(0.0125)	(0.0049)	(0.0054)	(0.0016)
α	0.0215	0.0396	0.0953	0.0388	0.0856	0.0590
	(0.0094)	(0.0058)	(0.0073)	(0.0084)	(0.0079)	(0.0056)
β	0.8906	0.9330	0.8561	0.9323	0.9010	0.9332
	(0.0577)	(0.0114)	(0.0105)	(0.0123)	(0.0078)	(0.0064)
$\frac{c}{1-\alpha-\beta}$	0.8106	0.3995	1.8574	0.4167	2.7748	1.1805
	(0.0477)	(0.0401)	(0.2200)	(0.0485)	(3.2069)	(0.2076)
Break Dates	12/30/1991	11/27/1996	03/14/2003	01/17/2007	—	—
(StD in days)	(21.0781)	(32.2836)	(50.0851)	(32.3907)	—	—
Durations (in days)	938	1265	1590	946	—	—

Table 6. The table reports the posterior means of the parameters of the selected model and the same specification without break ("No Break") for the S&P 500 application. Standard Deviations (StD) are reported in parentheses.

Results are easily analyzed graphically. Figure 3 displays the S&P 500 return series with the four break dates evaluated at their posterior mean. Finding justifications for these dates is certainly not an easy task. However, the graphic below shows quite unambiguously that these dates are plausible. Indeed, it seems that the last break corresponds to the beginning of the recent subprime crisis, the third break to the end of the "dot com" bubble, and the second to the beginning of the "irrational exuberance" period. The first break date indicates mainly a large decrease in the (local) unconditional variance of the series.

Figure 3: S&P 500 with Break Dates (Posterior Means)



The algorithm also yields pertinent information concerning the possible recurrent states in the index. Table 7 displays the probability of being either in a new state or in a previous one. It shows that the fourth regime is very similar to the second. Graphically (see Figure 3), the two regimes seem to model volatility during quiet, less volatile, periods. Also, the fifth regime has a probability of only 0.87% of being the same as a previous regime. Hence, the algorithm suggests creating a new regime for that last period, which is certainly relevant information for forecasting purposes. According to our results a Markov-switching GARCH model should then consider at least four regimes. Furthermore, our results predict the next expected regime to happen in April 2018. However, this result is based on an AR(1) structure fitted to only four durations which is a rather small set of observations.

Table 7: S&P 500 Recurrent states

	Regime 1	Regime 2	Regime 3	Regime 4	Regime 5
Regime 1	1	-	-	-	-
Regime 2	0	1	-	-	-
Regime 3	0	0	1	-	-
Regime 4	0	0.4939	0	0.5061	-
Regime 5	0	0	0.0087	0	0.9913

Table 7. The table reports the probability that a regime in column is either a new regime or the same as a regime in row. All lines add up to one.

6.2 Additional Series

We summarize the results obtained for eight commonly analyzed financial time series in stocks, commodities, and interest rates derivatives trading markets. Again, for each series we run 1250 MCMC iterations for each chain after convergence and set the number of chains equal to ten. We consider all values for the number of regimes from one up to the *ex post* chosen number of regimes plus one.

Table 8 displays the best specification between AR(0)-GARCH(1,1), AR(1)-GARCH(1,1), and AR(2)-GARCH(1,1) and the number of regimes for each series according to the MLL computed via the Bridge sampling method. All series seem to exhibit at least three breaks, which proves the relevance of taking breaks into account. The statistical fit of the series is never improved by adding autoregressive lags to the mean equation, which comes from the fact that we work with daily returns. The table also displays the next future break dates.

Table 8: D-DREAM Output for Various Financial Time Series

Series	Period (Daily Ret.)	Best Model	# regimes	Next exp. break
S&P 500	04/21/88-04/25/11	AR(0)-GARCH(1,1)	5	April 2018
DJIA	11/25/87-04/25/11	AR(0)-GARCH(1,1)	5	July 2018
NYSE	11/25/87-04/25/11	AR(0)-GARCH(1,1)	6	October 2017
JPMORG	06/19/91-04/25/11	AR(0)-GARCH(1,1)	7	February 2020
BOEING	11/25/87-04/25/11	AR(0)-GARCH(1,1)	5	February 2017
GOLD	06/19/91-04/25/11	AR(0)-GARCH(1,1)	5	September 2014
METALS	06/19/91-04/25/11	AR(0)-GARCH(1,1)	5	October 2015
Pound/US $	06/19/91-04/25/11	AR(0)-GARCH(1,1)	4	September 2019

Table 8. The table reports the best model and number of regimes according to the MLL computed via Bridge sampling for the S&P 500 Index, the Dow Jones Industrial Average Index, the New York Stock Exchange Index, the share price of Boeing, the price of Gold (in $), Metals (WCFI Base Metals Sub-Index), and the exchange rate British Pound/US Dollar.

7 Conclusion

Ignoring change-points generally produces poor forecasts and often gives the spurious impression of a nearly integrated behaviour of the time series. Change-points are generally accounted for following the method of Chib (1996) by introducing a hidden state variable in the model that indicates the regime which the observations belong to. However, this method suffers from the problem of path dependence.

In this paper, we introduce an MCMC algorithm that considers break dates as parameters to be estimated, which avoids the path dependence problem. We provide a proof of the convergence of the algorithm and we show how to select a model by computing marginal likelihoods using two different techniques, namely the method of Chib and Jeliazkov (2001) and the Bridge sampling method of Meng and Wong (1996). Another contribution of this paper is that we show how to allow for recurrent regimes and how to forecast the next break date. Recurrent regimes are considered by allowing a regime to be the same as a previous one. The model is then more flexible than a change-point model and gets closer to Markov-switching models. Future break dates are predicted by imposing a structure on the durations of the regimes that is left to the discretion of the user of the algorithm. In this work, we opted for an AR(1) model.

We illustrate the efficiency and rapidity of the algorithm by means of simulations and apply it to financial time series to highlight the presence of structural breaks. Indeed, all eight series studied exhibit at least three structural breaks. In particular, we find evidence of four structural breaks in the daily series of returns of the S&P 500 index at the following dates: December 30, 1991, November 27, 1996, March 14, 2003, and January 17, 2007.

Further research could focus on applying the same type of algorithm to multivariate time series. It is also of great importance to check whether detecting and forecasting structural breaks really enhances the predictions or not. An ambitious research work would also focus on the causes of structural breaks.

Appendix

Proof of Theorem 1

The proof is due to Vrugt, ter Braak, Diks, Robinson, Hyman, and Higdon (2009) and requires minor adjustments in the present case. It consists of four steps and is based on the fact that the chains do not converge separately of each other but rather converge together towards a stationary distribution. In the first step, we recall the conditional detailed balance condition required for the existence of a stationary distribution in a N-component Metropolis algorithm. In the second step, we prove the conditional detailed balance of the D-DREAM algorithm. In the third step, we show that the joint distribution of the break date vectors of the N chains is the product of the marginal distributions of each vector of break dates. In the fourth step, we prove the ergodicity of the chains and uniqueness of the stationary distribution. We give here the proof for the Γ-block only.

- Step 1. Recalling conditional detailed balance.

  In order to have the stationary distribution as target distribution one needs to show that if $(\Gamma_1^{(t)}, ..., \Gamma_N^{(t)}) \sim \tilde{\pi}(.)$, with $\tilde{\pi}(.)$ the joint pdf of $(\Gamma_1, ..., \Gamma_N)$ at iteration t, then $(\Gamma_1^{(t+1)}, ..., \Gamma_N^{(t+1)}) \sim \tilde{\pi}(.)$ at iteration $t+1$. That is, one needs the conditional detailed balance condition to be satisfied for each and every chain i such that, $\forall i$ (omitting Θ_i from the conditioning variables):

$$\begin{aligned}&\tilde{\pi}(\Gamma_i^{(t)}|\Gamma_1^{(t+1)}, ..., \Gamma_{i-1}^{(t+1)}, \Gamma_{i+1}^{(t)}, ..., \Gamma_N^{(t)}) \times K_i(\Gamma_i^{(t+1)}|\Gamma_1^{(t+1)}, ..., \Gamma_{i-1}^{(t+1)}, \Gamma_i^{(t)}, \Gamma_{i+1}^{(t)}, ..., \Gamma_N^{(t)})\\ &= \tilde{\pi}(\Gamma_i^{(t+1)}|\Gamma_1^{(t+1)}, ..., \Gamma_{i-1}^{(t+1)}, \Gamma_{i+1}^{(t)}, ..., \Gamma_N^{(t)}) \times K_i(\Gamma_i^{(t)}|\Gamma_1^{(t+1)}, ..., \Gamma_{i-1}^{(t+1)}, \Gamma_i^{(t+1)}, \Gamma_{i+1}^{(t)}, ..., \Gamma_N^{(t)}),\end{aligned}$$

with $K_i(.|.)$ the jumping kernel.

- Step 2. Proving the conditional detailed balance condition.

  Now, we show that the D-DREAM algorithm for the Γ_i-block satisfies the detailed balance condition *with respect to* $\pi(.)$ such that:

$$\pi(\Gamma_i^{(t)})K_i(\Gamma_i^{(t+1)}|\Gamma_1^{(t+1)}, ..., \Gamma_{i-1}^{(t+1)}, \Gamma_i^{(t)}, \Gamma_{i+1}^{(t)}, ..., \Gamma_N^{(t)})$$
$$= \pi(\Gamma_i^{(t+1)})K_i(\Gamma_i^{(t)}|\Gamma_1^{(t+1)}, ..., \Gamma_{i-1}^{(t+1)}, \Gamma_i^{(t+1)}, \Gamma_{i+1}^{(t)}, ..., \Gamma_N^{(t)}).$$

  Recall that the proposal Z_i for an update of chain i proceeds by random selection of pairs of other chains. Hence, the kernel of this update is a mixture of kernels which maintains conditional detailed balance with respect to $\pi(.)$ if each of its components does. Since ϵ follows a symmetric distribution and the pairs $\{\sum_{g=1}^{\delta}\Gamma_{r_1(g)}, \sum_{h=1}^{\delta}\Gamma_{r_2(h)}\}$ and $\{\sum_{h=1}^{\delta}\Gamma_{r_1(h)}, \sum_{g=1}^{\delta}\Gamma_{r_2(g)}\}$ are equally likely, this condition is fulfilled by accepting the proposal with probability $\min\{\frac{f(Y_T|Z_i)p(Z_i)}{f(Y_T|\Gamma)p(\Gamma)},1\}$. This also holds true with the *round*[.] operator since it does not alter the symmetry of the proposal distribution and with the binomial cross-over scheme used to modify only selected dimensions of Γ_i.[9] [10]

- Step 3. Showing that the joint is the product of the marginals.

  As showed in step 2, the kernel $K_i(.|.)$ for updating chain i satisfies the detailed balance condition of step 1 *with*, $\forall i = 1, ...N$, $\tilde{\pi}(\Gamma_i^{(t)}|\Gamma_1^{(t+1)}, ..., \Gamma_{i-1}^{(t+1)}, \Gamma_{i+1}^{(t)}, ..., \Gamma_N^{(t)}) = \pi(\Gamma_i^{(t)})$. Since the identity in step 1 is satisfied with the marginal distribution $\pi(\Gamma_i^{(t)})$, the D-DREAM algorithm is an N-component Metropolis-within-Gibbs algorithm with joint stationary distribution $\tilde{\pi}(\Gamma_1, ..., \Gamma_N)$, such that

$$\tilde{\pi}(\Gamma_1, ..., \Gamma_N) = \pi(\Gamma_1) \times ... \times \pi(\Gamma_N).$$

- Step 4. Proving ergodicity and uniqueness.

  For ergodicity, it is sufficient to show that the chains are aperiodic because of the

[9] As discussed above, the two possible issues in the proposal - that a break date falls outside the sample length [2,T] or that two break dates are inverted - are discarded by our prior distribution choice since the likelihood of the proposal Z_i would then be equal to zero.

[10] The round[.] operator must insure the symmetry for every possibilities. Hence a stochastic rounding has to be implemented for a particular case where one of the proposal parameter has exactly a fractional part equal to .5 even though it will virtually never happen in practice.

discreteness of Γ. The condition is satisfied thanks to the random walk component generated by ϵ in proposal 2. For uniqueness of the stationary distribution, it is sufficient to show that the chains are irreducible which is guaranteed by the unbounded support of the distribution of ϵ that necessarily covers the entire support [0,T].

■

Proposal Distribution for the Computation of the Marginal Likelihood via Bridge Sampling

We consider independence between Θ and Γ. Following Ardia, Hoogerheide, and van Dijk (2009) we work with a mixture of distributions. The proposal $q(\Theta)$ samples from a mixture of five normally distributed components. The means of the first three components are set to the posterior mean (mu_{pm}) of the MCMC draws and the most likely draw (μ_{max}) of the MCMC scheme defines the means of the last two components. The variance-covariance matrices of all components differ from the variance-covariance matrix of the MCMC draws by a scaling factor. The proposal $q(\Gamma)$ consists of a two-component mixture and follows the same ideas as for $q(\Theta)$. The weights were chosen arbitrarily. The information is summarized in table 9.

Table 9: Proposal Distribution Mixture

AR/GARCH parameters			Γ parameters		
Comp. #	Weight	Distribution	Comp. #	Weight	Distribution
1	0.50	N(μ_{pm},Σ)	1	0.75	N(μ_{pm},0.01Σ)
2	0.25	N(μ_{pm},0.25Σ)	2	0.25	N(μ_{pm},4Σ)
3	0.10	N(μ_{pm},10Σ)			
4	0.05	N(μ_{max},10Σ)			
5	0.10	N(μ_{max},Σ)			

Table 9. The table shows the components in the proposal mixture used for the computation of the MLL via Bridge sampling. μ_{pm} and μ_{max} stand respectively for the posterior mean of the MCMC draws and the most likely draw. Σ is the variance-covariance matrix obtained from the MCMC draws.

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