Regularity and stability of equilibria in an overlapping generations growth model

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Consider a small change in (lump sum transfer) policy in an overlapping generations model with production near a balanced growth equilibrium. Can one calculate the response of the equilibrium variables?
The challenges

- Infinite economy
  (Chichilnisky and Zhou (1997), Shannon and Zame (2002))
- Non-Pareto equilibria
- Previously established indeterminacy results
  (Brown and Geanakopolos 1980’s, Kehoe and Levine 1985)
The economy: individuals

- An individual can be born at any time \( x \in \mathbb{R} \).
- Population grows exponentially at rate \( \nu \).
- Individual life span is \([0, 1]=\)time endowment.
- Time-separable life-time utility with intertemporal elasticity \( \sigma \), time preference \( \beta \), and no disutility from labour:

\[
V(c) = \int_{0}^{1} e^{-\beta s}(c(s))^{1-\frac{1}{\sigma}} ds
\]

- Wealth: \( M_x = \int_{0}^{1} p_{x+s} \omega_{x,s} ds + \int_{0}^{1} w_{x+s} \zeta_s ds \).
  - \( w_t \) is the (per unit efficiency) wage rate at time \( t \)
  - \( \zeta \) is life-time productivity
  - \( \omega_{x,s} \) is the consumption endowment at age \( s \)
The economy: production

- Instantaneous production:
  \[ Y_t = A K_t^\alpha L_t^{1-\alpha}, \quad 0 < \alpha < 1, \ A > 0 \]

- Efficiency of labour changes by age: \( \zeta(s) \geq 0 \), labour productivity grows exponentially at rate \( \gamma \), so aggregate productive labour available at \( t \) equals:
  \[ L_t = N_0 e^{(\gamma+\nu)t} \int \zeta_s e^{-\nu s} ds \]

- Capital \( K_t \) accumulates as \( K_t' = l_t - \delta K_t \)
### Equilibrium equations in productive labour units

**Notation**

\[ k_t = \frac{K_t}{L_t}, \text{ etc.}, \quad E_{t,s} = \frac{N_0 e^{\nu (t-s)} \omega_t - s}{L_t}, \quad \Omega_t = \int E_{t,s} ds, \quad R = \gamma + \nu + \delta, \]

The fixed points of \( \Upsilon: (k, E) \mapsto \tilde{k} \), defined as a composition of

- \( k \mapsto y: \ y_t = Ak_t^\alpha \)
- \( k \mapsto f: \ f_t = R - \alpha Ak_t^{\alpha-1} (= \gamma + \nu + \frac{p'_t}{p_t}) \)
- \( (y, f, E) \mapsto c \)
- \( (y, E, c) \mapsto i: \ i_t = y_t + \Omega_t - c_t \)
- \( i \mapsto \tilde{k}: \ \tilde{k}_t = e^{-Rt} \int_{-\infty}^{t} e^{Rs} i_s ds > 0, \) characterise

**Int. Eq.:** all equilibria of the general model where \( 0 < i_t < y_t \) a.e., provided the solutions satisfy \( 0 < i_t < y_t \);

**BGE:** if \( K_t \) is exponential, all BGE of the general model with \( \omega = 0 \).
A **balanced growth equilibrium** (BGE) is an equilibrium with $E_t = 0$ and $k_t$ constant (and hence $i$, $y$, $c$). It is a **golden rule equilibrium** (GRE) if $\frac{i}{y} = \alpha$.

In a GRE $f = 0$ and $R = \alpha A k_t^{\alpha - 1}$. 
There are two types of ‘balanced growth’ equilibria

- Golden Rule
- ‘Non-monetary’ (no aggregate-debt) ones: total net savings of consumers = $p_t K_t$.

**Figure:** $R = 11$, $\sigma = .5$, $\eta = 2$, $a_\varphi = .2$, $b = .75$. Two equilibria $\forall \alpha$. 

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Regularity and Stability in OLG
Example 2

Figure: $R = 11$, $\sigma = .25$, $\eta = 2$, $a_\phi = .135$, $b = .5$. Two to four equilibria.
Figure: $R = 10, \sigma = .25, \eta = 2.5, a_\varphi = .25, b = .75$. 1 equilibrium $\forall \alpha$. 
Example 4

Figure: $R = 15, \sigma = .24, \eta = 1.9, a_\varphi = .24, b = .55$. 1 or 3 equilibria.
Generic Regularity: formulation

Recall an equilibrium \( k \) solves \( F(k, E) \overset{\text{def}}{=} \Upsilon(k, E) - k = 0 \), \( \Upsilon \): 

- \( k \mapsto y: \ y_t = A k_t^\alpha \)
- \( k \mapsto f: \ f_t = R - \alpha A k_t^{\alpha-1} \) (\( = \gamma + \nu + \frac{p_t'}{p_t} \))
- \( (y, f, E) \mapsto c \)
- \( (y, E, c) \mapsto i: \ i_t = y_t + \Omega_t - c_t \)
- \( i \mapsto \tilde{k}: \ \tilde{k}_t = e^{-R t} \int_{-\infty}^{t} e^{Rs} i_s ds > 0. \)

Fix a BGE \( \varpi(0) \). “IFT”: 

\[
\varpi'_k(\delta E) = - \left( \frac{\partial F}{\partial k} \right)^{-1} \circ \frac{\partial F}{\partial E}(\delta E)
\]

\( \frac{\partial F}{\partial k} \) is generically invertible (in the set of parameters).
Spectrum of the derivative \( (\partial \Upsilon(k, E) / \partial k) \) operator

Figure: BGE of fig. with example 2, \( \alpha Y/I = 3 \)
Spectrum of the derivative $(\partial \Upsilon(k, E)/\partial k)$ operator

Figure: BGE of fig. with example 4, $\alpha Y/I = 1/2$
Definitions

Notation

- $\|E\|_{\infty,1} = \sup_x \int_x^{x+1} \int_0^1 |E_{t,s}| \, ds \, dt$
- $\psi_{\lambda_-,\lambda_+}(z) = e^{\lambda_- z} + e^{\lambda_+ z}$
- $\psi_\Lambda = \psi_{\lambda_-,\lambda_+}$ for a compact interval $\Lambda = [\lambda_-, \lambda_+]$
- $\|k\|_{L^C_\Lambda} = \sup_x \sup_{s,t} \psi_\Lambda(x-t)|k(x; t, s)|$ for a kernel $k(x; t, s)$
- $\Lambda_\epsilon = [\max\{\min\{0, \lambda_- + \epsilon\}, \frac{-1}{\epsilon}\}, (\lambda_+ - \epsilon)^+]$
Solutions of the equilibrium system are locally unique

Theorem ("IFT 1")

For a generic economy, \( \exists \delta > 0 \) and there is an open ball \( B \) of endowment perturbations \( E \) s.t., for any \( BGE \) \( \varpi (0) \), there is \( \forall E \in B \) a unique solution \( \varpi (E) \) with \( \| \varpi_k (E) - \varpi_k (0) \|_\infty \leq \delta \).
Solutions in the neighbourhood of a BGE are regular and stable

**Theorem (“IFT 2”)**

For an open ball $B$ of endowment perturbations from theorem 1:

$\forall \varepsilon > 0 \exists$ an open ball $B_\varepsilon \subseteq B$ s.t., $\exists$ a compact interval $\Lambda_\varepsilon \subseteq \Lambda$, s.t., on $B_\varepsilon$:

1. $E \mapsto \varpi'(E)$ is Lipschitz from $\|\cdot\|_{\infty,1}$ to $\|\cdot\|_{L^\Lambda_\varepsilon}$ for the normalised values of capital, aggregate consumption, aggregate output, and inflation rate.

2. For the same components of $\varpi$,

$\exists \kappa : \forall \lambda \in \Lambda_\varepsilon, \|\varpi(E_1) - \varpi(E_2)\|_\infty^\lambda \leq \kappa \|E_1 - E_2\|_{\infty,1}^\lambda$
Illustrating the speed of convergence

\[ \lambda_+ \text{ and } \lambda_-; \text{ GRE of Fig. 2.} \]

\[ \lambda_+ \text{ and } \lambda_-; \text{ BGE of Fig. 2L.} \]
Equilibria are locally unique

Theorem

For any economy in the generic set of parameters, \( \forall B \in \mathcal{E} \exists \delta > 0 \) and the open ball \( B \) of endowment perturbations where the solution of the equilibrium system is unique, \( \exists \delta_0, \delta_1 > 0, \text{s.t. for any } E \in B \) satisfying

- \( \| \int E \cdot s ds \|_\infty \leq \delta_0 \) and
- \( \operatorname{ess sup}_x \int E_{x+s}^- s ds \leq \delta_1 \),

\( \varpi(E) \) is the unique equilibrium of the \( E \)-perturbed economy s.t. \( \| \varpi_k(E) - \varpi_k(0) \| \leq \delta \) and \( \| \varpi_c(E) - \varpi_c(0) \| \leq \delta \).
Policy analysis in OG models with time as $\mathbb{R}$ is feasible!

- We provide a complete characterization of interior equilibria of a classical OG model.
- We demonstrate generic regularity and stability of the BGE.
- ... and a way to easily compute the equilibrium response!