A probabilistic Reputation Model

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Abstract

This work introduces a probabilistic model of reputation. It is based on the following simple consumer-provider interaction model. Consumers are assumed to order items to providers, who each have some internal, latent, “quality of service” score. In the basic model, the providers supply the items with a quality following a normal law, centered on their internal “quality of service”. The consumers, after the reception of the item, rate it according to a linear function of its quality (a standard regression model). This regression model accounts for the bias of the consumer in providing ratings as well as his reactivity towards changes in item quality. Moreover, the constancy of the provider in supplying an equal quality level for delivering the items is estimated by the standard deviation of his normal law of item quality generation. Symmetrically, the constancy of the consumer in providing similar ratings for a given quality is quantified by the standard deviation of his normal law of ratings generation. Two extensions of this basic model are considered as well: a model accounting for truncation of the ratings and a Bayesian model assuming a prior distribution on the parameters. Expectation-maximization algorithms allowing to provide estimations of the parameters only based on the ratings are developed for all the models. The experiments suggest that these models are able to extract useful information from the ratings.

1 Introduction

1.1 General introduction

The Internet has created a lot of new opportunities to interact with strangers. These interactions involve various kinds of applications, such as on-line markets. After just
three years of business, eBay had already conducted over one billion dollars in transactions in 1998, and by 2007 [10] this figure had climbed to $59.4 billion. A total of 276.3 million users either bid or listed an item on eBay during 2007, and according to a July 2005 survey conducted for eBay by ACNielsen International Research, an estimated 724,000 Americans and 170,000 Europeans report eBay as a primary or secondary source of income [30].

With the growth of on-line markets comes an increasing need for bidders and sellers to engage in transactions with counterparts with whom they have had little or no previous interaction. This new type of markets has introduced a new risk dimension to traders: the winner of the auction might not deliver payment, the seller might not deliver the item, or the delivered item might not be as the seller described [18]. These risks are an important restraint to the growth of on-line markets. One of the principal means by which on-line auction sites try to mitigate these risks associated with exchange among strangers is to use electronic reputation or feedback mechanisms. Such mechanisms aim at providing the type of information available in more traditional close-knit groups, where members are frequently involved in one another’s dealings. Bolton et al compare in [4] trading in a market with on-line feedback to a market without feedback, as well as to a market in which the same people interact with one another repeatedly. They found that the feedback mechanism induces quite a substantial improvement in transaction efficiency. Using such mechanisms is a means to distinguish honest sellers from dishonest ones. Without a mechanism for sellers to develop a reputation, dishonest sellers might drive out honest ones, leading to a kind of market failure. The idea is that even if the consumer cannot try the product or service in advance [23], he can be confident that it will be as he expects as long as he trusts the seller. A seller with a good reputation score therefore has a significant advantage in case the product quality cannot be verified in advance. Moreover, as shown in several works [29, 30, 33], the reputation of the sellers (and consequently, the method used to compute the reputation scores) has much influence on auctions (for example, on the decision of a bidder to participate in an auction or on the price of an auction). Notice then that Livingston et al enumerated in [29] various reasons to doubt that on-line market’s (such as eBay) reputation mechanism should work (e.g., sellers could build a reputation by selling relatively inexpensive items and then cheat in auctions of more expensive items).

Before being studied in such on-line markets, reputation and trust have become important topics of research in many fields, as shown in this section.

Reputation has long been of interest to economists. Kreps et al use reputation to explain the cooperation observed in experimental studies of Prisoners’ dilemma game [26]. Economic theory indicates that there is a balance between the cost of establishing a good reputation and the financial benefit of having a good reputation, leading to an equilibrium [23]. Variations in the quality of services or goods can be a result of deliberate management decisions or uncontrolled factors, and whatever the cause, the changes in quality will necessarily lead to variations in reputation. Although a theoretic equilibrium exists, there will always be fluctuations, and it is possible to characterize the conditions under which oscillations can be avoided or converge towards the equilibrium [44].

Scientometrics [36], referring to the study of measuring research outputs such as journal impact factors, also used the notion of reputation. In this context, reputation refers to the number of cross citations that a given author or journal has accumulated over a period of time.

In the field of social science, reputation as a quantitative concept is often studied
as a network parameter associated with a society of agents [9, 46]. Reputation or prestige is often measured by various centrality measures. An example of such a measure is provided by Katz in [25], taking into account not only the number of direct links between agents but, also, the number of indirect links (going through intermediaries) between agents.

Another consideration, closely related to the work done in social sciences, about reputation systems is its collaborative aspect. Indeed, reputation systems could also be called collaborative filtering systems [23] to reflect their collaborative nature (notice that Katz’ measure has recently been rediscovered in the context of collaborative recommendation [19] and kernel methods where it is known as the von Neumann kernel [45]). Collaborative filtering systems (also called recommender systems) try to provide people with recommendations of items they will appreciate, based on their past preferences (evaluations), history of purchase, and demographic information. (see, for example, [2, 11, 17]). As stated by Josang in [23], collaborative filtering systems have similarities with reputation systems (in that both collect ratings from members in a community) as well as fundamental differences: (1) in recommender systems different people have different tastes, and rate things differently according to subjective taste while in reputation systems, all members in a community should judge the performance of a transaction partner or the quality of a product or service consistently; (2) recommender systems take ratings subject to taste as input, whereas reputation systems take ratings assumed insensitive to taste as input; and (3) recommender systems and reputation systems assume an optimistic and a pessimistic world view respectively (recommender systems assume all participants to be trustworthy and sincere while reputation systems, on the other hand, assume that some participants try to misrepresent the quality of services in order to make more profit).

This section has highlighted the two fundamental aspects of reputation systems: the presence of communication protocols (allowing participants to provide ratings about transaction partners as well as to obtain reputation scores of potential transaction partners) and a reputation computation method to derive aggregated scores for each participant based on received ratings, and possibly also on other information. Notice that, in his survey of trust and reputation systems [23], Josang also distinguishes centralised from distributed reputation systems.

Our work is devoted to the definition of new reputation computation models. Section 2 introduces a probabilistic model of reputation (based on a simple consumer-provider interaction model where consumers order items to providers and rate them while providers supply the items with a certain quality) as well as two extensions of this basic model. Expectation-maximization (EM) algorithms allowing to provide estimations of the parameters only based on the ratings are developed for all the models. Section 3 describes the experimental settings as well as the four experiments which are conducted. The results of all these experiments are also part of Section 3 while Section 4 is the conclusion.

1.2 Related work

Reputation is indicative of the confidence and trust placed in a system or entity’s ability to deliver desired results. It also represents the user perception and a reputation system is intended to aggregate and disseminate feedback about participants’ past behavior. These systems encourage trustworthiness and help people choose the right system for service request. This section describes various principles for computing reputation and trust measures, some of them are used in commercial applications. According to a
recent survey [23], trust and reputation systems can be classified into several categories, as described now (and inspired by [27]).

The simplest form of computing reputation scores is simply to sum the number of positive ratings and negative ratings separately, and to keep a total score as the positive score minus the negative score (used in eBay’s reputation forum as described in [39]). The advantage is that anyone can understand the principle behind the reputation score, the disadvantage is that it is primitive and therefore gives a poor picture on participants’ reputation score although this is also due to the way rating is provided. Computing the average of all ratings as reputation score is also used in the reputation systems of numerous commercial web sites (see, for example, [43]). Advanced models in this category compute a weighted average of all the ratings, where the rating weight can be determined by factors such as rater trustworthiness/reputation, age of the rating, distance between rating and current score, etc.

Discrete trust models have also been suggested in several works [1, 7, 8]. Humans are often better able to rate performance in the form of discrete verbal statements, than continuous measures. In the model of Abdul-Rahman et al [1], the trust concept is divided into direct and recommender trust. The agent’s belief in another agent’s trustworthiness (direct trust) is represented within a certain context to a certain degree (“very trustworthy”, “trustworthy”, “untrustworthy”, or “very untrustworthy”). Recommender trust can be derived from word-of-mouth recommendations, which they consider as reputation. In [7], Cahill et al investigate how entities that encounter each other in unfamiliar, pervasive computing environments can overcome initial suspicion to allow secure collaboration to take place. More precisely, their model focuses on a set of trust values (whose elements represent degrees of trust) with two orderings: the first one reflecting the fact a particular trust value may represent a higher level of trust than another whereas the second one reflects that a particular trust value may be more informative than another.

Probabilistic models [6, 27] directly model the statistical interaction between the consumers and the providers. For instance, in the context of food quality assessment, Brockhoff & Skovgaard [6] analyze sensory panel data where individuals (raters) evaluate different items. In their model, they assume that each rater evaluates each item through a linear regression of its quality. The models proposed in this work belong to probabilistic models category and can be considered as an extension of the Brockhoff & Skovgaard model, as detailed later. On the other hand, Laureti et al [27] propose a model for raters evaluating objects. Each object has a quality that is estimated by a weighted average of the ratings provided for this object. The weighting factor depends on the rater and is proportional to a reliability score of the rater, defined as the inverse of his variance.

Bayesian systems (see, for example, [22, 35, 36, 47]) are based on computing reputation scores by statistical updating of probability density functions. The a posteriori (i.e., the updated) reputation score is computed by combining the a priori (i.e., the previous) reputation score with the new rating. The advantage of Bayesian systems is that they provide a theoretically sound basis for computing reputation scores, whereas the main disadvantage is that it might be too complex and difficult to interpret.

Belief theory is a framework related to probability theory, but where the sum of probabilities over all possible outcomes not necessarily add up to 1, and the remaining probability is interpreted as uncertainty. Josang [20, 21] has proposed a belief/trust metric called opinion as well as a set of logical operators that can be used for logical reasoning with uncertain propositions. Yu and Singh [48] have proposed to use belief theory to represent reputation scores.
Trust and reputation can also be represented as linguistically fuzzy concepts, where membership functions describe to what degree an agent can be described as, for example, trustworthy or not trustworthy. Fuzzy logic provides rules for reasoning with fuzzy measures of this type. The methodology proposed by Manchala [32] as well as the REGRET reputation system proposed by Sabater and Sierra [40, 41, 42] fall in this category (see also [38]). In Sabater and Sierra’s scheme, individual reputation is derived from private information about a given member, social reputation is derived from public information about an agent, whereas context dependent reputation is derived from contextual information.

Flow models represent systems that compute trust or reputation by transitive iteration through looped or arbitrarily long chains. Some flow models assume a constant trust/reputation weight for the whole community, and this weight can be distributed among the members of the community. Participants can only increase their trust/reputation at the cost of others. Google’s PageRank [37], the Appleseed algorithm [49], and Advogato’s reputation scheme [28] belong to this category. In general, a participant’s reputation increases as a function of incoming flow, and decreases as a function of outgoing flow. In the case of Google, many hyperlinks to a web page contribute to an increased PageRank score for that web page. Flow models do not always require the sum of the reputation/trust scores to be constant. One such example is the EigenTrust model [24] which computes agent trust scores in P2P networks through repeated, iterative, multiplication, and aggregation of trust scores along transitive chains until the trust score of each member of the P2P community converge to a stable value.

2 Probabilistic models of reputation (PMR)

The proposed reputation models are based on the following simple consumer-provider interaction model. First, we assume a consumer (or buyer) is ordering an item to a provider (or seller), which has some internal “quality of service” score. He will supply the item with a quality following a normal law, centered on his internal “quality of service”. The consumer, after the reception of the item, rate it according to a linear function of its quality (a standard regression model). This regression model accounts for the bias of the consumer in providing ratings as well as its reactivity towards changes in quality. Moreover, the constancy of the provider in supplying a constant quality level for delivering the items is estimated by the standard deviation of the normal law. Symmetrically, the consistancy of the consumer in providing similar ratings for a constant quality is quantified by the standard deviation of the normal law of ratings generation. This is the framework for the basic model of consumer-provider interaction, called Probabilistic Model of Reputation (PMR1).

A second more sophisticated model is also investigated. It accounts for the fact that ratings are often constrained to a specific range of values. Consequently, the rating provided by the consumer is truncated in order to scale within a limited interval, for instance $[-3, +3]$. This second model, PMR2, leads to a more complex parameters updating scheme. Therefore, a third model, which is an approximation of the second one, is also introduced. It corresponds to a simplification of the second model that takes truncation into account while keeping the simplicity of the first model. It will be called PMR3. Experiments show that this model behaves well while keeping the implementation as simple as possible. Then, a fourth, further simplified, model with truncation (PMR4), reducing dramatically the number of variables to be estimated is studied. It assumes that the providers always supply the same quality for the items
Figure 1: Summary of the various models: 1 for truncated ratings, 2 for dropping the variance term, 3 for fixing the internal quality score deterministically, and 4 for introducing prior distributions on the parameters.

(the quality of the item is deterministic and no more a random variable). This model greatly simplifies the previous model; in particular it reduces significantly the number of variables to be estimated. We expect this model to be particularly useful when the number of providers is greater than the number of consumers, in which case the number of parameters to estimate would be too high.

The last extension (PMR5) consists in introducing a prior probability distribution on the reputation parameter as well as the parameters characterizing the consumer (a Bayesian framework). This allows to regularize the estimate and to take the number of ratings into account when computing the reputation score. Indeed, the uncertainty about the reputation estimate is certainly larger for a provider having received very few ratings than for a provider having a large number of ratings. Introducing a prior distribution on the reputation scores allows to balance the a priori, subjective, opinion about the provider of interest and the evidence provided by the ratings.

After having developed the different PMR models, we realized that the PMR models can be viewed as extensions of the Brockhoff & Skovgaard model [6] which was developed in the context of food quality and preference assessment. Brockhoff & Skovgaard’s model is in fact similar to a simplified PMR1 model. Indeed, it assumes deterministic providers that always provide the same quality level, $q_k$, for the supplied items (while the PMR1 model assumes a stochastic provider). It further assumes that each consumer rates once and only once each provider, as often considered in ANOVA models. Moreover, the estimation procedure proposed in [6] is different and is not based on the expectation-maximization algorithm.

A wide range of probabilistic reputation models has been investigated; the one that should be used depends, of course, of the problem at hand. Therefore, the main considerations that have to be taken into account for choosing the most relevant model are detailed in Section 2.6.

For all the models, the only observed data are the ratings; the other quantities being unobserved. A variant of the well-known Expectation-Maximization (EM) algorithm is used in order to estimate both the quality of service of each provider as well as the bias and the reactivity of each consumer. The estimated quality of service of the providers will be the suggested reputation score for the provider.

\footnote{We thank Professor Ritter for pointing us the relationships between the PMR models and Brockhoff & Skovgaard’s model.}
2.1 The basic model: PMR1

2.1.1 Description of the basic model

Assume we have a set of $n_x$ providers and $n_y$ consumers. Each provider (say provider number $k$) has a latent internal quality score, $q_k$, that is hidden to the external world. We define the reputation score associated to provider $k$ as the estimate of $q_k$ based on empirical data. Indeed, each time the provider $k$ sells an item (referred to transaction $i$; each sold item corresponds to a transaction) the internal quality of this item $x_{ki}$ is a random variable following a normal law centered on $q_k$. Thus, the quality of transaction $i$ generated by provider $k$ is given by

$$x_{ki} = q_k + \varepsilon_{ki}^x$$

where $x_{ki}$ is a realization of the random variable $x_k$, the noise random variable $\varepsilon_{ki}^x$ (the superscript $x$ means that the noise model involves the provider) is normal centered and $\varepsilon_{ki}^y$ is a realization of this random variable appearing in transaction number $i$, $\varepsilon_{ki}^y \sim N(0, \sigma_y^y)$. The total number of transactions is denoted by $N$. Therefore, each provider is characterized by two features, (i) his internal quality score $x_k$, and (ii) his stability in providing a constant quality $\sigma_k^y$.

On the other hand, the consumer $l$ who ordered the item rate it based on the inspection of its quality $x_{ki}$. Here, we assume that the consumer can be characterized by three different features: (i) his reactivity with respect to the quality of the provided item $q_k$, (ii) his bias $b_l$, and (iii) his stability in providing constant ratings for a fixed observed quality $\sigma_l^y$. A linear regression model taking all these features into account is assumed. The rating provided by consumer $l$ for transaction $i$ with provider $k$ is given by

$$y_{kli} = a_l x_{ki} + b_l + \varepsilon_{li}^y$$

where the random variable $\varepsilon_{li}^y$ (the superscript $y$ means that the noise model involves the consumer) is normal centered and $\varepsilon_{li}^y$ is a realization of this random variable appearing in transaction number $i$, $\varepsilon_{li}^y \sim N(0, \sigma_l^y)$. Since, as for a one-way analysis of variance, the $q_k$ are only defined up to an additive constant, we constrain the $q_k$ parameters to sum to zero. Thus, $\sum_{k=1}^{n_k} q_k = 0$ [6]. The $q_k$ are therefore normalized.

The quantity $e_l = -b_l/a_l$ is often called the expectation of the consumer in marketing research [5]. The model described by Equation (2) can be re-expressed in terms of this expectation as

$$y_{kli} = a_l(x_{ki} - e_l) + \varepsilon_{li}^y$$

which indicates that the rating provided by the consumer is directly proportional to the difference between the observed quality of the item $x_{ki}$ and his expectation $e_l$. The expectation of each consumer is an interesting feature that is provided as a by-product of the model.

Yet another interesting feature concerns the consumers, i.e., the raters who can be evaluated as well, as already proposed in [6]. A rater (or consumer) is considered as highly reliable if (i) his reactivity $a_l$ is close to 1 (he is fair), (ii) his bias $b_l$ is close to 0 (he is unbiased), and (iii) his standard deviation $\sigma_l^y$ is close to 0 (he is consistent). Therefore, the reliability $r_l$ of a consumer/rater $l$ could, for instance, be evaluated by

$$r_l = \{(a_l - 1)^2 + (b_l)^2 + (\sigma_l^y)^2\}^{1/2},$$

corresponding to the expectation of the squared error of the provided ratings when the input is an iid standardized signal (independent, zero-mean and unit variance), but other choices are, of course, possible.
One can easily show that the joint probability of \([x_k, y_{kl}]\) is also normal with mean and variance-covariance matrix

\[
m = [q_k, a_l q_k + b_l] \quad \text{and} \quad S = \begin{bmatrix}
    (\sigma_k^2)^2 & a_l (\sigma_k^2)^2 \\
    a_l (\sigma_k^2)^2 & a_l^2 (\sigma_k^2)^2 + (\sigma_l^2)^2
\end{bmatrix}
\]

(4)

Notice that this implies that the random variable \(y_{kl}\) relative to provider \(k\) is normally distributed \(y_{kl} \sim N(a_l x_k + b_l, \sqrt{a_l^2 (\sigma_k^2)^2 + (\sigma_l^2)^2})\), given that provider is \(k\). In the first model, we suppose that only the ratings \(y_{kli}\) are observed while the \(x_{ki}\) are unobserved.

### 2.1.2 The likelihood function of the model

We now consider the problem of estimating the different parameters of the model. For this purpose, the set of values is considered as incomplete. The complete set of variables is \(\{x_{ki}, y_{kli}\}, k = 1 \ldots n_x, l = 1 \ldots n_y, i = 1 \ldots N\), and since only the \(\{y_{kli}\}, k = 1 \ldots n_x, l = 1 \ldots n_y, i = 1 \ldots N\) are observed, all the other variables are considered as unobserved. Assuming independence between the observations, the likelihood for the complete (observed and unobserved) data is

\[
\mathcal{L}(\Theta) = \prod_{k=1}^{n_x} \prod_{l=1}^{n_y} \prod_{i \in (k,l)} P(x_{ki}, y_{kli}) = \prod_{k=1}^{n_x} \prod_{l=1}^{n_y} \prod_{i \in (k,l)} P(y_{kli}|x_{ki}) P(x_{ki})
\]

(5)

where \(\Theta\) is the vector containing all the parameters of the model, \(\Theta = \{q_k, a_l, b_l, \sigma_k^2, \sigma_l^2\}\), and \((k, l)\) denotes the set of transactions involving provider \(k\) and consumer \(l\), with \(n_{kl}\) being the total number of transactions \(i \in (k, l)\). Thus, in the previous equation, the product on \(i\) is taken on the set of \(n_{kl}\) transactions belonging to \((k, l)\). Taking the logarithm of both sides and denoting the log-likelihood by \(l = \log \mathcal{L}\) provides

\[
l(\Theta) = \sum_{k=1}^{n_x} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \left\{ \log(P(x_{ki})) + \log(P(y_{kli}|x_{ki})) \right\}
\]

(6)

\[
= - \sum_{k=1}^{n_x} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \left\{ \frac{1}{2} \left[ \frac{x_{ki} - q_k}{\sigma_k^2} \right]^2 + \frac{1}{2} \left[ \frac{y_{kli} - (a_l x_{ki} + b_l)}{\sigma_l^2} \right]^2 \right\} + \log(\sigma_k^2) + \log(2\pi)
\]

(7)

\[
+ \log(\sigma_l^2) + \log(2\pi)
\]

(8)

This likelihood function serves as basis for the parameters estimation algorithm.

### 2.1.3 Estimating the reputation scores

This section is based on Appendix A where we develop an EM-based algorithm (the so-called “one-step-late” algorithm ([13]; see also [34]) for the estimation of both the parameters of the model and the unobserved variables. We suppose that we observe the rating \(y_{kli}\) for each transaction. Notice that the estimates of the parameters of interest are denoted by a hat.

The estimate of the unobserved variable \(x_k\) depends on the consumer \(l\) (see Appendix A) and is therefore denoted by \(\hat{x}_{kl} = E[x_k|y_{kl}, \Theta]\), including a subscript \(l\).
stressing the dependence on the consumer. The obtained update of the estimate of the unobserved variable \( x_{kli} \) for transaction \( i \) is

\[
\hat{x}_{kli} \leftarrow \hat{q}_k + \frac{\hat{a}_i (\hat{\sigma}^x_k)^2}{\hat{\sigma}^x_k (\hat{\sigma}^x_k)^2 + (\hat{\sigma}^y_k)^2} \left[ y_{kli} - (\hat{a}_l \hat{q}_k + \hat{b}_l) \right]
\]

(9)

\[
\left( \hat{\sigma}^x_k \right)^2 \leftarrow \frac{1}{\hat{\sigma}^x_k} \left( \hat{\sigma}^x_k \right)^2 + (\hat{\sigma}^y_k)^2 \left[ \hat{q}_k + \hat{a}_l (\hat{\sigma}^x_k)^2 \left( y_{kli} - \hat{b}_l \right) \right]
\]

(10)

\[
\left( \hat{\sigma}^y_k \right)^2 \leftarrow \frac{1}{\hat{\sigma}^y_k} \left( \hat{\sigma}^y_k \right)^2 + (\hat{\sigma}^x_k)^2 \left( y_{kli} - \hat{b}_l \right)
\]

(11)

where we define \( \hat{\sigma}^2 = (\hat{\sigma}^x_k)^2 + (\hat{\sigma}^y_k)^2 \). The meaning of this updating formula is standard: for computing the update of \( \hat{x}_{kli} \) at time \( t \) (left-hand side), use the estimates at time \( t-1 \) (right-hand side).

On the other hand, we obtain the following reestimation formulas for the parameters of the model. The total number of transactions involving provider \( k \) is denoted by \( n_{k*} \), while \( n_{*l} \) denotes the total number of transactions involving consumer \( l \). In order to simplify the notations, we define the following intermediate variable, also updated at each iteration,

\[
(\hat{\sigma}_{kl}^x)^2 \leftarrow \frac{(\hat{\sigma}_k^x)^2}{\hat{\sigma}_k^x (\hat{\sigma}_k^x)^2 + (\hat{\sigma}_k^y)^2}
\]

(12)

Let us first consider the initialization of the parameters. At the first iteration step \( t=0 \), the quality levels are initialized to the average score and normalized \( \sum_{k=1}^{n_k} \hat{q}_k = 0 \):

\[
\begin{align*}
\hat{q}_k & \leftarrow \frac{1}{n_k} \sum_{i=1}^{n_k} \sum_{j \in (k,l)} y_{kli}, \text{ and normalize the } \hat{q}_k \\
\hat{x}_k & \leftarrow 1, \hat{a}_i \leftarrow 1, \hat{b}_l \leftarrow 0, \hat{\sigma}_k^y \leftarrow 1
\end{align*}
\]

(13)

The parameters associated to the providers are then updated at each iteration \( t \):

\[
\hat{q}_k \leftarrow \frac{1}{n_{k*}} \sum_{i=1}^{n_{k*}} \sum_{j \in (k,l)} \hat{x}_{kli}, \text{ and normalize the } \hat{q}_k
\]

(14)

\[
(\hat{\sigma}_k^x)^2 \leftarrow \frac{1}{n_{k*}} \sum_{i=1}^{n_{k*}} \sum_{j \in (k,l)} \left[ \hat{q}_k - \hat{x}_{kli} \right]^2 + (\hat{\sigma}_{kl}^x)^2 \]

(15)

For the parameters associated to the consumers, we obtain

\[
\hat{a}_i \leftarrow \frac{\sum_{k=1}^{n_k} \sum_{j \in (k,l)} \hat{x}_{kli} \left( y_{kli} - \hat{b}_l \right)}{\sum_{k=1}^{n_k} \sum_{j \in (k,l)} \hat{x}_{kli}^2 + (\hat{\sigma}_{kl}^x)^2}
\]

(16)

\[
\hat{b}_l \leftarrow \frac{1}{n_{*l}} \sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left( y_{kli} - \hat{a}_i \hat{x}_{kli} \right)
\]

(17)

\[
(\hat{\sigma}_k^y)^2 \leftarrow \frac{1}{n_{*l}} \sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left[ y_{kli} - \left( \hat{a}_i \hat{x}_{kli} + \hat{b}_l \right) \right]^2 + \hat{a}_l^2 (\hat{\sigma}_{kl}^y)^2 \]

(18)

Iterating these equations until convergence provides an estimation of the quality of each provider as well as of the other parameters.
2.2 A more sophisticated model accounting for truncation: PMR2

2.2.1 Description of the model

As already stated before, ratings are often expressed on a limited scale. This model assume that the ratings are truncated in order to obtain a final rating \( z_{kli} \) in the interval \([-c, +c]\). We assume, without lack of generality, that the ratings are normalized in the interval \([-1, +1]\), and thus \(0 \leq c \leq 1\). In other words, only the truncated ratings \( z_{kli} \) are observed while the \( y_{kli} \) are unobserved. Therefore,

\[
z_{kli} = \text{trunc}(y_{kli}, c)
\]

where \( \text{trunc} \) is the truncation operator defined as

\[
\text{trunc}(y, c) = \delta(y \geq 0) \min(y, c) + \delta(y < 0) \max(y, c)
\]

The function \( \delta(y \geq 0) \) is equal to 1 if the condition \( y \geq 0 \) is true and 0 otherwise. Thus, the truncation operator saturates the variable \( y \) by constraining its range in the interval \([-c, +c]\).

This model thus considers that we directly observe the truncated ratings for the \( N \) transactions, \( \{z_{kli}\} \), and the objective is to estimate the quality of the providers based on these ratings. As before, this estimate, \( \hat{q}_k \), will be the reputation score for provider \( k \).

2.2.2 The likelihood function of the model

This section considers the problem of estimating the different parameters of the model. In this case, the complete set of variables is \( \{x_{ki}, y_{kli}, z_{kli}; k = 1 \ldots n_x, l = 1 \ldots n_y, i = 1 \ldots N\} \), and since only the \( \{z_{kli}; k = 1 \ldots n_x, l = 1 \ldots n_y, i = 1 \ldots N\} \) are observed, all the other variables are considered as unobserved. Assuming independence between the observations, the likelihood of the observations for the complete data is

\[
\mathcal{L}(\Theta) = \prod_{k=1}^{n_x} \prod_{l=1}^{n_y} \prod_{i \in \{k,l\}} P(x_{ki}, y_{kli}, z_{kli})
\]

and the likelihood of the complete data has quite the same form as for the PMR1 model.

2.2.3 Estimating the reputation scores

A few notations are needed before stating the reestimation formulas. The standard normal distribution and the standard normal cumulative distribution function are denoted by

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du
\]

For the update of the unobserved variable \( x_{kli} \), we obtain the same form as for model PMR1,

\[
\hat{x}_{kli} \leftarrow \frac{1}{s_{kl}^2} \left[ (\hat{\sigma}_y^2) \hat{\sigma}_k + (\hat{\sigma}_k^2) \hat{\sigma}_l \left( \hat{y}_{kli} - \hat{b}_l \right) \right]
\]
with $\hat{s}_{kl}^2$ defined as
\[
\hat{s}_{kl}^2 = \hat{a}_l^2 \hat{a}_k^2 + (\hat{\sigma}_l^2)^2
\] (25)

The update of the unobserved variable $y_{kli}$ depends on the observed value of the corresponding rating, $z_{kli}$. Three cases have to be considered: $z_{kli} = -c$, $-c < z_{kli} < +c$ and $z_{kli} = +c$ (for more details, see Appendix B).

**First case:** $z_{kli} = -c$:
\[
\hat{y}_{kli} \leftarrow (\hat{a}_l \hat{q}_k + \hat{b}_l) + \hat{s}_{kl} \lambda(\hat{\gamma}_{kl})
\]
(26)
\[
\hat{V}_{kl}^- \leftarrow \hat{s}_{kl}^2 [1 + \hat{\gamma}_{kl} \lambda(\hat{\gamma}_{kl}) - \lambda^2(\hat{\gamma}_{kl})]
\]
(27)
with
\[
\hat{\gamma}_{kl} = -\frac{c - (\hat{a}_l \hat{q}_k + \hat{b}_l)}{\hat{s}_{kl}^2}
\]
\[
\lambda(\hat{\gamma}_{kl}) = -\frac{\varphi(\hat{\gamma}_{kl})}{\phi(\hat{\gamma}_{kl})}
\]
(28)

**Second case:** $-c < z_{kli} < +c$:
\[
\hat{y}_{kli} \leftarrow z_{kli}
\]
(29)
\[
\hat{V}_{kl}^- = \delta(z_{kli} = -c) \hat{V}_{kl}^- + \delta(z_{kli} = +c) \hat{V}_{kl}^+
\]
(33)

**Third case:** $z_{kli} = +c$:
\[
\hat{y}_{kli} \leftarrow (\hat{a}_l \hat{q}_k + \hat{b}_l) + \hat{s}_{kl} \lambda(\hat{\gamma}_{kl})
\]
(30)
\[
\hat{V}_{kl}^+ \leftarrow \hat{s}_{kl}^2 [1 + \hat{\gamma}_{kl} \lambda(\hat{\gamma}_{kl}) - \lambda^2(\hat{\gamma}_{kl})]
\]
(31)
with
\[
\hat{\gamma}_{kl} = \frac{c - (\hat{a}_l \hat{q}_k + \hat{b}_l)}{\hat{s}_{kl}^2}
\]
\[
\lambda(\hat{\gamma}_{kl}) = -\frac{\varphi(\hat{\gamma}_{kl})}{\phi(\hat{\gamma}_{kl})}
\]
(32)

At the first iteration step ($t = 0$), we initialize the parameters as before (see Equation (13)). For the parameters associated to the providers, we have
\[
\hat{q}_k \leftarrow \frac{1}{n_k} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \hat{x}_{kli}, \text{ and normalize the } \hat{q}_k
\]
(34)
\[
(\hat{\sigma}_k^2) = \frac{1}{n_k} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \left( \hat{q}_k - \hat{x}_{kli} \right)^2 + (\hat{\sigma}_k^2)^2 + \frac{\hat{\sigma}_l^2(\hat{\sigma}_k^2)^2}{\hat{s}_{kl}^2}
\]
(35)
And for the parameters associated to the consumers,

\[
\tilde{a}_l \leftarrow \frac{\sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left( \tilde{x}_{kli} \left( \tilde{y}_{kli} - \tilde{b}_l \right) + \frac{\tilde{a}_l (\tilde{\sigma}_k^2 \tilde{V}_{kli})}{\tilde{S}_{kl}^2} \right)}{\sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left[ \tilde{x}_{kli}^2 + (\tilde{\sigma}_{kl}^2) + \frac{\tilde{a}_l^2 (\tilde{\sigma}_k^2 \tilde{V}_{kli})^2}{\tilde{S}_{kl}^4} \right]}
\]  

\[\tilde{b}_l \leftarrow \frac{1}{n_{zkl}} \sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left( \tilde{y}_{kli} - \tilde{a}_l \tilde{x}_{kli} \right)
\]

\[(\tilde{\sigma}_l^y)^2 \leftarrow \frac{1}{n_{zkl}} \sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left[ \left( \tilde{y}_{kli} - (\tilde{a}_l \tilde{x}_{kli} + \tilde{b}_l) \right)^2 + \tilde{a}_l^2 (\tilde{\sigma}_{kl}^y)^2 \right] + \frac{(\tilde{\sigma}_l^2)^4 \tilde{V}_{kli}}{\tilde{S}_{kl}^4}
\]

with \(\tilde{\sigma}_{kl}^y\) defined as

\[(\tilde{\sigma}_{kl}^y)^2 = \frac{(\tilde{\sigma}_k^y)^2 (\tilde{\sigma}_l^y)^2}{\tilde{a}_l^2 (\tilde{\sigma}_k^2 + (\tilde{\sigma}_l^2)^2)}
\]

\[\text{2.3 A simplified model with truncation: PMR3}
\]

For this simplified model, called PMR3, we drop the variance term, \(V\). In this case, the model corresponds to a simple two-stage procedure: (i) compute the conditional expectations of the \(y_{kli}\) given the observed \(z_{kli}\) and (ii) compute the estimates of the \(x_k\) as well as the parameters by considering that the conditional expectations of the \(y_{kli}\) are the real observed values as in PRM1. Thus, this simplified model is equivalent to PMR2 where we drop the variance term, \(V\) (i.e., we use the update equations of PMR2 with \(V = 0\)).

What concerns the unobserved variable \(x_{kli}\), we obtain the same updating rules as for models PMR1 and PMR2. For the unobserved \(y_{kli}\), we still have to consider three cases: \(z_{kli} = -c\), \(-c < z_{kli} < +c\) and \(z_{kli} = +c\).

**First case: \(z_{kli} = -c\):**

\[\tilde{y}_{kli} \leftarrow (\tilde{a}_l \tilde{q}_k + \tilde{b}_l) + \tilde{s}_{kl} \lambda(\tilde{\gamma}_{kl})
\]

with

\[
\begin{align*}
\tilde{\gamma}_{kl} &= -c - (\tilde{a}_l \tilde{q}_k + \tilde{b}_l) \\
\lambda(\tilde{\gamma}_{kl}) &= \frac{\phi(\tilde{\gamma}_{kl})}{\varphi(\tilde{\gamma}_{kl})}
\end{align*}
\]

**Second case: \(-c < z_{kli} < +c\):**

\[\tilde{y}_{kli} \leftarrow z_{kli}
\]

**Third case: \(z_{kli} = +c\):**

\[\tilde{y}_{kli} \leftarrow (\tilde{a}_l \tilde{q}_k + \tilde{b}_l) + \tilde{s}_{kl} \lambda(\tilde{\gamma}_{kl})
\]

with

\[
\begin{align*}
\tilde{\gamma}_{kl} &= c - (\tilde{a}_l \tilde{q}_k + \tilde{b}_l) \\
\lambda(\tilde{\gamma}_{kl}) &= \frac{\tilde{s}_{kl}}{\phi(\tilde{\gamma}_{kl})} \frac{\varphi(\tilde{\gamma}_{kl})}{1 - \varphi(\tilde{\gamma}_{kl})}
\end{align*}
\]
At the first iteration step \((t = 0)\), we initialize the parameters as before. Then, for the parameters associated to the consumers, we have

\[
\hat{q}_k \leftarrow \frac{1}{n_{k*}} \sum_{i=1}^{n_k} \sum_{l \in (k, l)} \hat{x}_{kli}, \text{ and normalize the } \hat{q}_k \quad (44)
\]

\[
(\hat{\sigma}_k^2)^2 \leftarrow \frac{1}{n_{k*}} \sum_{i=1}^{n_k} \sum_{l \in (k, l)} \left[ (\hat{q}_k - \hat{x}_{kli})^2 + (\hat{\sigma}_k^2)^2 \right] \quad (45)
\]

For the parameters associated to the providers, we have

\[
\hat{a}_l \leftarrow \frac{\sum_{k=1}^{n_x} \sum_{i \in (k, l)} \left[ \hat{x}_{kli} \left( \hat{y}_{kli} - \hat{b}_l \right) \right]}{\sum_{k=1}^{n_x} \sum_{i \in (k, l)} \left[ \hat{x}_{kli}^2 + \left(\hat{\sigma}_k^2\right)^2 \right]} \quad (46)
\]

\[
\hat{b}_l \leftarrow \frac{1}{n_{l*}} \sum_{k=1}^{n_x} \sum_{i \in (k, l)} \left( \hat{y}_{kli} - \hat{a}_l \hat{x}_{kli} \right) \quad (47)
\]

\[
(\hat{\sigma}_l^2)^2 \leftarrow \frac{1}{n_{l*}} \sum_{k=1}^{n_x} \sum_{i \in (k, l)} \left[ \left( \hat{y}_{kli} - (\hat{a}_l \hat{x}_{kli} + \hat{b}_l) \right)^2 + \hat{a}_l^2 \left(\hat{\sigma}_k^2\right)^2 \right] \quad (48)
\]

Notice finally that the model without truncation is simply obtained by replacing the unobserved variables \(\hat{y}_{kli}\) by the observed ones \(y_{kli}\).

### 2.4 A further simplified model with truncation: PMR4

A further simplified model with truncation, PMR4, can be obtained by assuming that the provider \(k\) always supplies the same quality \(q_k\) for the items (the quality of the item is no more a random variable). This model simplifies greatly the previous model; in particular it reduces significantly the number of variables to be estimated since there is no need to introduce unobserved variables, \(\hat{x}_{kli}\). We expect this model to be particularly useful when the number of providers is greater than the number of consumers \((n_x > n_y)\), in which case the number of parameters to estimate would be too high.

There are essentially two main differences with the previous model: (1) the quality of the provided item is now deterministic and is equal to the latent quality score of the provider (in particular, this implies that the variable \(y_{kli}\) is now normally distributed with mean \(q_k\) and standard deviation \(\sigma_l^y\) (instead of \(s_{kli}\), \(y_{kli} \sim N(q_k, b_l, \sigma_l^y)\)), which results in a different update of the estimated \(\hat{y}_{kli}\), and (2) the simplified expected log-likelihood function, after the expectation step and after neglecting the extra variance term coming from the truncation in PMR3, is now

\[
E_{y|z} \left[ t \mid z, \Theta \right] = -\frac{1}{2} \sum_{k=1}^{n_x} \sum_{l=1}^{n_y} \sum_{i \in (k, l)} \left\{ \frac{\left[ \hat{y}_{kli} - \frac{(a_l q_k + b_l)}{\sigma_l^y} \right]^2}{\frac{1}{n_{l*}} \sum_{k=1}^{n_x} \sum_{i \in (k, l)} \left[ \left( \hat{y}_{kli} - \frac{(a_l q_k + b_l)}{\sigma_l^y} \right)^2 + \log((\sigma_l^y)^2) \right]} \right\} \quad (49)
\]

Instead of Equation (108).

It can be easily shown that the resulting update rules are the following. Let us first consider the three truncation cases: \(z_{kli} = -c, -c < z_{kli} < +c\) and \(z_{kli} = +c\). \(s_{kli}\) is simply replaced by \(\sigma_l^y\) in the updating formulas of PMR3 (Equations (41)-(43)): 
First case: \( z_{kli} = -c \):

\[
\tilde{y}_{kli} \leftarrow (\tilde{a}_l \tilde{q}_k + \tilde{b}_t) + \tilde{\sigma}_l^y \lambda(\tilde{\gamma}_{kl})
\]

with

\[
\begin{align*}
\tilde{\gamma}_{kl} &= -c - (\tilde{a}_l \tilde{q}_k + \tilde{b}_t) \\
\lambda(\tilde{\gamma}_{kl}) &= \frac{\psi(\tilde{\gamma}_{kl})}{\phi(\tilde{\gamma}_{kl})}
\end{align*}
\] (50)

Second case: \(-c < z_{kli} < +c\):

\[
\tilde{y}_{kli} \leftarrow z_{kli}
\] (51)

Third case: \( z_{kli} = +c \):

\[
\tilde{y}_{kli} \leftarrow (\tilde{a}_l \tilde{q}_k + \tilde{b}_t) + \tilde{\sigma}_l^y \lambda(\tilde{\gamma}_{kl})
\]

with

\[
\begin{align*}
\tilde{\gamma}_{kl} &= c - (\tilde{a}_l \tilde{q}_k + \tilde{b}_t) \\
\lambda(\tilde{\gamma}_{kl}) &= \frac{\psi(\tilde{\gamma}_{kl})}{1 - \phi(\tilde{\gamma}_{kl})}
\end{align*}
\] (52)

For the reputation parameters associated to the providers, when minimizing Equation (49) (M-step), we easily obtain

\[
\tilde{q}_k \leftarrow \frac{\sum_{t=1}^{n_t} \sum_{i \in (k,t)} (\tilde{a}_l \tilde{q}_k + \tilde{b}_t)}{\sum_{t=1}^{n_t} (n_{kl} \tilde{a}_l^2) / (\tilde{\sigma}_l^y)^2}, \quad \text{and normalize the } \tilde{q}_k
\] (53)

And for the parameters associated to the consumers,

\[
\begin{align*}
\tilde{a}_l &\leftarrow \frac{\sum_{k=1}^{n_k} \sum_{t \in (k,t)} \tilde{q}_k (\tilde{y}_{kli} - \tilde{b}_t)}{\sum_{k=1}^{n_k} n_{kl} \tilde{q}_k^2} \\
\tilde{b}_t &\leftarrow \frac{1}{n_{st}} \sum_{k=1}^{n_k} \sum_{t \in (k,t)} (\tilde{y}_{kli} - \tilde{a}_l \tilde{q}_k)
\end{align*}
\] (54) (55)

\[
(\tilde{\sigma}_l^y)^2 \leftarrow \frac{1}{n_{st}} \sum_{k=1}^{n_k} \sum_{t \in (k,t)} \left( \tilde{y}_{kli} - (\tilde{a}_l \tilde{q}_k + \tilde{b}_t) \right)^2
\] (56)

2.5 Introducing a Bayesian prior on the parameters: PMR5 and simple PMR5 (sPMR5)

Yet another extension consists in introducing a prior probability distribution on the reputation parameter \( q_k \). This allows to regularize the estimate and to take the number of ratings into account when computing the reputation score. Indeed, the uncertainty about the reputation estimate is certainly larger for a provider having very few ratings than for a provider having a large number of ratings. Introducing a prior distribution on the \( q_k \) allows to balance the a priori, subjective, opinion about the provider of interest and the evidence provided by the ratings.

We consider that the a priori reputation score is zero (a neutral rating), but this can be easily modified if some a priori information concerning the consumer is provided. In this case, the reputation score will be around zero at the beginning (no rating yet
recorded) and will progressively deviate from zero when the number of ratings for this customer becomes more significant. The same approach is applied in order to regularize the parameters $a_k$ and $b_l$.

Concretely, we introduce a normal prior on $q_k$, $q_k \sim N(0, \sigma_0^2)$ where $\sigma_0^2$ is typically set in order to obtain a 0.99 probability of observing $q_k \in [-c, +c]$, in which case $\sigma_0^2 = c/2.57$. The normal distribution is the natural conjugate prior for the location parameter of a normal distribution [12]. This extension aims to maximize the a posteriori distribution of $q_k$, and this can be done by observing (see [34]) that the maximum a posteriori estimate of $\Theta$ maximizes $\log(P(\Theta|z)) = \log(P(z|\Theta)) + \log(P(\Theta)) + \log(P(z))$ where $z$ is the set of observed data. Since $P(z)$ does not depend on the parameter, this is equivalent to maximize $\log(P(\Theta|z)) = \log(P(z|\Theta)) + \log(P(\Theta))$.

The PMR5 model: PMR3 with priors. It can easily be shown that the expectation step remains the same as for the computation of the maximum likelihood [34]. On the other hand, the maximization step differs in that the objective function for the maximization process is augmented by the log prior density, $\log(P(\Theta))$. A few calculus shows that the update formula for $q_k$ in PMR3 becomes

$$\hat{q}_k \leftarrow \frac{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \tilde{x}_{kli}}{n_k + (\sigma_{k}^2/\sigma_0^2)^2}, \text{ and normalize the } \hat{q}_k$$

(57)

Of course, prior distributions could be assigned to the other parameters $a_l$ and $b_l$ as well by following the same procedure. Here are the resulting update rules for $a_l, b_l$ extending the PMR3 model,

$$\hat{a}_l \leftarrow \frac{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \tilde{x}_{kli} \left( \tilde{y}_{kli} - \hat{b}_l \right)}{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \tilde{x}_{kli}^2 + (\sigma_{k}^2/\sigma_0^2)^2}, \text{ and normalize the } \hat{a}_l$$

(58)

$$\hat{b}_l \leftarrow \frac{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \tilde{y}_{kli} - \hat{a}_l \tilde{x}_{kli}}{n_l + (\sigma_{k}^2/\sigma_0^2)^2}$$

(59)

where $\sigma_0^2$ and $\sigma_k^2$ are the prior standard deviations, assuming a normal distribution.

The other update equations are not modified. The PMR3 model extended with this Bayesian framework is called PMR5 model. It consists of the update rules of PMR3 with the update rule for $q_k$ being replaced by (57).

The simple PMR5 model (sPMR5): PMR4 with priors. In exactly the same way, here is the adaptation of the simple PMR4 model:

$$\hat{y}_k \leftarrow \frac{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \left( \hat{a}_l \left( \tilde{y}_{kli} - \hat{b}_l \right) \right) \left( \sigma_{k}^2/\sigma_0^2 \right)^2}{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \left( \hat{a}_l \right)^2 \left( \sigma_{k}^2/\sigma_0^2 \right)^2 + 1}, \text{ and normalize the } \hat{y}_k$$

(60)

$$\hat{a}_l \leftarrow \frac{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \left( \hat{y}_{kli} - \hat{b}_l \right) \left( \sigma_{k}^2/\sigma_0^2 \right)^2}{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \left( \hat{y}_{kli} - \hat{a}_l \hat{q}_k \right) \left( \sigma_{k}^2/\sigma_0^2 \right)^2 + \sum_{i=1}^{n_k} \left( n_{k,l} \hat{a}_l^2 \right) \left( \sigma_{k}^2/\sigma_0^2 \right)^2}$$

(61)

$$\hat{b}_l \leftarrow \frac{\sum_{i=1}^{n_k} \sum_{l \in \{k,l\}} \left( \tilde{y}_{kli} - \hat{a}_l \hat{q}_k \right)}{n_l + (\sigma_{k}^2/\sigma_0^2)^2}$$

(62)
2.6 Some guidelines on the use of the probabilistic reputation models

A wide range of probabilistic reputation models has been investigated in the previous sections. The one that should be used depends of course of the problem at hand. There are three main considerations that have to be taken into account for choosing the most relevant model:

- Should we consider a deterministic generation of the quality of the items?
- Are the ratings truncated or not?
- Does it make sense to regularize the parameters of interest, namely the internal quality, the bias, and the reactivity?

Concerning these alternatives, the guidelines are the following:

- Are there more providers than customers? If this is the case, a simplified model considering a deterministic generation of the quality of the items (such as PMR4 or sPMR5) should be used.
- Do the ratings involve truncation? If yes, a model accounting for truncation should be used (PMR2, PMR3, or PMR4).
- Are there very few ratings? In this case, a Bayesian prior on the parameters should be considered (PMR5 or sPMR5).

In any case, when dealing with truncation, we recommend the use of the simplified model which avoids the complexity of the full model without a significant drop in performance.

3 Experiments

The experimental section aims to answer four important research questions: (1) Are the PMR models able to estimate the parameters of interest, namely the provider’s quality, the customer’s bias, and the customer’s reactivity, in an accurate way; (2) Do the suggested models compare favorably with respect to a simple average; (3) Do the suggested models provide better results than the Brockhoff Model (BKF) and Iterative Refinement (IR); (4) Which suggested model (PMR1, PMR2, PMR3, PMR4, PMR5, or sPMR5) provides the best results overall. In other words, does the PMR models show some added value in tasks involving reputation estimation. In order to investigate these questions, we performed four experiments that are now described.

3.1 Experimental settings

The experiments simulate a simple consumer-provider interaction model: a consumer is requesting a service to a provider while the provider has some score representing his internal quality of service. A transaction is the execution of a requested service by a provider for a consumer. The achievement of a transaction between a provider and a consumer brings the consumer to give a rating representing the quality of the provider in executing the service. Thus, the reputation score of a provider for a transaction
depends on one hand, on the service quality and on the other hand, on the behavior of the consumer.

In accordance with previous notations, a provider \( k \) is characterized by his internal quality score \( q_k \), and his stability in providing a constant quality \( \sigma^x_k \). A consumer \( l \) is characterized by his reactivity \( a_l \), his bias \( b_l \), and his stability in providing constant rates for a fixed observed quality \( \sigma^y_l \). These values are referred to as the parameters of the providers/consumers.

The experiments are performed on an artificial data set generated in order to simulate the interactions between \( n_x \) consumers and \( n_y \) providers. Each consumer-provider pair is connected through \( n_t \) links; each link representing a transaction \( i \) which is characterized by its quality \( x_{ki} \) depending on provider \( k \) and a rating \( y_{kli} \) provided by consumer \( l \). Each investigated model estimates the reputation score \( q_k \) and the stability \( \sigma^x_k \) of each provider \( k \), as well as the behavior \( (a_l, b_l, \sigma^y_l) \) of each consumer \( l \) from a data set containing \( n_k.n_l.n_i \) transactions. The parameters are estimated on the available ratings only.

The first step aims to generate a set of consumers and providers having each their own parameters. The way the parameters are generated and the number of providers and consumers differ from one experiment to another and are explained later.

The second step consists of generating a set of providers-consumers transactions characterized by their quality \( x_{ki} \) and their rating \( y_{kli} \) for each triplet \( (k, l, i) \). The quality and the ratings are generated from a normal distribution:

\[
\begin{align*}
x_{ki} &= N(q_k, \sigma^x_k) \\
y_{kli} &= N(a_l x_{ki} + b_l, \sigma^y_l)
\end{align*}
\] (63)

The ratings are then truncated in order to belong to the \([-1, 1]\) interval.

The aim of each model applied in the various experiments is of course to estimate at best the values of the parameters \( \{\hat{a}_l, b_l, \hat{\sigma}^y_l, \hat{\sigma}^x_k, \hat{q}_k\} \) from the ratings only. The estimated values are compared to the real generated values in order to evaluate the ability of the model to retrieve the real, generated parameters’ value (which are hidden to the model). Two performance indicators within this context are reported: the average absolute error between real and predicted values and the linear correlation between real and predicted values.

The estimated parameters are initialized as follows:

\[
\begin{align*}
\hat{q}_k &\leftarrow 0, \hat{\sigma}^x_k \leftarrow 1, \hat{\sigma}^y_l \leftarrow 0, \hat{\sigma}^y_l \leftarrow 1
\end{align*}
\] (64)

Moreover, each experiment averaged the results on 10 runs, each run consisting of (1) generating actual parameters values and (2) estimating these values by using specific models.

Three experiments, obeying these settings, were conducted. The first experiment (Section 3.2) compares the three main models: the basic model PMR1, a more sophisticated model PMR2 introducing truncation and a simplified model with truncation, PMR3 (as an intermediary model between PMR1 and PMR2). The second experiment (Section 3.3) analyzes the behavior and the robustness of these models (PMR1, PMR2, and PMR3) with respect to noise. The third experiment (Section 3.4) compares the performance of our best models (according to the first experiment) with three other models of reputation, namely, a simple average, Brockhoff et al.’s model (BKF), and the Iterative Refinement model (IR) which is described in this section. Finally, the last experiment (Section 3.5) executes and analyzes our models (i.e., PMR4 and sPMR5 providing the best results in the previous experiment) and BKF on a real data set.
Table 1: Comparison of PMR1, PMR2, PMR3, and SA in terms of the average absolute error and the linear correlation.

<table>
<thead>
<tr>
<th></th>
<th>PMR1</th>
<th>PMR2</th>
<th>PMR3</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average absolute error</td>
<td>0.085</td>
<td>0.082</td>
<td>0.084</td>
<td>0.276</td>
</tr>
<tr>
<td>Linear correlation</td>
<td>0.991</td>
<td>0.998</td>
<td>0.996</td>
<td>0.442</td>
</tr>
</tbody>
</table>

3.2 First preliminary, experiment: Comparing PMR1, PMR2 and PMR3

3.2.1 Description

In this experiment, values for all parameters (related to provider \( k \) and consumer \( l \)) have been uniformly generated within an interval given by:

\[
\begin{align*}
    a_l & \in [-0.15, 0.15] \\
    b_l & \in [-0.25, 0.25] \\
    \sigma_y^l, \sigma_k^l & \in [0.025, 0.25] \\
    q_k & \in [-0.5, 0.5]
\end{align*}
\]  

Notice that the values of \( a_l \) could be negative, corresponding to extreme conditions where the consumers are cheating about their ratings (i.e., allowing such behavior as providing a bad rating for a good item).

For this experiment, (1) the data set contains 50 consumers, 50 providers, and 20 transactions for each couple of provider-consumer (i.e., 50 \times 50 \times 20 transactions), and (2) a simple average (SA) as well as PMR1, PMR2, and PMR3 are compared. Each model updates the parameters associated to the providers and the consumers, at each iteration, until convergence of \( \hat{q}_k \). For SA, we simply average the ratings provided by the consumers as follows:

\[
\hat{q}_k \leftarrow \frac{1}{n_k} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} y_{kli}
\]  

3.2.2 Results

Figure 2 compares the real (generated, but hidden to the algorithm) and the predicted (estimated) reputation score for SA, PMR1, PMR2, and PMR3. The closer the dotted line to the continuous line \( \hat{q} = q \) (predicted = generated), the better the prediction is (i.e., the model predicts well the actual value of the parameters). Figures 3 and 4 show the real and predicted values of the reactivity \( a_l \) and the bias \( b_l \) for PMR1, PMR2, and PMR3. The performance of the PMR models are very similar and the estimated values of the parameters are close to the actual values. Moreover, we clearly observe that PMR1-3 outperform SA.

The average absolute error as well as the linear correlation between the real and the predicted reputation scores (\( q_k \)), averaged on 10 runs are provided in Table 1 for PMR1, PMR2, PMR3, and SA, confirming that PMR models provide better estimation than SA. The best model is PMR2 followed by PMR3 and PMR1. In order to test the difference between these models, a \( t \)-test has been performed, for the average absolute error, on the 10 runs, showing that the results of PMR2 are significantly \( (p < 10^{-2}) \) better than those provided by the other models.
Simple average (SA)

PMR1

-0.5 -0.3 -0.1 0.1 0.3 0.5

Generated $q$

-0.5 -0.3 -0.1 0.1 0.3 0.5

Estimated $q$

-0.7 -0.5 -0.3 -0.1 0.1 0.3 0.5

PMR2

-0.5 -0.3 -0.1 0.1 0.3 0.5

Generated $q$

-0.7 -0.5 -0.3 -0.1 0.1 0.3 0.5

Estimated $q$

-0.5 -0.3 -0.1 0.1 0.3 0.5

PMR3

-0.5 -0.3 -0.1 0.1 0.3 0.5

Generated $q$

-0.7 -0.5 -0.3 -0.1 0.1 0.3 0.5

Estimated $q$

-0.5 -0.3 -0.1 0.1 0.3 0.5

Figure 2: Comparison between the real (generated) and the predicted (estimated) reputation score represented by a dotted line, as computed by PMR1, PMR2, PMR3, and a simple average on the rating provided by the consumers.

<table>
<thead>
<tr>
<th>Average absolute error</th>
<th>PMR1</th>
<th>PMR2</th>
<th>PMR3</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear correlation</td>
<td>0.223</td>
<td>0.134</td>
<td>0.136</td>
<td>0.264</td>
</tr>
</tbody>
</table>

Table 2: Comparison of PMR1, PMR2, PMR3, and SA in terms of the average absolute error and the linear correlation.

3.3 Second experiment: Varying the stability $\sigma_y^l$ of the consumers

3.3.1 Description

The second experiment analyzes the behavior of SA, PMR1, PMR2, and PMR3 when increasing the noise of the ratings, i.e., $\sigma_y^l \in [0.25, 0.5]$. Thus, this experiment decreases the stability $\sigma_y^l$ of the consumers in providing constant rates for a fixed observed quality therefore also increasing the number of truncated values. The other parameters are initialized as defined in the first experiment.

3.3.2 Results

As in the first experiment, the linear correlation and the average absolute error between the real and the predicted scores have been averaged on 10 runs. These values are reported in Table 2. The best model is PMR2 followed by PMR3, PMR1, and finally by SA. A t-test confirms, for the average absolute error, the significant ($p < 10^{-2}$) differences between PMR2 and the other models (i.e., PMR2 outperforms the other models).
Since there is no substantial difference between PMR2 and PMR3, and since PMR3 is much simpler and less time-consuming than PMR2, we decided to only keep PMR3 for further investigations.

3.4 Third experiment: Comparisons with the Brockhoff and the Iterative Refinement models

3.4.1 Description

This experiment compares the behavior of SA, PMR3, PMR4, PMR5, and sPMR5 to two other probabilistic reputation models, the Brockhoff et al.’s model (BKF, as described in Section 1.2) [6] and the Iterative Refinement model (IR) [27].

For this experiment, 50 runs have been performed and, for each run, the settings, i.e., the number of providers ($n_x$), consumers ($n_y$) and transactions ($n_t$), are modified in the following way: Firstly, (i) we generate an original data set containing 20 providers, 20 consumers, and 20 transactions for each pair. Secondly, (ii) we extract 12 different data sets by sampling this original data set, selecting some of the consumers, providers, and transactions: 20 providers-20 consumers, 10 providers-10 consumers, 5 providers-5 consumers, and for each couple of provider-consumer a number of transactions equal to 20, 10, or 5. The results for each of these 12 settings are then averaged on 50 runs.

In order to compare the robustness of the models and to analyze their behavior in extreme environments, the values are uniformly generated with respect to the following

Figure 3: Comparison between the given and the predicted consumer reactivity $a_t$ represented by a dotted line, and computed by PMR1, PMR2, and PMR3.
Figure 4: Comparison between the given and the predicted consumer bias $b_l$ represented by a dotted line, and computed by PMR1, PMR2, and PMR3.

conditions:

$$\begin{cases} a_l \in [-1, 2] \\ b_l, q_k \in [-1, 1] \\ \sigma^q_y \in [0.1, 1] \\ \sigma^x_k \in [0.05, 0.5] \end{cases} \quad (67)$$

**IR model.** The IR model considers $n_y$ consumers rating $n_x$ providers. As for our models, each provider $k$ has a latent quality $q_k$ and each consumer $l$ has a latent judging power $1/\hat{V}_l$. For each transaction $i$, the consumer $l$ provides a rating $y_{kli}$. The latent quality $q_k$ of provider $k$ is estimated by a weighted average of the received ratings

$$\tilde{q}_k \leftarrow \frac{1}{n_k} \sum_{l=0}^{n_k} \hat{w}_l \sum_{i \in (k,l)} z_{kli} \quad (68)$$

where the $z_{kli}$ are the observed ratings.

The so-called inverse judging power $\hat{V}_l$ of consumer $l$ is estimated by

$$\hat{V}_l \leftarrow \frac{\sum_{k=0}^{n_k} \sum_{i \in (k,l)} (z_{kli} - \tilde{q}_k)^2}{n_{y_l}} \quad (69)$$

The unnormalized weights $\hat{w}_l$ take the general form

$$\hat{w}_l \leftarrow \frac{1}{\hat{V}_l^{1/\beta}} \quad (70)$$
with $\beta = 1$ corresponding to optimal weights, as explained in [27].

The IR algorithm solves Equations (68-70) through an iterative procedure:

1. Initialize $\hat{w}_l = 1/n$ for each consumer $l$;
2. Estimate $\hat{q}_k$ by Equation (68);
3. Estimate $\hat{V}_l$ by Equation (69);
4. Plug the estimated values in Equation (70) to compute the weights;
5. Repeat steps 2 to 4 until convergence.

### 3.4.2 Results

Table 3 and Figures 5-7 display the average absolute errors between the actual and estimated values of the reputation score $q_k$, the reactivity $a_l$, and the bias $b_l$ (remember that IR and SA models do not use any parameter representing the reactivity and the bias, and are therefore not included in Figures 6 and 7). We only show the results for 20 transactions, the other settings leading to the same behaviors. On the bar plots, the black bar represents the 5 providers-5 consumers setting, the dark gray the 10 providers-10 consumers setting, and the light gray the 20 providers-20 consumers setting.

<table>
<thead>
<tr>
<th>Absolute errors of $q_k$</th>
<th>5 - 5</th>
<th>10 - 10</th>
<th>20 - 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMR3</td>
<td>0.50</td>
<td>0.28</td>
<td>0.15</td>
</tr>
<tr>
<td>PMR4</td>
<td>0.49</td>
<td>0.27</td>
<td>0.13</td>
</tr>
<tr>
<td>PMR5</td>
<td>0.50</td>
<td>0.28</td>
<td>0.15</td>
</tr>
<tr>
<td>sPMR5</td>
<td>0.49</td>
<td>0.27</td>
<td>0.13</td>
</tr>
<tr>
<td>IR</td>
<td>0.52</td>
<td>0.31</td>
<td>0.19</td>
</tr>
<tr>
<td>SA</td>
<td>0.57</td>
<td>0.36</td>
<td>0.24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Absolute errors of $a_l$</th>
<th>5 - 5</th>
<th>10 - 10</th>
<th>20 - 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMR3</td>
<td>0.36</td>
<td>0.29</td>
<td>0.26</td>
</tr>
<tr>
<td>PMR4</td>
<td>0.32</td>
<td>0.24</td>
<td>0.21</td>
</tr>
<tr>
<td>PMR5</td>
<td>0.34</td>
<td>0.29</td>
<td>0.26</td>
</tr>
<tr>
<td>sPMR5</td>
<td>0.33</td>
<td>0.25</td>
<td>0.22</td>
</tr>
<tr>
<td>BKF</td>
<td>0.49</td>
<td>0.44</td>
<td>0.43</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Absolute errors of $b_l$</th>
<th>5 - 5</th>
<th>10 - 10</th>
<th>20 - 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMR3</td>
<td>0.20</td>
<td>0.11</td>
<td>0.06</td>
</tr>
<tr>
<td>PMR4</td>
<td>0.21</td>
<td>0.12</td>
<td>0.05</td>
</tr>
<tr>
<td>PMR5</td>
<td>0.20</td>
<td>0.11</td>
<td>0.06</td>
</tr>
<tr>
<td>sPMR5</td>
<td>0.20</td>
<td>0.12</td>
<td>0.05</td>
</tr>
<tr>
<td>BKF</td>
<td>0.21</td>
<td>0.18</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Table 3: Absolute errors of $q_k$, $a_l$, and $b_l$ by PMR3, PMR4, PMR5, sPMR5, BKF, IR, and SA when we increase the number of providers and consumers from 5-5, 10-10 until 20-20, the transactions number remaining at 20.

Figure 5 shows the absolute error for $q_k$. We clearly observe an influence of the provider and the consumer numbers on the absolute-error value: the more providers / consumers, the best estimations. Notice that this observation is also valid on the $a_l$ and $b_l$ results (see Figures 6 and 7).

Moreover, the smallest absolute-error values (i.e., the best estimations) for $q_k$ are provided, for the various settings, by PMR4 (0.49, 0.27, and 0.13) and sPMR5 (0.49, 0.27, and 0.13). The worse estimations of the score, are given by SA (0.57, 0.36, and 0.23), IR (0.52, 0.31, and 0.19), and BKF (0.51, 0.30, and 0.18), other PMR models providing in-between results.

The results for the absolute error for $a_l$ are shown in Figure 6. The smallest absolute-error values are, again, provided by PMR4 (0.32, 0.24, and 0.21) and sPMR5

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Figure 5: Absolute error of $q_k$ by PMR3, PMR4, PMR5, sPMR5, BKF, IR, and SA when the number of providers-consumers increase respectively from 5-5 (the left bar), 10-10 (the middle bar) until 20-20 (the right bar), with a fixed number of transactions (20) per consumer/provider pair.

(0.33, 0.25, and 0.22) for all the settings. The biggest errors are provided by BKF (0.49, 0.44, and 0.43).

When analyzing the absolute errors for $b_l$ provided by each model (see Figure 7), we observe that BKF (0.21, 0.18, and 0.17) provides the worst results while PMR models provide very close results (0.20 − 0.21, 0.11 − 0.12, and 0.05 − 0.06).

3.5 Fourth experiment: Application to a real data set

3.5.1 Description

This experiment compares the models PMR4, sPMR5, and BKF on a real data set containing the ratings of 12 professors on 99 students. Indeed, as an application of the proposed techniques, we decided to analyze the behavior of a set of professors (including one of the authors) teaching at the university of Louvain. For this purpose, the grades of 99 students (second-year students at the University of Louvain) were collected for their 12 courses.

More precisely, this data set includes, for each student, his grade for each of the 12 courses (with a total of $99 \times 12$ grades). In this framework, the students represent the suppliers of a service during course examination. On the other hand, the professor can be viewed as a consumer rating the student during the examination. The aim of this experiment is to compute the expectations of each professor, as defined in Equation (3), and compare his expectation to the average grade and the average number of failure of each course. Remember that the expectations $\hat{E}R_l$ of consumer (or professor) $l$ are given by (Section 2.1):

$$\hat{E}R_l \leftarrow -\frac{b_l}{a_l}, \quad (71)$$

We also decided to estimate the reliability of the professors. A professor is considered as highly reliable if (i) he is fair ($\hat{a}_l$ is close to 1), (ii) he is unbiased ($\hat{b}_l$ is close to 0), and (iii) he is consistent (his standard deviation $\hat{\sigma}^2_l$ is close to 0). Therefore, as
already stated in Section 2.1, the reliability $r_l$ of a professor $l$ can be evaluated by:

$$r_l = \left[ (\hat{\alpha}_l - 1)^2 + (\hat{\mu}^2 + (\hat{\sigma}_l^y)^2) \right]^{1/2}, \quad (72)$$

Both the expectation and the reliability will be reported.

### 3.5.2 Results

The results show that the expectations of each professor, $\hat{ER}_l$ are indeed highly correlated with the course average: we obtain a linear correlation of $-0.930$, $-0.931$, and $-0.930$ for PMR4, sPMR5, and BKF respectively. The linear correlation coefficients between $\hat{ER}_l$ and the number of failures are $0.851$, $0.778$, and $0.776$ for PMR4, sPMR5, and BKF respectively. Moreover, the results are also consistent with the “common knowledge” about the “difficulty” of each course. For instance, the fifth course (c5) is known to be a difficult one.

Figure 8 shows the reliability and the standardized expectation of each professor for the taught course. A professor with a high expectation means that he expects a lot from his students as a standard, i.e., a good exam. On the contrary, a professor with a low expectation means that his standards are lower.

What concerns the reliability, a professor with a reliability score close to 0 is highly reliable, meaning that his rating for a student examination is close to the intrinsic value of the student. It can be observed that the minimal reliability is quite high; this is due to the high variance of the students’ capabilities who show very different behaviors in function of the courses.

This analysis confirms that interesting information about the raters can be extracted from the ratings.

### 3.6 Discussion of the results

Let us now come back to our research questions. The experiments clearly show that (1) the PMR models provide a good approximation of the parameters of interest, even in
adverse conditions; (2-3) the PMR models outperform IR, BKF, and SA; (4) the PMR models providing the best results overall are PMR4 and sPMR5, whatever the number of providers and consumers (i.e., augmenting the number of providers and consumers clearly decrease the absolute errors but does not change the ranking of the models). Moreover, we observe that PMR3, which is an easy-to-implement simplified version of PMR2, provides results very close to the ones of PMR2, therefore showing that the simplified version provides a good approximation of the original model. Thus, assuming that the provider \( k \) always supplies the same quality \( q_k \) for the items (remember Figure 1) leads to the two best models overall: PMR4 and sPMR5.

4 Conclusions

This paper proposed a general procedure allowing to estimate reputation scores characterizing a provider’s quality of service, based on transactions between consumers (raters) and providers. As a by-product, some essential features of the raters, such as their reactivity and their bias, are also estimated.

The procedure is based on a probabilistic model of consumer-provider interactions whose parameters are estimated by a variant of the expectation-maximization algorithm. Computer simulations show that the model is able to accurately retrieve the correct parameters, much more accurately than simply taking, as a measure of reputation, the average of the ratings for each provider.

Further work will investigate the adaptation and the application of the proposed models to the computation of importance scores in social networks or web pages. Indeed, the PMR models could be adapted in order to provide importance scores instead of reputation scores. One key difference between these two features is the crucial impact of the number of ratings on the importance score. A node is considered as important if both the received ratings and the number of ratings are high. We will therefore have to integrate the influence of the number of ratings in our PMR models.
Figure 8: Reliability and expectation for the courses (c1 to c12) taught by a professor.

References


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**Appendix: Proof of the Main Results**
A Updating rules for the basic model PMR1

In this appendix, the EM algorithms allowing to estimate the various parameters of the basic model are detailed.

A.1 A study of the likelihood function

Let us remember the complete log-likelihood of the data,

\[
 l = \sum_{k=1}^{n_x} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \left\{ \log(P(x_{ki})) + \log(P(y_{kl} | x_{ki})) \right\} 
\]  

(73)

\[
 = - \sum_{k=1}^{n_x} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \left\{ \frac{1}{2} \left[ \frac{x_{ki} - q_k}{\sigma_k^x} \right]^2 + \frac{1}{2} \left[ \frac{y_{kl} - (a_l x_{ki} + b_l)}{\sigma_l^y} \right]^2 + \log(\sigma_k^x \sigma_l^y) + \log(2\pi) \right\} 
\]  

(74)

It can easily be shown that the joint probability of \([x_k, y_{kl}]\) is normal, 

\[
[x_k, y_{kl}] \sim N(m_{kl}, \Sigma_{kl}), 
\]

with mean vector and variance-covariance matrix

\[
m_{kl} = \begin{bmatrix} q_k \\ a_l q_k + b_l \end{bmatrix} \quad \text{and} \quad \Sigma_{kl} = \begin{bmatrix} (\sigma_k^x)^2 & a_l (\sigma_k^x)^2 \\ a_l (\sigma_k^x)^2 & a_l^2 (\sigma_k^x)^2 + (\sigma_l^y)^2 \end{bmatrix} 
\]  

(75)

The inverse of this variance-covariance matrix is

\[
\Sigma_{kl}^{-1} = \begin{bmatrix} \frac{a_l^2 (\sigma_k^x)^2 + (\sigma_l^y)^2}{\sigma_l^y} & -a_l \\ -a_l & \frac{(\sigma_l^y)^2}{\sigma_l^y} \end{bmatrix} 
\]  

(76)

Notice that the marginal distributions are

\[
P(x_k) = \frac{1}{\sqrt{2\pi \sigma_k^x}} \exp \left\{ \frac{(x_k - q_k)^2}{2(\sigma_k^x)^2} \right\} 
\]  

(77)

\[
P(y_{kl}) = \frac{1}{\sqrt{2\pi (a_l^2 (\sigma_k^x)^2 + (\sigma_l^y)^2)}} \exp \left\{ \frac{(y_{kl} - (a_l q_k + b_l))^2}{2(a_l^2 (\sigma_k^x)^2 + (\sigma_l^y)^2)} \right\} 
\]  

(78)

The complete log-likelihood of the data (Equation (74)) can be rewritten as

\[
l = - \sum_{k=1}^{n_x} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \left\{ \frac{1}{2} \left[ \begin{bmatrix} x_{ki} - q_k \\ y_{kl} - (a_l q_k + b_l) \end{bmatrix} \right]^T \Sigma_{kl}^{-1} \left[ \begin{bmatrix} x_{ki} - q_k \\ y_{kl} - (a_l q_k + b_l) \end{bmatrix} \right] + \log(\sigma_k^x \sigma_l^y) + \log(2\pi) \right\} 
\]  

(79)

The first step (the expectation step) in the EM algorithm aims to compute the expectation of the unobserved variables \(x_k\) given the observed variables, \(y_{kl}\), and the current estimate of the parameters, \(\Theta\). For the sake of simplicity, the sets of variables \(\{x_k\}\), \(\{y_{kl}\}\) are denoted by \(x\) and \(y\). Moreover, the set of parameters \(\{q_k, \sigma_k^x, a_l, b_l, \sigma_l^y\}\) is denoted by \(\Theta\).
Let us first evaluate \( E_{x|y} [x_k | y, \Theta] \) and \( V_{x|y} [x_k | y, \Theta] \) which will be used in order to compute the EM updates. In order to compute the conditional expectation given \( y \), we need some standard results from applied statistics [34]. Consider a random vector \( w \) which is partitioned into two subvectors such that \( w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T \) and the mean, together with the variance-covariance matrix, are partitioned correspondingly as

\[
\begin{align*}
\mathbf{m} &= \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix} \\
\mathbf{\Sigma} &= \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}
\end{align*}
\] (80)

It is well-known [34] that if \( w \) is normal, i.e., \( w \sim \mathcal{N}(\mathbf{m}, \mathbf{\Sigma}) \), then the conditional distribution of \( w_1 \) given \( w_2 \) is also normal \( w_{1|2} \sim \mathcal{N}(\mathbf{m}_{1|2}, \mathbf{\Sigma}_{1|2}) \) with mean vector and variance-covariance matrix

\[
\begin{align*}
\mathbf{m}_{1|2} &= \mathbf{m}_1 + \mathbf{\Sigma}_{12} \mathbf{\Sigma}^{-1}_{22} (w_2 - \mathbf{m}_2) \\
\mathbf{\Sigma}_{1|2} &= \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12} \mathbf{\Sigma}^{-1}_{22} \mathbf{\Sigma}_{21}
\end{align*}
\] (81)

This result is used in the next section when computing the expectation step.

A.2 The expectation step

If we apply this result to our problem and use the current estimate of the parameter vector \( \Theta \) in order to perform the expectation, we obtain, from Equations (75) and (81)

\[
\begin{align*}
\hat{x}_{kl} &= E_{x|y} [x_k | y_{kl}, \hat{\Theta}] = \hat{q}_k + \frac{a_l}{\hat{\sigma}_k^2} \left( \hat{\sigma}_k^2 \right)^2 + \frac{a_l}{\hat{\sigma}_l^2} \left( \hat{\sigma}_l^2 \right)^2 \\
\hat{\sigma}_{kl}^2 &= V_{x|y} [x_k | y_{kl}, \hat{\Theta}] = \frac{\left( \hat{\sigma}_k^2 \right)^2 + \left( \hat{\sigma}_l^2 \right)^2}{\left( \hat{\sigma}_k^2 \right)^2 + \left( \hat{\sigma}_l^2 \right)^2}
\end{align*}
\] (82) (83)

where we defined the estimates of the conditional expectation of \( x_k \) and of the conditional variance as \( \hat{x}_{kl} \) and \( \left( \hat{\sigma}_{kl}^2 \right)^2 \) respectively. These equations provide the expectation and the variance of the unobserved variables given the observed variables.

By dropping the constant term (i.e., \( \log(2\pi) \)) in Equation (79), taking the expectation of the log-likelihood \( l \), using Equations (82) and (83), and going through a little calculus\(^2\), we obtain

\[
E_{x|y} [l | y, \hat{\Theta}] = -\frac{1}{2} \sum_{k=1}^{n_x} \sum_{l=1}^{n_y} \sum_{i \in \{k,l\}} \left\{ \frac{y_{kli} - (a_i \hat{x}_{kli} + b_i)}{\hat{\sigma}_l^2} \right\}^2 + \left[ \frac{\hat{x}_{kli} - q_k}{\hat{\sigma}_k^2} \right]^2
\]

\[
+ \frac{\left( \hat{\sigma}_{kl}^2 \right)^2}{\left( \hat{\sigma}_k^2 \right)^2 + \left( \hat{\sigma}_l^2 \right)^2} \left( \frac{\left( \hat{\sigma}_k^2 \right)^2 + \left( \hat{\sigma}_l^2 \right)^2}{\left( \hat{\sigma}_k^2 \right)^2 + \left( \hat{\sigma}_l^2 \right)^2} \right) + \log(\hat{\sigma}_l^2 \hat{\sigma}_l^2)^2
\]

(84)

This expected likelihood has to be maximized with respect to the parameters.

\(^2\)The computations have been performed with the help of the Mathematica system from Wolfram Research.
A.3 The maximization step

The maximization step consists in maximizing the expectation of the complete likelihood given the observed ratings, \( E_{q_k | y, \Theta} \), in terms of the parameters of the model. For this purpose, we take the derivative of \( E_{q_k | y, \Theta} \) (Equation (84)) and solve the resulting system of equations with respect to the parameters. Furthermore, the \( q_k \) are constrained to sum to zero (normalization, \( \sum_k q_k = 0 \)). Thus, taking the derivative with respect to \( q_k, \sigma_k^2 \), and isolating the parameter yields

\[
q_k = \frac{1}{n_{k*}} \sum_{i=1}^{n_y} \sum_{l \in (k,l)} \tilde{x}_{kli}, \quad \text{and normalize the } q_k \tag{85}
\]

\[
\sigma_k^2 = \frac{1}{n_{k*}} \sum_{i=1}^{n_y} \sum_{l \in (k,l)} \left[ (q_k - \tilde{x}_{kli})^2 + (\tilde{\sigma}_{k|i}^y)^2 \right] \tag{86}
\]

For \( a_l, b_l \) and \( \sigma_l^y \), we obtain

\[
a_l = \frac{\sum_{k=1}^{n_y} \sum_{i \in (k,l)} \tilde{x}_{kli} (y_{kli} - b_l)}{\sum_{k=1}^{n_y} \sum_{i \in (k,l)} (\tilde{x}_{kli}^2 + (\tilde{\sigma}_{k|i}^y)^2)} \tag{87}
\]

\[
b_l = \frac{1}{n_{k*}} \sum_{k=1}^{n_y} \sum_{i \in (k,l)} (y_{kli} - a_l \tilde{x}_{kli}) \tag{88}
\]

\[
(\sigma_l^y)^2 = \frac{1}{n_{k*}} \sum_{k=1}^{n_y} \sum_{i \in (k,l)} \left[ (y_{kli} - (a_l \tilde{x}_{kli} + b_l))^2 + a_l^2 (\tilde{\sigma}_{k|i}^y)^2 \right] \tag{89}
\]

Normally, this set of equations should be solved with respect to the parameters but, in our case, this is not an easy task because of a strong coupling between the different equations. Therefore, instead of using a standard EM algorithm, we rely on the so-called “one-step-later” algorithm introduced by Green [13, 14]; see also [34]. Instead of solving the system of equations, Green proposed to replace the parameters on the right-side of the equation by their current value, that is, by their current estimate rather than the new estimate. This procedure, quite similar to coordinate descent [31], provides the following set of updating rules

\[
\widehat{q}_k \leftarrow \frac{1}{n_{k*}} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \tilde{x}_{kli}, \quad \text{and normalize the } \widehat{q}_k \tag{90}
\]

\[
(\tilde{\sigma}_k^2)^2 \leftarrow \frac{1}{n_{k*}} \sum_{l=1}^{n_y} \sum_{i \in (k,l)} \left[ (\widehat{q}_k - \tilde{x}_{kli})^2 + (\tilde{\sigma}_{k|i}^y)^2 \right] \tag{91}
\]

\[
\widehat{a}_l \leftarrow \frac{\sum_{k=1}^{n_y} \sum_{i \in (k,l)} \tilde{x}_{kli} (y_{kli} - \widehat{b}_l)}{\sum_{k=1}^{n_y} \sum_{i \in (k,l)} (\tilde{x}_{kli}^2 + (\tilde{\sigma}_{k|i}^y)^2)} \tag{92}
\]

\[
\widehat{b}_l \leftarrow \frac{1}{n_{k*}} \sum_{k=1}^{n_y} \sum_{i \in (k,l)} (y_{kli} - \widehat{a}_l \tilde{x}_{kli}) \tag{93}
\]

\[
(\tilde{\sigma}_l^y)^2 \leftarrow \frac{1}{n_{k*}} \sum_{k=1}^{n_y} \sum_{i \in (k,l)} \left[ (y_{kli} - (\widehat{a}_l \tilde{x}_{kli} + \widehat{b}_l))^2 + \widehat{a}_l^2 (\tilde{\sigma}_{k|i}^y)^2 \right] \tag{94}
\]
This algorithm is much easier to compute. What is lost, however, in comparison with the true EM algorithm, is the guarantee that the method converges, and in particular that the iteration always increase the likelihood. Green [13, 14] did not observe any convergence problem nor did we in our experiments.

B Updating rules for the model PMR2 involving truncation

B.1 The expectation step

We consider now a model involving a truncation of the ratings, which therefore lie in the interval \([-c, +c]\). The first step in the EM algorithm involves the expectation of the likelihood, which is a function of the unobserved variables, \(x_k, y_{kl}, z_{kl}\) given the observed variables, \(z_{kl}\), and the current estimate of the parameters, \(\hat{\Theta}\). As before, the sets of variables \(\{x_k\}, \{y_{kl}\}, \{z_{kl}\}\) are denoted by \(x, y, z\); moreover, the set of parameters \(\{q_k, \sigma_{x_k}^2, a_l, b_l, \sigma_y^2\}\) is denoted by \(\Theta\).

First, observe that

\[
E_{xy|z} \left[ l | z, \hat{\Theta} \right] = E_{y|z} \left[ E_{x|yz} \left[ l | y, z, \hat{\Theta} \right] | z, \hat{\Theta} \right]
\]

and

\[
E_{y|z} \left[ l | y, \hat{\Theta} \right] = E_{x|y} \left[ l | y, \hat{\Theta} \right] | z, \hat{\Theta}
\]

(95)

(96)

because \(l\) is independent of \(z\) given \(y\). The computation of \(\tilde{L}_y = E_{x|y} [l|y, \hat{\Theta}]\) is exactly the same as for the basic model PRM1 and is not repeated here (see Equation (84)).

The second expectation, \(E_{y|z}[\tilde{L}_y | z, \hat{\Theta}]\), involves the conditional expectation of a set of independent normal-distributed variables \(y\) given the observed truncated variables, \(z\). Truncated variables are common in econometrics and have therefore been widely investigated. For instance, a regression model for which the dependent variable is truncated is called a tobit model (see, e.g., [15]). By examining the form of the likelihood \(\tilde{L}_y\) in Equation (84), we immediately see that we have to compute the expectations of the following functions, depending on the variable \(y_{kl}\): \(\left[(y_{kl} - (a_l \hat{x}_{kli} + b_l))/\sigma_y^2\right]^2\) and \(\left[(\hat{x}_{kli} - q_k)/\sigma_{x_k}^2\right]^2\). Notice that the \(\hat{x}_{kli}\) depend on the \(y_{kl}\) (see Equation (82)).

In order to compute these expectations, we make use of a well-known decomposition related to the bias-variance decomposition; see [3] for a similar computation for estimating the parameters of a tobit model with the EM algorithm:

\[
E \left[(\alpha y + \beta)^2\right] = (\alpha E[y] + \beta)^2 + \alpha^2 E \left[(y - E[y])^2\right]
\]

(97)

\[
= (\alpha E[y] + \beta)^2 + \alpha^2 \sigma_y^2
\]

(98)

where \(\sigma_y^2\) is the variance of the random variable \(y\). The first term consists in replacing the random variable \(y\) by its expectation \(E[y]\). This amounts, when computing the expectation of the likelihood in Equation (84), to simply replace the unobserved values of \(y_{kl}\) by their conditional expectations. The second term involves the variance of the random variable, \(\sigma_y^2\).

Thus, we have to evaluate both \(E_{y|z}[y_{kl} | z, \hat{\Theta}]\) and \(\sigma_y^2 E_{y|z} \left[ y_{kl} | z, \hat{\Theta} \right]\); that is, the conditional expectation and variance given the truncated values \(z\). Since \(y_{kl}\) only depends on \(y_{kli}\), the expectations reduce to \(E[y_{kli}|z_{kli}, \hat{\Theta}]\) and \(\sigma_y^2 E_{y_k|z_{kli}} \left[ y_{kli} | z_{kli}, \hat{\Theta} \right]\) Now,
it is well-known [15, 16] that if a normal random variable \( u \) is \( u \sim N(\mu, \sigma) \), the expectation and the variance can easily be computed for the two truncation cases. If the truncation is from the left, we have

\[
\begin{align*}
E[u | u \leq -c] &= \mu + \sigma \lambda(\gamma) \\
V[u | u \leq -c] &= \sigma^2 \left[ 1 + \gamma \lambda(\gamma) - \lambda^2(\gamma) \right]
\end{align*}
\]

with

\[
\begin{align*}
\gamma &= \frac{-c - \mu}{\sigma} \\
\lambda(\gamma) &= \frac{\varphi(\gamma)}{\phi(\gamma)}
\end{align*}
\]

(99)

where \( \varphi \) and \( \phi \) are the normal distribution and the normal cumulative function, as defined in Equation (23).

Now, if the truncation is from the right,

\[
\begin{align*}
E[u | u \geq c] &= \mu + \sigma \lambda(\gamma) \\
V[u | u \geq c] &= \sigma^2 \left[ 1 + \gamma \lambda(\gamma) - \lambda^2(\gamma) \right]
\end{align*}
\]

with

\[
\begin{align*}
\gamma &= \frac{c - \mu}{\sigma} \\
\lambda(\gamma) &= \frac{\varphi(\gamma)}{1 - \phi(\gamma)}
\end{align*}
\]

(100)

In our case, from Equation (78), \( y_{kli} \sim N(a_l q_k + b_l, s_{kl}) \) with \( s_{kl} = (a_l^2 (\sigma^2_k)^2 + (\sigma^2)^2)^{1/2} \). Thus, for computing the \( E[y] \) term in Equation (98), we define a new variable, \( \hat{y}_{kli} = E[y_{kli}|y_{kli}, \hat{\Theta}] \) whose value depends on the three cases: \( z_{kli} = -c \) (in other words, \( y_{kli} \leq -c \)), \( -c < z_{kli} < +c \) (in other words, \( y_{kli} = z_{kli} \)) and \( z_{kli} = +c \) (in other words, \( y_{kli} \geq c \)). Equations (99)-(100) yield:

**First case:** \( z_{kli} = -c \):

\[
\hat{y}_{kli} = (\hat{\alpha}_l \hat{q}_k + \hat{b}_l) + \hat{s}_{kl}\lambda(\hat{\gamma}_{kl})
\]

with

\[
\begin{align*}
\hat{\gamma}_{kl} &= \frac{-c - (\hat{\alpha}_l \hat{q}_k + \hat{b}_l)}{\hat{s}_{kl}} \\
\lambda(\hat{\gamma}_{kl}) &= \frac{\varphi(\hat{\gamma}_{kl})}{\phi(\hat{\gamma}_{kl})}
\end{align*}
\]

(101)

**Second case:** \( -c < z_{kli} < +c \):

\[
\hat{y}_{kli} = z_{kli}
\]

(102)

**Third case:** \( z_{kli} = +c \):

\[
\hat{y}_{kli} = (\hat{\alpha}_l \hat{q}_k + \hat{b}_l) + \hat{s}_{kl}\lambda(\hat{\gamma}_{kl})
\]

with

\[
\begin{align*}
\hat{\gamma}_{kl} &= \frac{c - (\hat{\alpha}_l \hat{q}_k + \hat{b}_l)}{\hat{s}_{kl}} \\
\lambda(\hat{\gamma}_{kl}) &= \frac{\varphi(\hat{\gamma}_{kl})}{1 - \phi(\hat{\gamma}_{kl})}
\end{align*}
\]

(103)

In the same way, we easily obtain for the variance (i.e., the \( V[y] \) term in Equation (98)): 33
First case: $z_{kli} = -c$

$$\hat{V}_{kli} = \mathbb{V}[y_{kli}|z_{kli} = -c, \hat{\Theta}] = s_{kli}^2 \left[ 1 + \hat{\gamma}_{kli} \lambda(\hat{\gamma}_{kli}) - \lambda^2(\hat{\gamma}_{kli}) \right]$$

with

$$\begin{cases} 
\hat{\gamma}_{kli} = -c - (\hat{a}_l \hat{q}_k + \hat{b}_l) \\
\lambda(\hat{\gamma}_{kli}) = -\frac{\varphi(\hat{\gamma}_{kli})}{\phi(\hat{\gamma}_{kli})} 
\end{cases}$$

...(104)

Second case: $-c < z_{kli} < +c; \mathbb{V}[y_{kli}| -c < z_{kli} < +c, \hat{\Theta}] = 0$

Third case: $z_{kli} = +c$

$$\hat{V}_{kli} = \mathbb{V}[y_{kli}|z_{kli} = +c, \hat{\Theta}] = s_{kli}^2 \left[ 1 + \hat{\gamma}_{kli} \lambda(\hat{\gamma}_{kli}) - \lambda^2(\hat{\gamma}_{kli}) \right]$$

with

$$\begin{cases} 
\hat{\gamma}_{kli} = \frac{c - (\hat{a}_l \hat{q}_k + \hat{b}_l)}{s_{kli}} \\
\lambda(\hat{\gamma}_{kli}) = \frac{\varphi(\hat{\gamma}_{kli})}{1 - \phi(\hat{\gamma}_{kli})} 
\end{cases}$$

...(105)

Now that the $\mathbb{E}[y]$ and $\mathbb{V}[y]$ terms in Equation (98) are found, the last term that remains to be computed is $\alpha$. From Equation (84), replacing the $\tilde{x}_{kli}$ by their value (Equation (82)) in the $\left[(y_{kli} - (a_l \tilde{x}_{kli} + b_l))/\sigma_k^2 \right]^2$ term and identifying $\alpha$ yields

$$\alpha = \frac{s_{kli}^2 - \hat{a}_l (\hat{\sigma}_k^2)^2 a_l}{s_{kli}^2 \sigma_k^2}$$

...(106)

while the $\left[(\tilde{x}_{kli} - q_k)/\sigma_k^2 \right]^2$ term provides

$$\alpha = \frac{\hat{a}_l (\hat{\sigma}_k^2)^2 a_l}{s_{kli}^2 \sigma_k^2}$$

...(107)

Therefore, by replacing the values of $y_{kli}$ by their expected values $\mathbb{E}[y] = \tilde{y}_{kli}$ and by adding the extra variance term, $\alpha^2 \mathbb{V}[y]$ in the likelihood function (84), the expected likelihood can be rewritten as

$$\mathbb{E}_{y|x \mid \Theta} \left[ \begin{array}{c} \mathbb{E} \\ \mathbb{V} \\
\end{array} \right] = \mathbb{E}_{y|x \mid \Theta} \left[ \begin{array}{c} \mathbb{E} \\ \mathbb{V} \\
\end{array} \right] = \frac{-1}{2} \sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \sum_{i(k,l)} \left( \frac{y_{kli} - (\hat{a}_l \tilde{x}_{kli} + b_l)}{\sigma_k^2} \right)^2 + \left( \tilde{x}_{kli} - q_k \right)^2$$

$$+ \left( \frac{(\hat{\sigma}_k^2)^2}{\left(\frac{\hat{\sigma}_k^2}{\sigma_k^2}\right)^2 + \left(\frac{\sigma_k^2}{\hat{\sigma}_k^2}\right)^2} \right) \log(\sigma_k^2 \sigma_k^2)$$

$$+ \delta(z_{kli} = -c) \tilde{V}_{kli}^+ \left( \frac{\hat{a}_l (\hat{\sigma}_k^2)}{\sigma_k^2} \right)^2 + \left( \frac{\hat{\sigma}_k^2 - \hat{a}_l (\hat{\sigma}_k^2)}{\sigma_k^2} \right)^2$$

$$+ \delta(z_{kli} = +c) \tilde{V}_{kli}^+ \left( \frac{\hat{a}_l (\hat{\sigma}_k^2)}{\sigma_k^2} \right)^2 + \left( \frac{\hat{\sigma}_k^2 - \hat{a}_l (\hat{\sigma}_k^2)}{\sigma_k^2} \right)^2$$

...(108)

where $\tilde{x}_{kli}$ and $\hat{\sigma}_k^2$ were defined in Equations (82)-(83):

$$\tilde{x}_{kli} = \hat{q}_k + \hat{a}_l (\hat{\sigma}_k^2) \left[ y_{kli} - (\hat{a}_l \hat{q}_k + \hat{b}_l) \right]$$

...(109)

$$\hat{\sigma}_k^2 = \frac{\hat{a}_l^2 (\hat{\sigma}_k^2)^2 + \sigma_k^2}{\hat{a}_l^2 (\hat{\sigma}_k^2)^2 + \sigma_k^2}$$

...(110)

$$\left(\frac{\hat{\sigma}_k^2}{\sigma_k^2}\right)^2 = \frac{\hat{a}_l^2 (\hat{\sigma}_k^2)^2 + \sigma_k^2}{\hat{a}_l^2 (\hat{\sigma}_k^2)^2 + \sigma_k^2}$$

...(111)
The next step aims to maximize the expected likelihood with respect to the parameters.

### B.2 The maximization step

Taking the derivative with respect to \( q_k \), \( \sigma^x_k \), defining \( \hat{V}_{kli} = \delta(z_{kli} = -c) \hat{V}^-_{kl} + \delta(z_{kli} = +c) \hat{V}^+_{kl} \), and isolating the parameter yields

\[
q_k = \frac{1}{n_{k*}} \sum_{i=1}^{n_x} \sum_{l \in (k,l)} \hat{x}_{kli}, \quad \text{and normalize the } q_k
\]

\[
(\sigma^x_k)^2 = \frac{1}{n_{k*}} \sum_{i=1}^{n_x} \sum_{l \in (k,l)} \left[ (q_k - \hat{x}_{kli})^2 + (\hat{\sigma}^x_k)^2 \right] \hat{V}_{kli}
\]

For \( a_l \), \( b_l \) and \( \sigma^y_l \), we obtain

\[
a_l = \frac{\sum_{k=1}^{n_k} \sum_{i \in (k,l)} \hat{x}_{kli} (y_{kli} - b_l) + \hat{a}_l (\hat{\sigma}^x_k)^2 \hat{V}_{kli}}{\sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left[ \hat{x}_{kli}^2 + (\hat{\sigma}^x_k)^2 \right] \hat{V}_{kli}}
\]

\[
b_l = \frac{1}{n_{l*}} \sum_{k=1}^{n_k} \sum_{i \in (k,l)} (y_{kli} - a_l \hat{x}_{kli})
\]

\[
(\sigma^y_l)^2 = \frac{1}{n_{l*}} \sum_{k=1}^{n_k} \sum_{i \in (k,l)} \left[ (\hat{g}_{kli} - (a_l \hat{x}_{kli} + b_l))^2 + a_l^2 (\hat{\sigma}^x_k)^2 \right] + \frac{\hat{V}_{kli}}{\hat{s}_{kl}^2} \left( \hat{a}_l (\hat{\sigma}^x_k)^2 \right) \left( \hat{a}_l - \hat{s}_{kl}^2 \right)^2
\]
As before, we rely on the so-called “one-step-later” algorithm [13, 14], which yields

\[
\hat{q}_k \leftarrow \frac{1}{n_k} \sum_{l=1}^{n_k} \sum_{i \in (k,l)} \hat{x}_{kli}, \text{ and normalize the } \hat{q}_k
\]  

(117)

\[
(\hat{\sigma}_k^x)^2 \leftarrow \frac{1}{n_k} \sum_{l=1}^{n_k} \sum_{i \in (k,l)} \left[ (\hat{q}_k - \hat{x}_{kli})^2 \right.
\]

\[
\left. + (\hat{\sigma}_{kli}^x)^2 + \frac{\hat{\sigma}_k^x (\hat{\sigma}_k^x)^4 \hat{V}_{kli}}{S_{kl}} \right]
\]  

(118)

\[
\hat{a}_l \leftarrow \frac{\sum_k \sum_{i \in (k,l)} \hat{x}_{kli} \left( \hat{y}_{kli} - \hat{a}_l \hat{x}_{kli} \right) + \frac{\hat{\sigma}_k^x (\hat{\sigma}_k^x)^4 \hat{V}_{kli}}{S_{kl}}}{\sum_k \sum_{i \in (k,l)} \hat{x}_{kli}^2 + (\hat{\sigma}_{kli}^x)^2 + \frac{\hat{\sigma}_k^x (\hat{\sigma}_k^x)^4 \hat{V}_{kli}}{S_{kl}}}
\]  

(119)

\[
\hat{b}_l \leftarrow \frac{1}{n^l} \sum_k \sum_{i \in (k,l)} \left( \hat{y}_{kli} - \hat{a}_l \hat{x}_{kli} \right)
\]  

(120)

\[
(\hat{\sigma}_l^y)^2 \leftarrow \frac{1}{n^l} \sum_k \sum_{i \in (k,l)} \left[ (\hat{y}_{kli} - \hat{a}_l \hat{x}_{kli} + \hat{b}_l)^2 \right.
\]

\[
\left. + \frac{(\hat{\sigma}_l^y)^4 \hat{V}_{kli}}{S_{kl}} \right]
\]  

(121)

Of course, the remarks about the convergence of the method made for model PMR1 can be repeated.