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How to share joint liability: A cooperative game approach

by

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Abstract

Sharing damage that has been caused jointly by several tortfeasors is analyzed from a normative point of view. We show how damage can be apportioned on two distinct bases, causation and degree of misconduct. Our analysis uses the concept of *potential damage* on the basis of which we define a transferable utility game. Its core defines acceptable judgments as allocations of the total damage against which no group of tortfeasors can object. We show that weighted Shapley values define acceptable judgments and, vice versa, acceptable judgments reveal weights. Our paper illustrates how the cooperative approach may bring useful insights into legal questions. In particular, the Shapley value appears of special interest being founded on axioms that are in line with fundamental principles of tort law.

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1. Introduction

Sharing damage that has been caused by several individuals is a difficult problem that courts often face. Even if there exist basic principles and rules to apportion damages among them (like for instance in the third *Restatement of Torts* promulgated in May 1999), legal scholars are still looking for a systematic method.[1] Consider the following example due to Landes and Posner (1980). A car driver hurts a pedestrian who has his leg broken. The victim is then taken to a hospital. There, because of a fault of the surgeon, the leg has to be amputated. Both the driver and the surgeon have a responsibility in the damage. They have jointly caused the damage since, without the accident, the fault of the surgeon would not have occurred and, without the fault of the surgeon, the pedestrian would not have lost his leg. How should a judge determine the compensation to be paid by each injurer? Should he/she consider that the driver is liable for the entire damage insofar as without the driver's the damage would not have occurred? Or that each of them is liable for half of it? Or that one of them is more liable than the other and to what extent? An apportionment rule is needed to correctly share the damage among them. Such litigations occur whenever two or more individuals have jointly caused damage and it is easy to think about the different fields of law concerned by this issue: nuisance and environmental law, accident law, medical malpractices, products liability, securities law, antitrust law, *and so on*.

Historically, common law did not accept any apportionment among joint tortfeasors: there was no compensation among those who were regarded as "joint tortfeasors", a rule that had its origin in the famous *Merryweather v. Nixan* case (1799): the victim had a claim against each tortfeasor for the full damage and no tortfeasor could have a claim against the other ones. The evolution of common law in the 19th and the 20th centuries, and the debates between legal scholars about the unfairness of that jurisprudence led to it being overruled. The *Uniform Contribution among Tortfeasors Act*, first drafted in 1939 and revised in 1955, opened the possibility of apportionment. The successive *Restatements of Torts*, from the first one promulgated in 1939 to the third one promulgated in 1999, followed this evolution by providing principles to apportion damage among joint tortfeasors.[2]

From an economic point view, some attention has been devoted to this topic since the beginning of the 1980's with for instance the contributions of Landes and Posner (1980), Rizzo and Arnold (1980, 1986), Shavell (1983), Cooter (1987), Kornhauser and Revesez (1989) or Parisi and Singh (2010). However, no general and clear theoretical agreement has

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[1] In US jurisprudence, the *Restatements of the Law* are treatises published by the *American Law Institute* to inform judges and lawyers of the evolution of American common law and statutes.

[2] See Landes and Posner (1980) for a short and clear history of this topic and Boston (1996) for a detailed study of American common law and statutes evolution.

emerged among economists of law. The aim of the paper is to reconsider the issue of apportionment among multiple tortfeasors in the light of basic economic principles. In contrast to the existing models in law and economics that focus on incentives in a non-cooperative setting, we model the apportionment of damage as a cooperative game where players are the tortfeasors who have jointly created an economic loss that is a priori indivisible. We therefore retain the first objective of tort law, compensation and fairness, and we neglect the second one, incentives. Fairness, as related to the causal role of each tortfeasor, deserves to be studied within a law and economics framework. Furthermore, incentives concern *ex ante* causation (how legal rules create incentives for future behaviors) while fairness concerns *ex post* causation (what tortfeasors have actually caused). We indeed concentrate on *ex post* causation.

Our method applies to any situation involving multiple causation within a *sequence* of events: after a first injury caused by A to the victim, the damage is aggravated by a second tortfeasor B, possibly together with a third one C, and so on.[3] We call a *judgment* the specification of the compensation paid by each tortfeasor to the victim. A judgment should be *acceptable*. There is a *minimum* compensation: each tortfeasors should pay at least the damage that he or she *would have caused alone*. This is the notion of "potential" damage that is sometimes used in the legal literature.[4] There is also a *maximum* compensation: each tortfeasor should pay at most the *additional* damage that he or she *has caused*. The additional damage is measured by the difference between the total damage and the damage that would have resulted without the participation of that individual tortfeasor. These two inequalities can actually be found in tort law.[5] Here we go further and extend them from individual tortfeasors to *subsets of tortfeasors,* leading to the following two conditions:

C1 *the contribution of any subset of tortfeasors should be at least equal to the damage they would have caused without the intervention of the others*.

C2 *the contribution of any subset of tortfeasors should not exceed the additional damage resulting from their participation*.

Combined, these two conditions define *acceptable judgments* by imposing bounds on what subsets of tortfeasors may be asked to contribute. We shall see that C1 and C2 are actually

[3] That means that we mix what Landes and Posner (1980) call *successive joint tort* and *simultaneous joint tort*.

[4] See Keeton (1984, p.353).

[5] The first section of the *Uniform Contribution among Tortfeasors Act* (1939) states that "no tortfeasor is compelled to make contribution beyond his *pro rata* share" and the third *Restatement of Torts* states that "no party should be liable for harm it did not cause" (1999, Topic 5). See also Boston (1996).

equivalent whenever judgments are required to specify an *exact* allocation of the total damage among the tortfeasors, a restriction that is actually imposed by tort law.[6]

One way of looking at condition C2 is in terms of objections. It says that a judgment should be *free from objections*. Objections could indeed come from an individual tortfeasor who is asked to contribute more than the additional damage he or she has caused. Objections could also come from a group of tortfeasors who are together asked to contribute more than the additional damage they have *jointly* caused.

To apprehend the notion of acceptable judgment, we construct a game with transferable utility, called *liability game*, whose characteristic function reflects the *potential* damage caused by any subset of tortfeasors without the intervention of the others. We will show that the *core* of a liability game defines the set of *all* acceptable judgments and that the (symmetric) *Shapley value* defines an acceptable judgment that is a fair compromise in which tortfeasors differ only in the damage they have caused. A judge may depart from that fair compromise by assigning weights to tortfeasors to reflect misconduct or negligence. We will show that the resulting *asymmetric* Shapley values also define acceptable judgments and that, vice versa, weights can be associated to any acceptable judgment. Hence acceptable judgments rely on two distinct elements: the causal distribution and the weights.

The causal distribution is an objective element that results from the work of experts. The weights instead are subjective elements that result from a decision of the judge. In this sense, our approach fits the two-step method advocated by the third *Restatement of Torts*: apportionment by causation and apportionment by responsibility.

Liability games have a sequential structure. Like airport games (Littlechild and Owen, 1973) or river games (Ambec and Sprumont, 2002), they are "consecutive games" as defined by Greenberg and Weber (1986). Actually, it is shown that liability games are duals of airport games.

The remainder of the paper is organized as follows. The general definitions of liability games and acceptable judgments are introduced in Section 2. Section 3 is devoted to the definition of the characteristic function that emerges from a sequential situation and its consequence in terms of the core and the Shapley values. Concluding remarks are offered in the last section and the Appendix gathers some technical results.

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[6] In game-theoretic terms, this condition is called *efficiency*.

2. Liability games: generalities

Someone suffers damage. A number of persons are considered to be co-responsible, possibly including the victim who is however entitled to compensation. The problem is to determine the contribution of each of these persons. In what follows, we will use the term "player" to designate all the actors the judiciary considers to be causally involved in the damage.

A *liability game* is a transferable utility game (N, v) where $N = \{1, ..., n\}$ is the set of players and v is the characteristic function that associates to each coalition $S \subset N$ the monetary value $v(S)$ of its *potential damage*.[7] It corresponds to the damage that would have resulted if those outside the coalition had followed a non-tortious behavior. This is consistent with the legal reasoning that compares the actual consequences of a behavior with the hypothetical consequences of the "normal" course of events. We take for granted that the judiciary has identified those who are implied in the damage. This is the difference between a legal cause and a material cause. Referring to the example of Landes and Posner (1980), suppose that the pedestrian has been transported to the hospital by an ambulance. Then, even if the driver of the ambulance appears as a *material* cause in the aggravation, he/she will not be considered by the law as causally implied and will therefore not be considered as an actor in the game.

For a given game (N, v), an *allocation* is a n-dimensional vector $(x_1, ..., x_n)$ that specifies an amount for each player, x_i for player i, such that $x(N) = v(N)$.[8] Applied to a liability game, x_i is the contribution of player i and it is assumed that the total damage $v(N)$ is *exactly* covered. We will consider two solution concepts, namely the core and the Shapley value, in its symmetric and asymmetric versions.

Applied to surplus sharing, the core (Gillies, 1953) is the set of allocations that no coalition can improve upon:

$$C(N, v) = \left\{ x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N \right\} \quad (1)$$

Using the identity $x(N) = v(N)$ it is easily verified that core allocations satisfy the following *equivalent* inequalities:

$$x(S) \leq v(N) - v(N \backslash S) \text{ for all } S \subset N$$

where the right-hand side is the additional damage attributable to coalition S.[9]

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[7] By convention $v(\varnothing) = 0$.

[8] **Notation**: For any vector x, $x(S) = \sum_{i \in S} x_i$. For coalitions S, T,… lower-case letters s, t ,… denote their size.

[9] The equivalence is lost if one assumes $x(N) \geq v(N)$. Under that weaker assumption, C2 implies C1. The inverse implication is verified if the reverse inequality holds.

Hence the core of a liability game is the set of acceptable judgments, as defined above:

C1 no coalition of players contributes *less* than its potential damage:
$x(S) \geq v(S)$ for all $S \subset N$;

C2 no coalition of players contributes *more* than its additional damage:
$x(S) \leq v(N) - v(N \backslash S)$ for all $S \subset N$.

Notice that the above conditions apply to individual players as well:

$$v(i) \leq x_i \leq v(N) - v(N \backslash i) \text{ for all } i \in N$$

that is, each individual player contributes an amount that is at least equal to his or her potential damage but at most equal to his or her additional damage.

A characteristic function v is *convex* if and only if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subset N.$$

It is *superadditive* if the above inequalities apply only to disjoint subsets. Convexity can be equivalently defined in terms of marginal contributions: a characteristic function v is *convex* if and only if marginal contributions are non-increasing with coalition size:

$$S \subset T \;\Rightarrow\; v(S) - v(S \setminus i) \leq v(T) - v(T \setminus i)$$

Being defined by linear inequalities, the core is a *convex polyhedron* (or *polytope*), possibly empty, whose dimension is at most n–1. Superadditivity is not sufficient to ensure non-emptiness of the core.[10] However, liability games will be shown to be *convex* and they therefore have a nonempty core. One then needs a *rule* (a mapping) φ that associates a particular core allocation $\varphi(N, v)$ to any given liability game (N, v). When restricted to convex games, the Shapley value is such a rule. It is defined as the average marginal contribution vector:

$$SV_i(N, V) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \mu_i(\pi) \quad i = 1, ..., n \tag{2}$$

where Π_N is the set of all players' orderings and $\mu(\pi)$ is the *marginal contributions vector* associated to the players' ordering $\pi = (i_1, ..., i_n) \in \Pi_N$:

[10] Necessary *and* sufficient conditions for non-emptiness of the core have been obtained independently by Bondareva (1963) and Shapley (1967).

$$\mu_{i_1}(\pi) = v(i_1) - v(\varnothing) = v(i_1)$$
$$\mu_{i_k}(\pi) = v(i_1,...,i_k) - v(i_1,...,i_{k-1}) \quad k = 2,...,n \tag{3}$$

Shapley (1953) has shown that it is the only *linear* allocation rule that is *symmetric* (equals are treated equally) and satisfies the *null player property* (null players get zero).[11] Alternative axiomatizations have been given. We will rely on the one proposed by Young (1985) who shows that the Shapley value is the only symmetric and *marginalist* allocation rule: what a player receives exclusively depends upon his or her marginal contributions $v(S) - v(S \backslash i)$.

Dropping symmetry opens the possibility for equal players to be treated differently. The *asymmetric* version of the value is obtained by introducing exogenous weights to cover asymmetries not included in the characteristic function.[12] For given positive weights $w = (w_1,...,w_n)$, the Shapley value is the average marginal contribution vector

$$SV_i(N,v,w) = \sum_{\pi \in \Pi_N} P_w(\pi)\, \mu_i(\pi)$$

where

$$P_w(i_1,i_2,...,i_n) = \frac{w_{i_n}}{w_{i_1} + ... + w_{i_n}} \frac{w_{i_{n-1}}}{w_{i_1} + ... + w_{i_{n-1}}} ... \frac{w_{i_2}}{w_{i_1} + w_{i_2}}$$

is the probability distribution induced by w on Π_N. With normalized weights, $w(N) = 1$ and w_i is the probability that player i be *last* in an arbitrary ordering. The symmetric value corresponds to equal weights, and the set of *all* weighted values is obtained by considering the limits for some but not all weights tending to zero. The weighted Shapley value can be obtained by replacing symmetry by the property that the weight w_i defines the share of player i in the *unanimity game*:

$$\varphi_i(N,u_N) = \frac{w_i}{w(N)}$$

where (N,u_N) is defined by

$$\begin{aligned} u_N(S) &= 1 \quad \text{if } S = N \\ &= 0 \quad \text{otherwise} \end{aligned}$$

[11] Two players are *equal* in a game if they contribute equally to all coalitions to which they both belong. A player is *null* in a game if he or she contributes to no coalition.

[12] It was introduced by Shapley (1953). See Kalai and Samet (1987) for a complete characterization. See Dehez (2011) for an axiomatization in a cost-sharing framework.

In general, the core is a subset of the Weber set defined as the convex hull of the marginal contributions vectors, a result due to Weber (1988), and the two sets coincide on the cone of convex games.[13] The set of weighted values is clearly a subset of the Weber set. Monderer, Samet and Shapley (1992) have shown that, in general, the core is a subset of the set of weighted values. Consequently, *the core coincides with the set of weighted values*. This is a key property for liability games. It means that an acceptable judgment consists in assigning weights to players: the resulting allocation defines an acceptable judgment and weights can be associated to any acceptable judgment. The characteristic function is computed in terms of potential damage on the basis of objective elements. The choice of weights instead forms the subjective part of a judgment that is based on the severity of the misconduct or negligence.

3. Liability games: characterization

3.1 The characteristic function

We will analyze situations that are *sequential* in the sense that the sequence of wrongful acts that has resulted in the injury has been identified, knowing that, along the sequence, some players may be jointly responsible for the aggravation of the damage. By jointly we mean that no additional damage would have occurred if any one of them had not been present. More precisely, a liability situation is described by the set of players $N = \{1,...,n\}$ that may include the victim, a partition $\{T_1,...,T_m\}$ of N into m non-empty successive sets, $1 \le m \le n,$ and a vector of damages $(d_1,\dots,d_m)$ where $d_h > 0$ is the additional damage associated to T_h. By successive we mean that if $i \in T_h$ and $j \in T_k$ for some $k > h$, then $j > i$.[14] Given a damage vector $d = (d_1,...,d_m),$ a *cumulative* damage (a cost) can be associated to each step:

$$c_k = \sum_{h=1}^{k} d_h$$

and, alternatively, additional damages $(d_1,...,d_m)$ can be associated to costs $(c_1,...,c_m)$:

$$d_h = c_h - c_{h-1}$$

with the convention that $c_0 = 0.$ These data result in the characteristic function defined by:

$$v(S) = c_k \text{ where } k \text{ is the largest integer such that } R_k = \bigcup_{h=1}^{k} T_h \subset S$$

In particular, $v(N) = c_m$ is the total damage and $v(S) = 0$ once $T_1 \not\subset S.$ Furthermore,

$$i \in T_k \;\Rightarrow\; v(N \setminus i) = c_{k-1}$$

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[13] It is actually a necessary and sufficient condition for convexity. See Shapley (1971) and Ichiishi (1981).

[14] An individual is included as a player in the sequence if he or she is considered to have caused an aggravation of the damage (not necessarily immediate) resulting from a wrongdoing. If a player has caused no immediate aggravation, that player is included in the T_h preceding him or her in the sequence.

There are two polar cases: the pure sequential case where $m = n$ i.e. $T_i = \{i\}$ for all $i = 1,\ldots,n$ and the simultaneous case where $m = 1$ i.e. $T_1 = N$. The latter is then the unanimity game.

Example 1 Consider a pure sequential situation involving three players i.e. $T_i = \{i\},\ i = 1, 2, 3.$ The associated characteristic function is then defined by[15]

$$v(1) = v(12) = v(13) = d_1$$

$$v(12) = d_1 + d_2 \text{ and } v(123) = d_1 + d_2 + d_3$$

Example 2 Consider the mixed situation involving four players with $T_1 = \{1\}, T_2 = \{2,3\}$ and $T_3 = \{4\}.$ The associated characteristic function is then defined by

$$v(1) = v(12) = v(13) = v(14) = v(124) = v(134) = d_1$$

$$v(123) = d_1 + d_2 \text{ and } v(1234) = d_1 + d_2 + d_3$$

Lemma 1 Liability games are *convex* (and thereby *superadditive*).

Consider a player *i* and coalitions $S, S' \subset N$ such that $i \in S \subset S'.$ Assume that $i \in T_k$ for some *k*. There are two possible cases:

$$v(S) < c_k \;\Rightarrow\; \begin{array}{l} v(S\backslash i) = v(S) \\ v(S') \geq v(S'\backslash i) \end{array}$$

and

$$v(S) \geq c_k \;\Rightarrow\; \begin{array}{l} v(S\backslash i) = v(S'\backslash i) = c_{k-1} \\ v(S') \geq v(S) \end{array}$$

In both cases, $v(S) - v(S \setminus i) \leq v(T) - v(T \setminus i),$ confirming convexity. ♦

As mentioned above, liability games belong to the class of consecutive games. We actually have the following proposition.

Lemma 2 Liability games are duals of airport games.

The proof is given in Appendix A1 for the pure sequential case.

3.2 The core

The problem is to allocate the total damage among the *n* players, which means finding a vector $x = (x_1, ..., x_n)$ where x_i is the contribution of player *i*, such that:

$$\sum_{i=1}^{n} x_i = \sum_{k=1}^{m} d_k \equiv c_m$$

[15] We only list the coalition having a positive worth.

If the victim is held partially liable for the damage incurred, he or she is one of the players somewhere along the sequence. His or her contribution is then simply deducted from the total damage to define the actual compensation received and the total compensation paid by the tortfeasors is then less than the damage.

Recalling the definition $R_k = \bigcup_{h=1}^{k} T_h$, the core as defined in (1) can be written as:

$$C(N,v) = \left\{ x \in \mathbb{R}_+^n \;\middle|\; x(N) = c_m \text{ and } \sum_{i \in R_k} x_i \geq c_k \text{ for all } k = 1,...,m \right\} \tag{4}$$

We observe that the core imposes that the players in T_1 pay *at least* their immediate damage:

$$\sum_{i \in T_1} x_i \geq d_1 = c_1$$

Actually, the core includes the allocation that imposes that the players in T_1 pay for the *entire* damage:

$$\sum_{i \in T_1} x_i = c_m$$

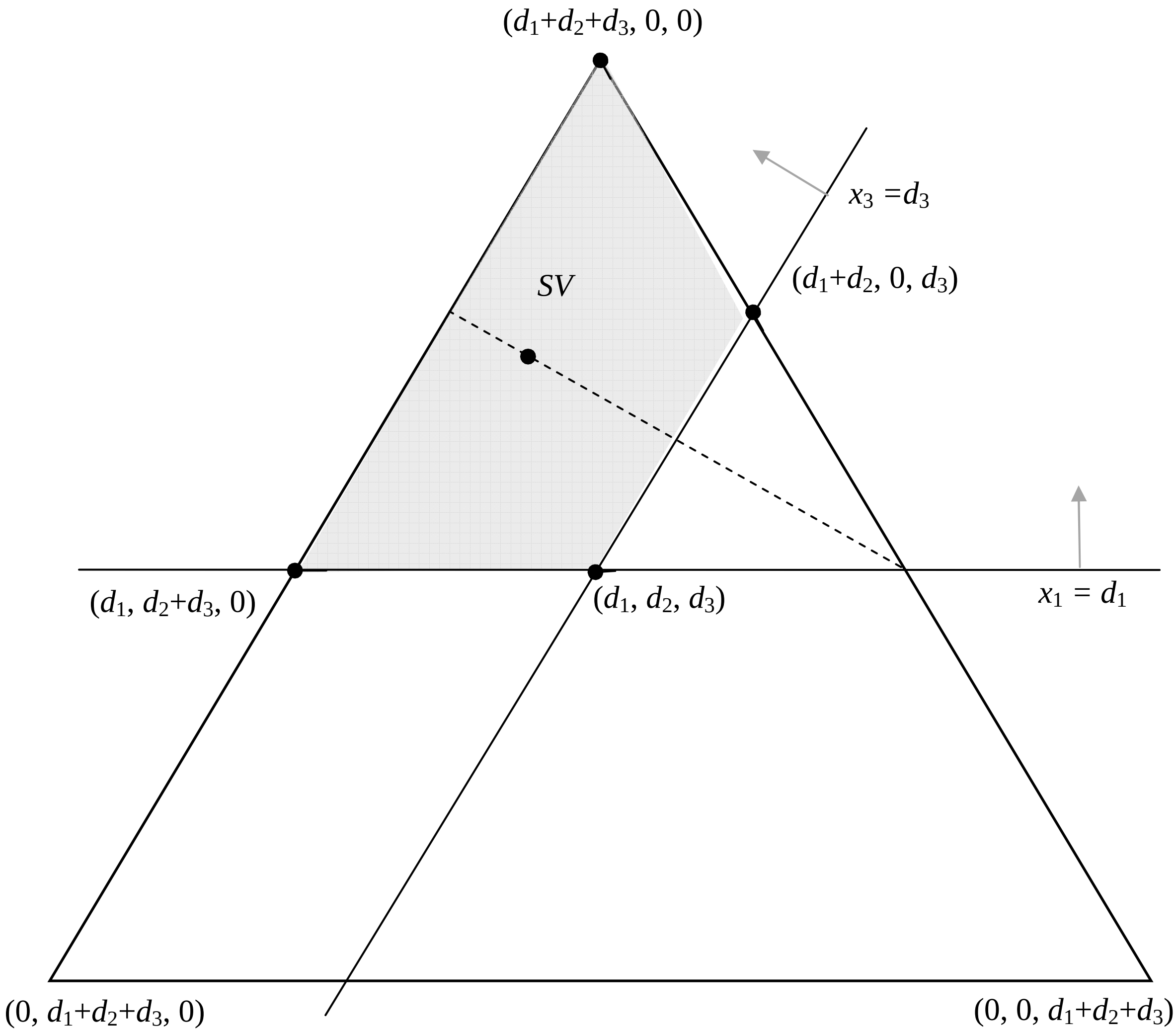
It also includes the allocation that imposes that each group of players pay for its additional damage:

$$\sum_{i \in T_h} x_i = d_h \;\; (h = 1,...,m)$$

By convexity, the core is the polyhedron whose vertices are the marginal contribution vectors defined by (3). There may be up to n! distinct such vectors. The core of liability games however has fewer vertices. In Example 1, there are four distinct marginal contribution vectors:

$$\mu^1 = (d_1, d_2, d_3)$$
$$\mu^2 = (d_1, d_2 + d_3, 0)$$
$$\mu^3 = (d_1 + d_2, 0, d_3)$$
$$\mu^4 = (d_1 + d_2 + d_3, 0, 0)$$

This is illustrated in the following figure. Each vector allocates the increase in damage to subsets of players in a coherent way, always including player 1 for at least the initial damage. All configurations are possible in terms of exemption of player 2 and/or 3. In pure sequential games, there are 2^{n-1} distinct vectors. Each exemption configuration is associated to one or more players' ordering though there is *one and only one* vector for each exemption configuration. For instance, in a six-player case, $(d_1+d_2, 0, d_3+d_4, 0, d_5+d_6, 0)$ is the only vector where players 2, 4 and 6 are exempt of contribution. The orderings (2,1,4,3,6,5) and (2,1,4,6,3,5) are two possible players' ordering that produce that vector.



In general, at most one player can be exempted in any subset T_h and players in T_1 cannot be all exempted. Hence there are $t_1 \prod_{h=2}^{m} (t_h + 1)$ exemption configurations. This is the number of vertices of the core. It reduces to 2^{n-1} when $t_h = 1$ for all h. In Example 2, there are six distinct marginal cost vectors:

$$\mu^1 = (d_1 + d_2 + d_3, 0, 0, 0)$$
$$\mu^2 = (d_1 + d_2, 0, 0, d_3)$$
$$\mu^3 = (d_1, d_2 + d_3, 0, 0)$$
$$\mu^4 = (d_1, 0, d_2 + d_3, 0)$$
$$\mu^5 = (d_1, d_2, 0, d_3)$$
$$\mu^6 = (d_1, 0, d_2, d_3)$$

3.3 The Shapley value

Instead of computing the symmetric Shapley value separately, we go straight to the computation of the weighted value.

Proposition The weighted Shapley value associates to any given liability situation $(N, T_1, ..., T_m, d)$ and positive weights $(w_1, ..., w_n)$ the allocation defined by:

$$SV_i(N,v,w) = \sum_{k=h}^{m} \frac{w_i}{w(R_k)} d_k \quad \text{for all } i \in T_h \quad (h = 1,...,m) \tag{5}$$

Proof The characteristic function can be decomposed as a sum of *m* elementary games:

$$v(S) = \sum_{k=1}^{m} v_k(S)$$

where

$$v_k(S) = d_k \quad \text{if} \quad R_k \subset S$$
$$= 0 \quad \text{if not}$$

In the elementary game (N, v_k), players outside R_k are null players:

$$SV_i(N, v_k, w) = 0 \quad \text{for all } i \notin R_k$$

and we can restrict the game (N, v_k) to the players in R_k where each one is allocated an amount proportional to his or her weight:

$$SV_i(N, v_k, w) = \frac{w_i}{w(R_k)} d_k \quad \text{for all } i \in R_k$$

It then remains to apply additivity to get the result. ♦

It is important to notice that, as a rule, the weighted Shapley value is such that no one is liable for damage caused *downstream* in the liability sequence: what a player $i \in T_h$ contributes depends only on $(d_h, ..., d_m)$. Furthermore,

$$\sum_{i \in T_1} SV_i(N,v,w) = d_1 + \sum_{i \in T_1} SV_i(N,v',w)$$

$$SV_i(N,v,w) = SV_i(N,v',w) \quad \text{for all } i \in T_h, \; h = 2,...n$$

where v' is the characteristic function obtained by setting $d_1 = 0$. In the symmetric case, the weights are equal and (5) reduces to

$$SV_i(N,v,w) = \sum_{k=h}^{m} \frac{1}{r_k} d_k \quad \text{for all } i \in T_h \quad \text{where } r_k = \sum_{j=1}^{k} t_j$$

Players in any T_h are equal and therefore contribute equally in the symmetric Shapley value. In the examples 1 and 2, the weighted value are given by:

$$x_1 = d_1 + \frac{w_1}{w_1 + w_2} d_2 + \frac{w_1}{w_1 + w_2 + w_3} d_3$$
$$x_2 = \frac{w_2}{w_1 + w_2} d_2 + \frac{w_2}{w_1 + w_2 + w_3} d_3$$
$$x_3 = \frac{w_3}{w_1 + w_2 + w_3} d_3$$

and

$$x_1 = d_1 + \frac{w_1}{w_1 + w_2 + w_3} d_2 + \frac{w_1}{w_1 + w_2 + w_3 + w_4} d_3$$
$$x_2 = \frac{w_2}{w_1 + w_2 + w_3} d_2 + \frac{w_2}{w_1 + w_2 + w_3 + w_4} d_3$$
$$x_3 = \frac{w_3}{w_1 + w_2 + w_3} d_2 + \frac{w_3}{w_1 + w_2 + w_3 + w_4} d_3$$
$$x_4 = \frac{w_4}{w_1 + w_2 + w_3 + w_4} d_3$$

The symmetric versions are given by:

$$x = \left(d_1 + \frac{1}{2} d_2 + \frac{1}{3} d_3,\ \frac{1}{2} d_2 + \frac{1}{3} d_3,\ \frac{1}{3} d_3 \right)$$

and

$$x = \left(d_1 + \frac{1}{3} d_2 + \frac{1}{4} d_3,\ \frac{1}{3} d_2 + \frac{1}{4} d_3,\ \frac{1}{3} d_2 + \frac{1}{4} d_3,\ \frac{1}{4} d_3 \right)$$

Players 2 and 3 are indeed equal and, as a consequence, they contribute an equal amount in the symmetric case.

The set of all weighted values is obtained by a limit argument, letting some of the w_i's go to zero. Zero and non-zero weight players can be treated separately. What a non-zero-weight player receives coincides with the weighted value of the game restricted to the set of non-zero-weight players. What remains is then allocated among zero-weight players. That allocation may however not be uniquely defined in the case of more than one zero-weight players.[16]

[16] See Dehez (2011).

We know that under convexity there is an equivalence between core allocation and weighted values, i.e. weighted values are acceptable judgments and weights can be associated to any acceptable judgment. Using (5), it is easily seen that a *unique* set of normalized weights can be associated to any "interior" allocations, that is allocations x satisfying $x_i > 0$ for all i. Consider Example 1. The simple average of the core's vertices is the interior allocation given by:

$$x_1 = d_1 + \frac{1}{2}d_2 + \frac{1}{4}d_3$$
$$x_2 = \frac{1}{2}d_2 + \frac{1}{4}d_3$$
$$x_3 = \frac{1}{2}d_3$$

It corresponds to the (normalized) weights $w_1 = w_2 = 1/4$ and $w_3 = 1/2$. This extends to any $n \geq 2$:

$$w_i = 1/2(n-1) \text{ for all } i = 1,...,n-1$$
$$w_n = 1/2$$

As shown in Appendix A2, the *nucleolus* (Schmeidler, 1969) is given by the allocation:

$$\begin{aligned} x &= \left(d_1 + \frac{d_2}{2} + \frac{d_3}{4}, \frac{d_2}{2} + \frac{d_3}{4}, \frac{d_3}{2} \right) && \text{if } d_3 \leq 2d_2 \\ &= \left(d_1 + \frac{d_2 + d_3}{3}, \frac{d_2 + d_3}{3}, \frac{d_2 + d_3}{3} \right) && \text{if } d_3 \geq 2d_2 \end{aligned} \tag{6}$$

In the first case, it is average of core's vertices we have just computed. In the second case, it is the *equal loss* allocation where the corresponding weights depend upon individual damages. Moreover, this example shows that the nucleolus, as an apportionment rule, violates the "downstream" condition: in the second case, what player 4 contributes depend upon damage caused by player 3.

4. Concluding remarks

The central concept that we have introduced is the notion of *acceptable judgment*, which restricts what subsets of tortfeasors are asked to contribute. It corresponds to the core of the transferable utility game based on potential damage. The weighted Shapley value offers an apportionment rule that defines acceptable judgments, the choice of the weights being left to the court. Furthermore, any acceptable judgment reveals weights. These properties rely on the sequential nature of the game. More complex situations with a tree structure could should be studied, for instance along the lines of Estévez-Fernàndez (2012).

We do not propose a single rule, leaving room for a court to possibly weight tortfeasors. The symmetric Shapley value stands however as a particular and attractive rule whose use is justified when there is no compelling reason to differentiate between tortfeasors beyond the damage they have caused. It is a rule that is uniquely defined by properties that do not contradict fundamental principles of law. We rely on the axiomatization proposed by Young (1985) because it fits our legal framework better. *Efficiency* retains only the rules that allocate the total damage among the tortfeasors exactly (except of course when the victim is co-responsible). *Symmetry* retains only the rules that allocate an identical amount to players who contribute equally to the potential damage of any coalition to which they belong. *Marginalism* retains only the rule that imposes that players pay an amount that depends exclusively on their contributions to potential damages, independently of the way the other players contribute.

In tort law, the apportionment of damages among multiple tortfeasors is one of the most difficult problems that judges face. The cooperative game approach followed here turns out to give useful insights into that problem by eliciting the rationality of legal decisions and making clear what are the normative properties that underlie those judgments. In particular, it fits the two-step method advocated by third *Restatement of Tort Law*: apportionment by causation and apportionment by responsibility. In this way we have illustrated how the axiomatic approach may be useful in solving legal problems and, more generally, understanding the law.

Appendix

A.1 Liability games are dual of airport games[17]

Airport games were introduced by Littlechild and Owen (1973).[18] They form a particular class of cost-sharing games. The problem is to allocate the cost of building a facility (e.g. a runway) capable of meeting all players' needs, knowing the cost q_i of a facility meeting player *i*'s needs. It is assumed that a facility of a given cost covers the needs of players with a *lower* cost. The associated cost game is then defined by the following characteristic function:

$$C(S) = Max_{i \in S}\ q_i$$

Consider the pure sequential case and define the variable $q_i = \sum_{j=i}^{n} d_j = c_n - c_{i-1}$ (recalling the convention $c_0 = 0$). It is the increase in the total damage due to player *i*'s presence in the

[17] We are thankful to Sylvain Béal for suggesting this section and the results it contains.
[18] See Thomson (2007) for a overview on airport games.

sequence. We have to show that $v(S) = C(N) - C(N \setminus S)$ for all $S \subset N$. We have, successively,

$$v(S) = Max\left(0, Min_{i \notin S}\ c_{i-1} - Min_{i \in S}\ c_{i-1}\right)$$
$$= Max\left(0, (c_n - Min_{i \in S}\ c_{i-1}) - (c_n - Min_{i \notin S}\ c_{i-1})\right)$$
$$= Max\left(0, Max_{i \in S}\ q_i - Max_{i \notin S}\ q_i\right) = Max\left(0, C(S) - C(N \setminus S)\right)$$

If $C(S) \geq C(N \setminus S)$ we have $C(N) = C(S)$ and therefore $v(S) = C(N) - C(N \setminus S)$. If, instead, $C(S) < C(N \setminus S)$ we have $v(S) = 0$ and $C(N) = C(N \setminus S)$. ♦

Because the value of a game coincides with the value of its dual, that proposition allows for an immediate calculation of the (symmetric) Shapley value.

A.2 The nucleolus of a 3-player liability game

Like the Shapley value, the nucleolus is translation invariant. Hence we may assume that $d_1 = 0$. It is also a *symmetric* rule. That makes the computation of the nucleolus easier because it defines an allocation that can be written in terms of a single parameter $\alpha \in [0, d_2 + d_3]$:

$$x_1 = x_2 = \frac{d_2 + d_3 - \alpha}{2} \text{ and } x_3 = \alpha$$

The first two players are indeed equal when $d_1 = 0$. We shall first construct the *least core* and then show that it reduces to a single allocation, defining the nucleolus.[19] Defining $e(x, S) = v(S) - x(S)$ as the excess associated to coalition $S\ (S \neq \emptyset, N)$ and allocation x, the least core is the set of allocations that minimizes the largest excess. In the present framework, it means solving the following problem:

$$\text{Min}_{\alpha \in [0, d_2 + d_3]}\ \text{Max}_{\substack{S \subset N \\ S \neq \emptyset, N}}\ e(\alpha, S)$$

where the excesses associated to coalitions $S \neq \emptyset, N$ are given by

$$e(\alpha, 1) = e(\alpha, 2) = -\frac{d_2 + d_3}{2} + \frac{\alpha}{2}$$

$$e(\alpha, 3) = -\alpha$$

$$e(\alpha, 12) = -d_3 + \alpha$$

$$e(\alpha, 13) = e(\alpha, 23) = -\frac{d_2 + d_3}{2} - \frac{\alpha}{2}$$

[19] The least core, formally introduced by Maschler, Peleg and Shapley (1979), extends the notion of ε-core introduced earlier by Shapley and Shubik (1966).

We can disregard $e(\alpha,13)$ which always falls below $e(\alpha,1)$ and the solution depends on the relative position of $e(\alpha,12)$. If $d_3 < 2d_2$, the solution is given by the intersection between $e(\alpha,12)$ and $e(\alpha,3)$. Otherwise, it is given by the intersection between $e(\alpha,1)$ and $e(\alpha,3)$. This defines a unique solution that defines (6) up to d_1:

$$\alpha = \frac{d_3}{2} \quad \text{if} \quad d_3 \leq 2d_2$$

$$\alpha = \frac{d_2 + d_3}{3} \quad \text{if} \quad d_3 \geq 2d_2$$

References

Ambec, S. & Sprumont, Y. (2002). Sharing a river. *Journal of Economic Theory* 107, 453-462.

Bondareva, O.N. (1963). Some applications of linear programming methods to the theory of cooperative games (in Russian), *Problemy Kybernetiki* 10, 119–139.

Boston, G.W. (1996). Apportionment of harm in tort law: A proposed restatement. *University of Dayton Law Review* 21, 267-372.

Cooter, R.D. (1987). Torts as the union of liberty and efficiency: An essay on causation. *Chicago-Kent Law Review* 63, 522-551.

Dehez, P. (2011). Allocation of fixed costs: Characterization of the (dual) weighted Shapley value. *International Game Theory Review* 13, 1-16.

Estévez-Fernàndez, A. (2012). A game theoretical approach to sharing penalties and rewards in projects. *European Journal of Operational Research* 216, 647-657.

Gillies, D.B. (1953). *Some theorems on n-person games*, Ph.D. Thesis, Princeton University.

Greenberg, J. & Weber, S. (1986). Strong tiebout equilibrium under restricted preferences domain. *Journal of Economic Theory* 38, 101–117.

Ichiishi, T. (1981). Super-modularity: Applications to convex games and to the greedy algorithm. *Journal of Economic Theory* 25, 283-286.

Kalai, E. & Samet, D. (1987). On weighted Shapley values. *International Journal of Game Theory* 16, 205-222.

Keeton, W. P. (1984), *Prosser and Keeton on Torts*, (5th ed.), West Group, Saint Paul.

Kornhauser, L. & Revesez, R.L. (1989). Sharing damages among multiple tortfeasors. *The Yale Law Journal* 98, 831-884.

Landes, W.M. & Posner, R.A. (1980). Joint and multiple tortfeasors: An economic analysis. *Journal of Legal Studies* 9, 517-555.

Littlechild, S.C. & Owen, G. (1973). A simple expression for the Shapley value in a special case. *Management Science* 20, 370-372.

Maschler, M., Peleg, B., & Shapley, L.S. (1979). Geometric properties of the kernel, nucleolus and related solution concepts. *Mathematics of Operations Research*, 4, 303-338.

Monderer D., Samet, D. & Shapley, L.S. (1992). Weighted values and the core. *International Journal of Game Theory* 21, 27-39.

Parisi, F. & Singh, R. (2010). The Efficiency of Comparative Causation," *Review of Law and Economics* 6, 219-245.

Rizzo, M.J. & Arnold, F.S. (1980). Causal apportionment in the law of torts: An economic theory. *Columbia Law Review* 80, 1399-1429.

Rizzo, M.J. & Arnold, F.S. (1986). Causal apportionment: Reply to the critics. *Journal of Legal Studies* 15, 219-226.

Shapley, L.S. (1953). A value for n-person games. In H. Kuhn, and A.W. Tucker, (Eds.), *Contributions to the Theory of Games II* (pp. 307-317), Annals of Mathematics Studies 24, Princeton University Press, Princeton.

Shapley, L.S. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly* 14, 453–460.

Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory* 1, 11-26.

Shapley, L.S. and Shubik, M. (1966). Quasi-cores in a monetary economy with nonconvex preferences. *Econometrica* 34, 805-827.

Shavell, S. (1983). Torts in which victim and injurer act sequentially. *Journal of Law and Economics* 26, 589-612.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal of Applied Mathematics* 17, 1163-1170.

Thomson, W. (2007). Cost allocation and airport problems. *Rochester Center for Economic Research Working Paper* 538.

Weber, R.J. (1988). Probabilistic values for games. In A. Roth (Ed.), *The Shapley value. Essays in honor of Lloyd Shapley* (pp. 101-119). Cambridge University Press, Cambridge.

Young, P.Y. (1985). Monotonic solutions of cooperative games. *International Journal of Game Theory* 14, 65-72.