DISTRIBUTIONS FOR WHICH $\text{div } v = F$ HAS A CONTINUOUS SOLUTION

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Abstract. The equation $\text{div } v = F$ has a continuous weak solution in an open set $U \subset \mathbb{R}^m$ if and only if the distribution $F$ satisfies the following condition: $F(\phi_i)$ converge to zero for each sequence $\{\phi_i\}$ of test functions such that the supports of $\phi_i$ are contained in a fixed compact subset of $U$, and in the $L^1$ norm, $\{\phi_i\}$ converges to zero and $\{\nabla \phi_i\}$ is bounded.

If $F$ is a distribution in $\mathbb{R}^m$, then a vector field $v \in L^1(\mathbb{R}^m; \mathbb{R}^m)$ is a solution of the equation $\text{div } v = F$ whenever

$$F(\phi) = -\int_{\mathbb{R}^m} v(x) \cdot \nabla \phi(x) \, dx$$

for each test function $\phi \in \mathcal{D}(\mathbb{R}^m)$. If such a $v$ is continuous and $\varepsilon > 0$, we can find a $w \in C^1(\mathbb{R}^m; \mathbb{R}^m)$ so that $|v(x) - w(x)| < \varepsilon$ for each $x$ in the ball $B(1/\varepsilon)$ of radius $1/\varepsilon$ about the origin. Selecting $\phi$ supported in $B(1/\varepsilon)$ and integrating by parts, we obtain

$$|F(\phi)| \leq \left| \int_{B(1/\varepsilon)} \phi \text{div } w \right| + \left| \int_{B(1/\varepsilon)} (w - v) \cdot \nabla \phi \right|$$

$$\leq |\phi|_1 \sup_{x \in B(1/\varepsilon)} |\text{div } w(x)| + \varepsilon |\nabla \phi|_1,$$

which implies a stronger continuity of $F$. In other words, the following continuity property of $F$ is necessary for the equation $\text{div } v = F$ to have a continuous solution.

Continuity. Given $\varepsilon > 0$ there is a $\theta > 0$ such that

$$|F(\phi)| \leq \theta |\phi|_1 + \varepsilon |\nabla \phi|_1 \quad (*)$$

for each $\phi \in \mathcal{D}(\mathbb{R}^m)$ with $\text{supp } \phi \subset B(1/\varepsilon)$.

Our main result is Theorem 3.7 below, which asserts that this necessary continuity property is also sufficient. For historical reasons (see below), a distribution $F$ satisfying the above continuity property is called a strong charge.

An example of a strong charge is the distribution associated with a function $f \in L^m_{loc}(\mathbb{R}^m)$ (Proposition 2.9 below). H. Brezis and J. Bourgain [1, Proposition 1] proved that a continuous solution of $\text{div } v = f$ exists for a $\mathbb{Z}^m$ periodic function $f \in L^m_{loc}(\mathbb{R}^m)$. The continuity of $v$ is the main point — establishing the existence of a solution $v \in L^\infty(\mathbb{R}^m; \mathbb{R}^m)$ is appreciably easier (Proposition 2.11 below). In
general, neither a continuous nor essentially bounded solution is obtainable by solving the Poisson equation \( \triangle u = f \) and letting \( v := \nabla u \); a pertinent example is due to L. Nirenberg [1, Remark 7]. The absence of such a solution is related to the role of \( p = m \) as the critical exponent for representing elements of \( W^{1,p} \) by continuous functions [7, Chapter 5, Theorem 5].

We outline the proof of Theorem 3.7, which is inspired by the above mentioned proof of Brezis and Bourgain. The linear spaces \( S \) of all strong charges, and \( C \) of all continuous vector fields \( v : \mathbb{R}^m \to \mathbb{R}^m \), are equipped with the Fréchet topologies of locally uniform convergence. For a \( v \in C \), we define a strong charge \( F_v \) by

\[
F_v(\varphi) := -\int_{\mathbb{R}^m} v(x) \cdot \nabla \varphi(x) \, dx
\]

for each \( \varphi \in \mathcal{D}(\mathbb{R}^m) \), and observe that the linear map \( \Gamma : v \mapsto F_v \) from \( C \) to \( S \) is continuous. Showing that

(i) \( \Gamma(C) \) is a dense subspace of \( S \) (Lemma 3.1 below),

(ii) if \( \Gamma^* : S^* \to C^* \) is the adjoint map of \( \Gamma \), then \( \Gamma^*(S^*) \) is closed in the strong topology of \( C^* \) (Proposition 3.6 below),

completes the argument: (ii) and the Closed Range Theorem imply that \( \Gamma(C) \) is closed in \( S \), and hence \( \Gamma(C) = S \) by (i).

Because the space \( S \) is topologized so that its dual \( S^* \) is isomorphic to the linear space \( BV_c \) of all compactly supported BV functions in \( \mathbb{R}^m \) (Proposition 3.2 below), the adjoint map \( \Gamma^* \) of \( \Gamma \) has an intuitive geometric meaning. Indeed, interpreting the continuous vector fields as \((m-1)\)-forms and strong charges as \( m \)-forms, we can think of \( \Gamma \) as the exterior derivative; note that by definition, \( \Gamma \) is a weak divergence operator. Thus \( \Gamma^* \) is a boundary operator which maps \( g \in BV_c \) to a compactly supported Radon measure \( Dg \) in \( \mathbb{R}^m \); see equality (3.2) below. Clearly, \( Dg \) belongs to the dual space \( C^* \) of \( C \).

In the obvious way, the balls \( B(i), \ i = 1, 2, \ldots \), determine seminorms \( s_i \) and \( c_i \) which define the topologies of locally uniform convergence in \( S \) and \( C \), respectively. Theorem 3.8 below shows that given a strong charge \( F \) and an integer \( i \geq 1 \), we can find a solution \( v \in C \) of \( \text{div} \, v = F \) so that \( c_i(v) \) is as close to \( s_i(F) \) as we wish.

If \( F \) is a strong charge, then the set \( \Gamma^{-1}(F) \) of all continuous solutions of the equation \( \text{div} \, v = F \) has many elements. In Section 4 we consider continuous vector fields and strong charges that are invariant with respect to the orthogonal group \( O(m) \), and produce constructively an isometry \( \Upsilon : S_{\text{inv}} \to C_{\text{inv}} \) that is a right inverse of \( \Gamma \) (Proposition 4.3 below). The construction depends on showing that a rotation invariant strong charge on the sphere \( S^{m-1} \) is a multiple of a strong charge induced on \( S^{m-1} \) by the Hausdorff measure \( H^{m-1} \) in \( \mathbb{R}^m \) (Proposition 4.2 below).

A strong charge is a special case of a charge, i.e., of a distribution \( F \) with the above continuity property where inequality (*) is replaced by the inequality

\[
|F(\varphi)| \leq \theta |\varphi|_1 + \varepsilon (|\nabla \varphi|_1 + |\varphi|_{\infty}).
\]

Every distribution associated with an \( f \in L^1_{\text{loc}}(\mathbb{R}^m) \) is a charge, called an absolutely continuous charge. Elaborating on the proof of Theorem 3.7, we show that each charge is the sum of a strong charge and an absolutely continuous charge (Theorem 5.2 below).
The concepts of charges and strong charges originate from our previous work on generalized Riemann integrals and the Gauss-Green theorem [12, 3, 2, 13, 5, 4]. In Section 6, we indicate how a substantial generalization of the classical Gauss-Green theorem (Theorem 6.5 below) can be obtained by means of charges and their derivatives. This version of the Gauss-Green theorem admits further generalizations that can be applied to removable sets of PDEs in divergence form [5, 4, Sections 4].

1. Preliminaries

The set of all real numbers is denoted by \( \mathbb{R} \). In the Cartesian product \( \mathbb{R}^n \) where \( n \geq 1 \) is an integer, we denote by \( x \cdot y \) the usual inner product, which induces the norm \( |x| \). The zero vector in \( \mathbb{R}^n \) is denoted by \( 0 \). All functions we consider are real valued. For a map \( f : A \to B \) and an \( x \in A \), we use the symbols \( f(x) \) and \( (f, x) \) interchangeably; both denote the value of \( f \) at \( x \).

The ambient space of this paper is \( \mathbb{R}^m \) where \( m \geq 2 \) is a fixed integer. Restricting to dimensions larger than one merely eliminates trivialities. The closure, interior, and diameter of a set \( E \subset \mathbb{R}^m \) are denoted by \( \text{cl} E \), \( \text{int} E \), and \( d(E) \), respectively. The open and closed balls of radius \( r > 0 \) centered at \( x \in \mathbb{R}^m \) are denoted by \( B(x, r) \) and \( B[x, r] \), respectively. We write \( B(r) \) instead of \( B(0, r) \), and \( B[r] \) instead of \( B(0, r] \).

In \( \mathbb{R}^m \) we use Lebesgue measure \( \mathcal{L} := \mathcal{L}^m \) and the Hausdorff measure \( \mathcal{H} := \mathcal{H}^{m-1} \). For \( E \subset \mathbb{R}^m \), we write \( |E| \) instead of \( \mathcal{L}(E) \), and define the restricted measures \( \mathcal{L} \upharpoonright E \) and \( \mathcal{H} \upharpoonright E \) in the usual way [8, Section 1.1.1]. Unless specified otherwise, the words “measure,” “measurable,” and “negligible,” as well as the expressions “almost everywhere” and “almost all” refer to Lebesgue measure \( \mathcal{L} \). Symbols \( \int f \) and \( \int f(x) \, dx \) denote the Lebesgue integral \( \int f \, d\mathcal{L} \).

Throughout, \( U \subset \mathbb{R}^m \) is a fixed nonempty open set. For \( 1 \leq p \leq \infty \) and an integer \( n \geq 1 \), we give \( \mathbb{L}^p_{\text{loc}}(U; \mathbb{R}^n) \) a topology induced by the seminorms

\[
|f|_{p,K} := \left| f \upharpoonright K \right|_p
\]

where \( f \in \mathbb{L}^p_{\text{loc}}(U; \mathbb{R}^n) \) and \( K \subset U \) is a compact set. As there is an increasing sequence of compact subsets of \( U \) whose interiors cover \( U \), the space \( \mathbb{L}^p_{\text{loc}}(U; \mathbb{R}^n) \) is a Fréchet space. Clearly, \( C(U; \mathbb{R}^m) \) topologized as a subspace of \( \mathbb{L}^\infty_{\text{loc}}(U; \mathbb{R}^m) \) is a Fréchet space as well. We write \( \mathbb{L}^p_{\text{loc}}(U) \) instead of \( \mathbb{L}^p_{\text{loc}}(U; \mathbb{R}) \), and denote by \( \mathbb{L}^p_{\text{loc}}(U) \) the linear space of all functions in \( \mathbb{L}^p_{\text{loc}}(U) \) whose support is a compact subset of \( U \).

We denote by \( \mathbb{BV}(U) \) the linear space of all BV functions in \( U \), and let

\[
\mathbb{BV}_c(U) := \mathbb{BV}(U) \cap \mathbb{L}^1_{\text{loc}}(U) \quad \text{and} \quad \mathbb{BV}^\infty(U) := \mathbb{BV}(U) \cap \mathbb{L}^\infty_{\text{loc}}(U).
\]

If \( g \in \mathbb{BV}(U) \), then \( \|g\| \) is the total variation of the distributional gradient \( Dg \) of \( g \).

The essential boundary, perimeter and exterior normal of a BV set \( E \subset U \) are denoted by \( \partial_\text{e} E, \|E\| \) and \( \nu_E \), respectively. Note that \( \|E\| = \mathcal{H}(\partial_\text{e} E) = \|\chi_E\| \)

where \( \chi_E \) is the indicator of \( E \) in \( U \).

As usual, \( \mathcal{D}(U) \) and \( \mathcal{D}'(U) \) are the linear spaces of all test functions and all distributions in \( U \), respectively. In accordance with the notation of the previous paragraph, we let \( \|\varphi\| := |\nabla \varphi|_1 \) for each \( \varphi \in \mathcal{D}(U) \).
2. Definitions and basic properties

**Definition 2.1.** A distribution $F \in \mathcal{D}'(U)$ is called *fluxing*, or simply a *flux*, if the equation $\text{div} \ v = F$ has a continuous solution, i.e., if there is a vector field $v \in C(U; \mathbb{R}^m)$ such that for each $\varphi \in \mathcal{D}(U)$,

$$F(\varphi) = -\int_U v(x) \cdot \nabla \varphi(x) \, dx \quad (2.1)$$

The linear space of all fluxing distributions in $U$ is denoted by $\mathcal{F}(U)$. A distribution $F$ defined by equality (2.1) is called the *flux* of $v$, denoted by $F_v$.

We say a sequence $\{f_i\}$ of functions defined on $U$ is *compactly supported* if there is a compact set $K \subset U$ such that $\{f_i \neq 0\} \subset K$ for $i = 1, 2, \ldots$. If the compact set $K$ is specified a priori, we say that $\{f_i\}$ is supported in $K$. A sequence $\{A_i\}$ of subsets of $U$ is called *compactly supported*, or *supported* in a compact set $K \subset U$, whenever the sequence $\{\chi_{A_i}\}$ has the respective property.

**Observation 2.2.** If $F \in \mathcal{D}'(U)$ is a flux, then $\lim F(\varphi_i) = 0$ for every compactly supported sequence $\{\varphi_i\}$ in $\mathcal{D}(U)$ for which

$$\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup \|\varphi_i\| < \infty. \quad (2.2)$$

**Proof.** Let $F = F_v$ for a $v \in C(U; \mathbb{R}^m)$, and let $\{\varphi_i\}$ be a sequence in $\mathcal{D}(U)$ supported in a compact set $K \subset U$ and satisfying conditions (2.2). Find a sequence $\{w_j\}$ in $C_c^1(\mathbb{R}^m; \mathbb{R}^m)$ converging to $v$ uniformly in $K$, and observe

$$|F(\varphi_i)| \leq \int_K |v(x) - w_j(x)| \cdot |\nabla \varphi(x)| \, dx + \left| \int_K \varphi_i(x) \text{div} w_j(x) \, d(x) \right|$$

$$\leq \left( \sup \|\varphi_n\| \right) \sup_{x \in K} |v(x) - w_j(x)| + |\varphi_i|_1 \sup_{x \in K} |\text{div} w_j(x)|$$

for $i, j = 1, 2, \ldots$. Choosing a sufficiently large $j$ and then a sufficiently large $i$, we can make $F(\varphi_i)$ arbitrarily small. $\square$

Observation 2.2 motivates in part the following definition.

**Definition 2.3.** A linear functional $F : \mathcal{D}(U) \to \mathbb{R}$ is called

(i) a *charge* if $\lim F(\varphi_i) = 0$ for every compactly supported sequence $\{\varphi_i\}$ in $\mathcal{D}(U)$ for which

$$\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup \|\varphi_i\| + |\varphi_i|_{\infty} < \infty;$$

(ii) a *strong charge* (abbreviated as *s-charge*) if $\lim F(\varphi_i) = 0$ for every compactly supported sequence $\{\varphi_i\}$ in $\mathcal{D}(U)$ for which

$$\lim |\varphi_i|_1 = 0 \quad \text{and} \quad \sup \|\varphi_i\| < \infty.$$
are compact subsets of $L^1(U)$ [8, Section 5.2, Theorem 4]. Give $BV_c^\infty(U)$ and $BV_c(U)$, respectively, the largest topology $\mathcal{T}$ and $\mathcal{T}_s$ for which all inclusion maps $BV(K,n) \hookrightarrow BV_c^\infty(U)$ and $BV_s(K,n) \hookrightarrow BV_c(U)$ are continuous. Since $U$ is the union of an increasing sequence of compact sets, it follows from [13, Proposition 1.2.2] that the topologies $\mathcal{T}$ and $\mathcal{T}_s$ are locally convex, sequential, and sequentially complete. Moreover $\mathcal{T}_s \subset \mathcal{T}$, and $\mathcal{D}(U)$ is a dense subset of both $(BV_c^\infty(U), \mathcal{T})$ and $(BV_c(U), \mathcal{T}_s)$ [8, Section 5.2, Theorem 2].

**Observation 2.4.** A linear functional $F : \mathcal{D}(U) \rightarrow \mathbb{R}$ is, respectively, a charge or an s-charge if and only if it is $\mathcal{T}$ or $\mathcal{T}_s$ continuous. In particular, each charge has a unique $\mathcal{T}$ continuous extension to $BV_c^\infty(U)$, and each s-charge has a unique $\mathcal{T}_s$ continuous extension to $BV_c(U)$. These extensions are linear.

**Remark 2.5.** Observe that the flux $F_v$ of a locally bounded Borel vector field $v : U \rightarrow \mathbb{R}^m$, which need not be a charge, still extends to

$$F_v : g \mapsto - \int_U v \cdot d(Dg) : BV_c(U) \rightarrow \mathbb{R}.$$ 

In view of Observation 2.4, we always think of charges as defined on $BV_c^\infty(U)$, and of s-charges as defined on $BV_c(U)$. If $F$ is a charge and $E$ is a bounded BV set whose closure is contained in $U$, we let $F(E) := F(\chi_E)$. Note that

$$F_v(E) = - \int_U v \cdot d(D\chi_E) = \int_{\partial E} v \cdot H d\mathcal{H}.$$ 

**Proposition 2.6.** If $F : BV_c(U) \rightarrow \mathbb{R}$ is a linear functional, then the following properties are equivalent.

(i) The functional $F$ is an s-charge.

(ii) Given $\varepsilon > 0$ and compact set $K \subset U$, there is a $\theta > 0$ such that

$$|F(g)| \leq \theta |g|_1 + \varepsilon ||g||$$

for each $g \in BV_c(U)$ with $\{g \neq 0\} \subset K$.

(iii) For each compactly supported sequence $\{B_i\}$ of BV sets in $U$,

$$\lim \frac{F(B_i)}{||B_i||} = 0 \quad \text{whenever} \quad \lim |B_i| = 0.$$

**Proof.** (i) $\Rightarrow$ (ii). Suppose $F$ is an s-charge, and choose an $\varepsilon > 0$ and a compact set $K \subset U$. There is an $\eta > 0$ such that $|F(g)| < \varepsilon/2$ for each $g \in BV_c(U)$ with $|g|_1 < \eta$, $||g|| < 1$, and $\{g \neq 0\} \subset K$. Let $\theta := \varepsilon/(2\eta)$ and select a $g \in \mathcal{D}(U)$ with $\{g \neq 0\} \subset K$. With no loss of generality, we may assume $g \geq 0$; see [13, Theorem 1.8.12].

Let $p$ and $q$ be the smallest positive integers with $|g|_1/p < \eta$ and $||g||/q < 1$. Note $p \leq |g|_1/\eta + 1$ and $q \leq ||g|| + 1$. Since

$$s \mapsto \int_0^s |g > t| dt \quad \text{and} \quad s \mapsto \int_0^s ||g > t|| dt$$

are continuous increasing functions which map $[0, \infty]$ onto $[0, |g|_1]$ and $[0, ||g||]$, respectively, there are points $0 = a_0 < \cdots < a_p = \infty$ and $0 = b_0 < \cdots b_q = \infty$ such
that
\[ \int_{a_{i-1}}^{a_i} |\{g > t\}| \, dt = \frac{1}{p} |g|_1 < \eta \quad \text{and} \quad \int_{b_{j-1}}^{b_j} \|g > t\| \, dt = \frac{1}{q} \|g\| < 1 \]
for \( i = 1, \ldots, p \) and \( j = 1, \ldots, q \). Order the set \( \{a_0, \ldots, a_p, b_0, \ldots, b_q\} \) into a sequence \( 0 = c_0 < \cdots < c_r = \infty \). Then \( r \leq p + q - 1 \), and
\[ g_k := \max \{ \min \{g, c_k\}, c_{k-1} \} - c_{k-1}, \quad k = 1, \ldots, r, \]
are BV functions vanishing outside \( K \). As each \([c_{k-1}, c_k]\) is contained in some \([a_{i-1}, a_i] \cap [b_{j-1}, b_j]\), the previous inequalities imply \( |g_k|_1 < \eta \) and \( \|g_k\| < 1 \). Since
\[ g = \sum_{k=1}^{r} g_k, \]
we obtain
\[ |F(g)| \leq \sum_{k=1}^{r} |F(g_k)| < \frac{\varepsilon}{2} (p + q - 1) \]
\[ \leq \frac{\varepsilon}{2} \left( \frac{1}{\eta} |g|_1 + \|g\| + 1 \right) = \theta |g|_1 + \frac{\varepsilon}{2} \|g\| + \frac{\varepsilon}{2}, \]
and inequality (2.3) follows whenever \( \|g\| \geq 1 \). If \( 0 < \|g\| < 1 \), we apply the previous result to \( h := g/\|g\| \):
\[ |F(g)| = \|g\| \cdot |F(h)| \leq \|g\| (\theta |h|_1 + \varepsilon \|h\|) = \theta |g|_1 + \varepsilon \|g\|. \]
As the case \( \|g\| = 0 \) is trivial, the desired inequality is established.

(iii) \( \Rightarrow \) (ii). By [13, Proposition 2.1.7], given \( \varepsilon > 0 \) and a compact set \( K \subset U \), there is a \( \theta > 0 \) such that
\[ |F(B)| \leq \theta |B| + \varepsilon \|B\| \]
for each BV set \( B \subset K \). Now it follows from [13, Proposition 2.2.6 and Section 4.1] that \( F \) satisfies (ii).

The implications (ii) \( \Rightarrow \) (i) and (ii) \( \Rightarrow \) (iii) are obvious.

**Remark 2.7.** Charges in \( U \) are characterized by an inequality similar to (2.3). Indeed, it follows from [13, Proposition 2.2.6 and Section 4.1] that a linear functional \( F : BV^\infty_c(U) \to \mathbb{R} \) is a charge if and only if given \( \varepsilon > 0 \) and a compact set \( K \subset U \), there is a \( \theta > 0 \) such that
\[ |F(g)| \leq \theta |g|_1 + \varepsilon (\|g\| + |g|_\infty) \]
for each \( g \in BV^\infty_c(U) \) with \( \{g \neq 0\} \subset K \). A direct proof of this fact is analogous to that of Proposition 2.6; see also [5, Proposition 2.4].

**Remark 2.8.** It follows from [13, Section 4.1] that charges are uniquely determined by their values on the indicators of bounded BV sets. As bounded BV sets can be approximated by finite unions of nondegenerate compact intervals [13, Proposition 1.10.3], charges, and a fortiori s-charges, are uniquely determined by their values on the indicators of nondegenerate compact intervals.

The linear spaces of all charges in \( U \) and all s-charges in \( U \) are denoted by \( CH(U) \) and \( CH_s(U) \), respectively. By Observation 2.2,
\[ \mathcal{F}(U) \subset CH_s(U) \subset CH(U) \subset D'(U). \]
If \( f \in L^1_{\text{loc}}(U) \), then the distribution \( \Lambda(f) \) in \( \mathcal{D}'(U) \) defined by

\[
\langle \Lambda(f), \varphi \rangle := \int_U f(x) \varphi(x) \, dx
\]

for each \( \varphi \in \mathcal{D}(U) \) is a charge, called an \textit{absolutely continuous charge} (abbreviated as \textit{ac-charge}). Denoting by \( CH_{\text{ac}}(U) \) the linear space of all ac-charges in \( U \), we have a linear isomorphism

\[
\Lambda : f \mapsto \Lambda(f) : L^1_{\text{loc}}(U) \to CH_{\text{ac}}(U)
\]

In particular, each \( F \in CH_{\text{ac}} \) has an obvious linear extension to \( L^\infty(U) \).

While easy examples show that neither of the spaces \( CH_s(U) \) and \( CH_{\text{ac}}(U) \) contains the other, they have a sizable intersection.

**Proposition 2.9.** \( \Lambda[L^m_{\text{loc}}(U)] \subset CH_s(U) \).

**Proof.** Choose an \( f \in L^m_{\text{loc}}(U) \), and let \( F := \Lambda(f) \). For \( g \in BV_c(U) \) and a measurable set \( B \subset U \), the H"older and Poincaré inequalities imply

\[
\int_B |fg| \leq \left( \int_B |f|^m \right)^{\frac{1}{m}} \left( \int_U |g| \frac{m}{m-1} \right)^{\frac{m-1}{m}} \leq \kappa \|g\| \left( \int_B |f|^m \right)^{\frac{1}{m}}
\]

where \( \kappa \) is a positive constant depending only on the dimension \( m \) [8, Section 5.6, Theorem 1.(i)]. In particular

\[
|F(g)| \leq \kappa \|g\| \left( \int_{\{g \neq 0\}} |f|^m \right)^{\frac{1}{m}} < \infty,
\]

and it follows that \( F \) is a linear functional on \( BV_c(U) \). To show that \( F \) is an s-charge, select a sequence \( \{g_i\} \) in \( BV_c(U) \) supported in a compact set \( K \subset U \), and assume that \( \lim |g_i| = 0 \) and \( \sup \|g_i\| < \infty \). Applying inequality (2.4) to the set \( B_\theta := \{ x \in K : |f(x)| > \theta \} \) with \( \theta \geq 0 \), we obtain

\[
|F(g_i)| \leq \int_{K-B_\theta} |fg_i| + \int_{B_\theta} |fg_i| \leq \theta |g_i|_1 + \kappa |g_i| \left( \int_{B_\theta} |f|^m \right)^{\frac{1}{m}}
\]

\[
\leq \theta |g_i|_1 + \kappa (\sup_n \|g_n\|) \left( \int_{B_\theta} |f|^m \right)^{\frac{1}{m}}.
\]

As \( \lim_{\theta \to \infty} (\int_{B_\theta} |f|^m)^{1/m} = 0 \), choosing a sufficiently large \( \theta \) and then a sufficiently large \( i \), we can make \( F(g_i) \) arbitrarily small. \( \square \)

**Note.** We proved Proposition 2.9 directly from the definition of s-charges. Using Proposition 2.6, the second part of the proof can be simplified by choosing a compactly supported sequence \( \{B_i\} \) of BV sets in \( U \), and applying inequality (2.5) to \( g := \chi_{B_i} \). Indeed, we obtain

\[
|F(B_i)| \leq \kappa \|B_i\| \left( \int_{B_i} |f|^m \right)^{1/m}
\]

for \( i = 1, 2, \ldots \), and hence \( \lim [F(B_i)/\|B_i\|] = 0 \) whenever \( \lim |B_i| = 0 \).

The next example shows that the inclusion \( \Lambda[L^m_{\text{loc}}(U)] \subset CH_s(U) \cap CH_{\text{ac}}(U) \) established in Proposition 2.9 is generally proper.
Example 2.10. Assume $m = 2$, and let $f(\xi, \eta) := \xi^{-\eta} + \eta^{-\xi}$ for each $(\xi, \eta)$ in $U := (0, 1)^2$. If $p \geq 1$ then

$$\xi^{-p\eta} + \eta^{-p\xi} \leq \left[ f(\xi, \eta) \right]^p \leq 2^p (\xi^{-p\eta} + \eta^{-p\xi})$$

for each $(\xi, \eta) \in U$. Since for every $0 < a \leq 1/p$

$$\int_{[0,a]^2} (\xi^{-p\eta} + \eta^{-p\xi}) d\xi d\eta = \frac{2}{p} \int_{1-a}^1 t^{-1} a^t dt,$$

we see that $f \in L^p_{\text{loc}}(U)$ if and only if $p = 1$. On the other hand, the formula

$$v(\xi, \eta) := \left( \frac{\xi^{1-\eta}}{1-\eta}, \frac{\eta^{1-\xi}}{1-\xi} \right)$$

for $(\xi, \eta) \in U$ defines a $v \in C^\infty(U; \mathbb{R}^2)$ with $\operatorname{div} v = f$. Integration by parts shows that $\Lambda(f)$ is the flux of $v$, and hence an $s$-charge according to Observation 2.2.

Proposition 2.11. Given $f \in L^m(U)$, there is a $v \in L^\infty(U; \mathbb{R}^m)$ such that $\Lambda(f)$ is the flux $F_v$ of $v$, and $|v|_\infty \leq \kappa |f|_m$ where $\kappa$ is a constant depending only on the dimension $m$.

Proof. Since it suffices to prove the proposition in each connected component of $U$, we may assume $U$ is connected. Let $X := \{ \nabla \varphi : \varphi \in \mathcal{D}(U) \}$, and for $w \in X$, let

$$G(w) := \int_U f(x) \varphi(x) \, dx$$

where $\varphi$ is the unique element of $\mathcal{D}(U)$ with $\nabla \varphi = w$. By the Hölder and Poincaré inequalities, there is a constant $\kappa$ depending only on the dimension $m$ such that

$$|G(w)| \leq |f|_m |\varphi|_{\frac{m}{m-1}} \leq \kappa |f|_m |w|_1,$$

for each $w \in X$. Applying Hahn-Banach theorem, extend $G$ to a linear functional $H : L^1(U; \mathbb{R}^m) \to \mathbb{R}$ so that $|H(w)| \leq \kappa |f|_m |w|_1$ for each $w \in L^1(U; \mathbb{R}^m)$. Using the duality of $L^p$ spaces, find a $v \in L^\infty(U; \mathbb{R}^m)$ so that $|v|_\infty \leq \kappa |f|_m$, and

$$H(w) = \int_U v(x) \cdot w(x) \, dx$$

for each $w \in L^1(U; \mathbb{R}^m)$. In particular, for each $\varphi \in \mathcal{D}(U)$,

$$\langle \Lambda(f), \varphi \rangle = \int_U f(x) \varphi(x) \, dx = G(\nabla \varphi) = H(\nabla \varphi)$$

$$= \int_U v(x) \cdot \nabla \varphi(x) \, dx = \langle F_v, \varphi \rangle.$$

Remark 2.12. It follows from Proposition 2.11 that for each $f \in L^m(U)$, the equation $\operatorname{div} v = f$ has a solution in $L^\infty(U; \mathbb{R}^m)$. We included this result because it has a simple proof. Using a more elaborate argument, Brezis and Bourgain established the existence of a bounded continuous solution [1, Proposition 1]. The same, and more, follows from Section 3 below.
3. S-charges

A Lipschitz domain is an open set $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary [8, Section 4.2.1]. Note that each Lipschitz domain is a locally BV set. If $\Omega \subset U$ is a Lipschitz domain and $g \in BV_c(U)$ with support in $B(r)$, then it follows from [16, Remark 5.10.2 and Lemma 5.10.4] that $g\chi_\Omega \in BV_c(U)$, and that

$$
\|g\chi_\Omega\| \leq \kappa(\|g\|_1 + \|g\|)
$$

where $\kappa > 0$ depends only on $\Omega \cap B(r)$.

Let $F$ be an s-charge in $U$, and let $\Omega \subset U$ be a Lipschitz domain. In view of the previous paragraph and Proposition 2.6, the linear functional

$$
F \restriction \Omega : g \mapsto F(g\chi_\Omega) : BV_c(U) \to \mathbb{R}
$$

is an s-charge in $U$. If $c\Omega \subset U$, we view $F \restriction \Omega$ as an s-charge in $\mathbb{R}^m$; since $F(g\chi_\Omega)$ is defined for each $g \in BV_c(\mathbb{R}^m)$.

If $f : U \to \mathbb{R}$ is locally Lipschitz and $g \in BV_c(U)$, then $fg \in BV_c(U)$ and

$$
\|fg\| \leq L\|g\|_1 + c\|g\|
$$

where $L := \text{Lip}(f \mid \text{supp } g)$ and $c := |f \mid \text{supp } g|_{\infty}$. Thus by Proposition 2.6,

$$
F \restriction f : g \mapsto F(fg) : BV_c(U) \to \mathbb{R}
$$

is an s-charge in $U$ whenever $F$ is an s-charge in $U$.

We give $CH_s(U)$ a Fréchet topology induced by the seminorms

$$
\|F\|_{s,K} := \sup\{F(g) : g \in BV(U), \{g \neq 0\} \subset K, \text{ and } \|g\| \leq 1\}
$$

where $F \in CH_s(U)$ and $K \subset U$ is a compact set. In view of Observation 2.2, there is a linear map

$$
\Gamma : v \mapsto F_v : C(U; \mathbb{R}^m) \to CH_s(U),
$$

which is continuous. Indeed, given a compact set $K \subset U$, we have

$$
|F_v(g)| \leq \|g\| \cdot |v|_{\infty,K}
$$

for every $g \in BV(U)$ with $\{g \neq 0\} \subset K$; thus $\|F_v\|_{s,K} \leq |v|_{\infty,K}$ for each compact set $K \subset U$. Note that $\mathcal{F}(U)$ is the image of $\Gamma$.

By Proposition 2.9, the restriction $\Lambda_s := \Lambda \restriction L^m_{\text{loc}}(U)$ maps $L^m_{\text{loc}}(U)$ to $CH_s(U)$. It follows from inequality (2.5) that there is a constant $\kappa$, depending only on the dimension $m$, such that for each $f \in L^m_{\text{loc}}(U)$ and each compact set $K \subset U$,

$$
\|\Lambda(f)\|_{s,K} \leq \kappa|f|_{m,K}.
$$

(3.1)

In particular, the map $\Lambda_s : L^m_{\text{loc}}(U) \to CH_s(U)$ is continuous.

**Lemma 3.1.** Given $F \in CH_s(U)$, there is a sequence $\{v_i\}$ in $C^\infty(U; \mathbb{R}^m)$ such that the support of each $\text{div } v_i$ is a compact subset of $U$, and

$$
\lim\|F - \Gamma(v_i)\|_{s,K} = 0
$$

for every compact set $K \subset U$. In particular, the spaces $\mathcal{F}(U)$ and $\Lambda[\mathcal{D}(U)]$ are dense in $CH_s(U)$. 


Proof. There are bounded Lipschitz domains \( \Omega_i \) such that \( \text{cl} \Omega_i \subset \Omega_{i+1} \) and \( U = \bigcup_{i=1}^{\infty} \Omega_i \). Since every compact set \( K \subset U \) is contained in some \( \Omega_i \), we have

\[
\lim \| F - F \restriction \Omega_i \|_{s,K} = 0
\]

for each compact set \( K \subset U \). Thus it suffices to prove the lemma for an \( s \)-charge \( F \) such that \( F = F \restriction \Omega_0 \) for a bounded Lipschitz domain \( \Omega \) with \( \text{cl} \Omega \subset U \). Select a bounded Lipschitz domain \( \Omega_0 \) with \( \text{cl} \Omega_0 \subset \Omega \). There is a convergent (in the distributional sense) sequence \( \{ \eta_i \} \) of standard mollifiers such that the convolutions \( \varphi_i := F * \eta_i \) have support in \( \Omega_0 \). By [15, Theorems 6.30 and 6.32], each \( \varphi_i \) belongs to \( C^\infty(\mathbb{R}^m) \), the \( s \)-charges \( F_i := \Lambda(\varphi_i) \) converge to \( F \) in the distributional sense, and \( F_i(g) = F(\eta_i * g) \) for every \( g \in BV_c(\mathbb{R}^m) \).

For \( i = 1, 2, \ldots \) and \( x = (\xi_1, \ldots, \xi_m) \) in \( U \), let

\[
f_i(\xi_1, \ldots, \xi_m) := \int_{-\infty}^{\xi_i} \varphi_i(t, \xi_2, \ldots, \xi_m) \, dt.
\]

Since \( v_i := (f_i, 0, \ldots, 0) \) belongs to \( C^\infty(U; \mathbb{R}^m) \) and \( \text{div} v_i = \varphi_i \), integration by parts shows that \( F_i = \Gamma(v_i) \). As \( F = F \restriction \Omega_0 \) and \( F_i = F_i \restriction \Omega_0 \), it remains to prove \( \lim \| F - F_i \|_{s,K} = 0 \) for \( K := \text{cl} \Omega_0 \). To this end choose an \( \varepsilon > 0 \) and use Proposition 2.6 to find a \( \theta > 0 \) so that

\[
|F(g)| \leq \theta |g|_1 + \varepsilon \| g \|
\]

for each \( g \in BV(\mathbb{R}^m) \) with \( \{ g \neq 0 \} \subset K \). Select such a \( g \), and let \( g_x(z) := g(x - z) \) for all \( x, z \in \mathbb{R}^m \). By an argument identical to the proof of [13, Lemma 4.2.1],

\[
|g_x - g_y|_1 \leq |x - y| \cdot \| g \|
\]

for all \( x, y \in \mathbb{R}^m \). This and Fubini’s theorem yield

\[
|g - g \ast \eta_i|_1 = \int_{\mathbb{R}^m} \left| g(x) \int_{\mathbb{R}^m} \eta_i(y) \, dy - \int_{\mathbb{R}^m} g_x(y) \eta_i(y) \, dy \right| \, dx
\]

\[
\leq \int_{\mathbb{R}^m} \eta_i(y) \left( \int_{\mathbb{R}^m} |g_0(-x) - g_{-y}(-x)| \, dx \right) \, dy
\]

\[
\leq \int_{\mathbb{R}^m} \eta_i(y) |g_0 - g_{-y}|_1 \, dy \leq \int_{B(1/i)} \eta_i(y) |y| \cdot \| g \| \, dy \leq \frac{1}{i} \| g \|.
\]

Combining the above inequalities, we obtain

\[
|F(g) - F_i(g)| = |F(g - g \ast \eta_i)| \leq \theta |g - g \ast \eta_i|_1 + \varepsilon \| g - g \ast \eta_i \|
\]

\[
\leq \frac{\theta}{i} \| g \| + \varepsilon (\| g \| + \| g \ast \eta_i \|) = \| g \| \left( \frac{\theta}{i} + 2\varepsilon \right)
\]

for \( i = 1, 2, \ldots \), and the lemma follows from the arbitrariness of \( \varepsilon \).

The dual space of a topological vector space \( X \) is denoted by \( X^* \). Aside from the \( w^* \)-topology on \( X^* \), we will also use the strong topology defined by the uniform convergence on the family of all bounded subsets of \( X \) [6, Section 1.8.7 and 8.4].

**Proposition 3.2.** There is a linear bijection \( \Phi : BV_c(U) \to CH_s(U)^* \) defined by

\[
\langle \Phi(g), F \rangle := \langle F, g \rangle
\]

for each \( g \in BV_c(U) \) and each \( F \in CH_s(U) \).

\[\text{For traditional reasons we still write } \mathcal{D}'(U) \text{ rather than } \mathcal{D}(U)^*.\]
Proof. Clearly \( \Phi \) is a linear map. Since
\[
\left| \langle \Phi(g), F \rangle \right| = |\langle F, g \rangle| \leq \|g\| : \|F\|_{s,K}
\]
for each \( F \in CH_s(U) \), each compact set \( K \subset U \), and each \( g \in BV_c(U) \) with \( \{ g \neq 0 \} \subset K \), we see that \( \Phi \) maps \( BV_c(U) \) to \( CH_s(U)^* \). If \( g \in BV_c(U) \) and \( \Phi(g) = 0 \), then
\[
\int_B g(x) \, dx = \int_U \chi_B(x) g(x) \, dx = \langle \Lambda(\chi_B), g \rangle = \langle \Phi(g), \Lambda(\chi_B) \rangle = 0
\]
for each bounded measurable set \( B \subset U \). Consequently \( \Phi \) is injective.

Let \( T \in CH_s(U)^* \). As \( \Lambda_s : L^m_{\text{loc}}(U) \to CH_s(U) \) is continuous, \( T \circ \Lambda_s \in L^m_{\text{loc}}(U)^* \). Using the duality of \( L^p \) spaces [14, Theorem 6.16], find a \( g \in L^m_{\text{loc}}(U)^* \) so that
\[
\langle T, \Lambda(f) \rangle = \langle T \circ \Lambda_s, f \rangle = \int_U f(x) g(x) \, dx
\]
for every \( f \in L^m_{\text{loc}}(U) \). Now choose a \( v \in C^1(U; \mathbb{R}^m) \) with \( |v|_{\infty} \leq 1 \). Since
\[
\langle \Lambda(\text{div} \, v), h \rangle = \int_U h(x) \text{div} \, v(x) \, dx \leq \|h\|
\]
for each \( h \in BV_c(U) \), we infer \( \|\Lambda(\text{div} \, v)\|_{s,C} \leq 1 \) for every compact set \( C \subset U \). By the continuity of \( T \), there are a \( c > 0 \) and a compact set \( K \subset U \) such that
\[
\int_U g(x) \text{div} \, v(x) \, dx = \langle T, \Lambda(\text{div} \, v) \rangle \leq c \|\Lambda(\text{div} \, v)\|_{s,K} \leq c.
\]
Thus \( g \in BV_c(U) \) by the arbitrariness of \( v \), and for each \( f \in L^m_{\text{loc}}(U) \),
\[
\langle \Phi(g), \Lambda(f) \rangle = \langle \Lambda(f), g \rangle = \int_U f(x) g(x) \, dx = \langle T, \Lambda(f) \rangle.
\]
As \( \Lambda[L^m_{\text{loc}}(U)] \) is dense in \( CH_s(U) \) by Lemma 3.1, we conclude \( T = \Phi(g) \). \( \square \)

A set \( C \subset U \) is called amiable if it is compact, and for each connected component \( V \) of \( U - C \), either \( d(V) = \infty \) or \( \partial V \cap \partial U \neq \emptyset \).

**Lemma 3.3.** Each compact set \( K \subset U \) is contained in an amiable set \( C \subset U \).

**Proof.** Denote by \( W \) the collection of all bounded connected components \( W \) of \( U - K \) with \( \partial W \subset K \), and by \( V \) the collection of all other connected components of \( U - K \). Given \( W \in \mathcal{W} \), observe that
\[
\text{dist} \, (W, \partial U) = \text{dist} \, (\partial W, \partial U) \geq \text{dist} \, (K, \partial U),
\]
and as \( W \) is bounded, also \( d(W) = d(\partial W) \leq d(K) \). Consequently
\[
d(\bigcup \mathcal{W}) \leq 3d(K) \quad \text{and} \quad \text{dist} \, (\bigcup \mathcal{W}, \partial U) \geq \text{dist} \, (K, \partial U).
\]
Since \( \bigcup \mathcal{W} \) is a relatively closed subset of \( U - K \), the previous inequalities imply that \( C := K \cup \bigcup \mathcal{W} \) is a compact subset of \( U \). If \( V \in \mathcal{V} \) is bounded, then \( \partial V \) is a subset of \( \partial (U - K) = \partial U \cup \partial K \), but not a subset of \( K \). Thus \( V \) is either unbounded, or its boundary meets the boundary of \( U \). But \( \mathcal{V} \) is the collection of all connected components of \( U - C \), and hence \( C \) is amiable. \( \square \)

**Observation 3.4.** Let \( g \in BV_c(U) \), and let the support of \( Dg \) be contained in an amiable set \( C \subset U \). Then the support of \( g \) is contained in \( C \).
Proof. Observe that \( g \) is constant in each connected component of \( U - C \). If the support of \( g \) meets a connected component \( V \) of \( U - C \), then \( V \cap (\text{supp } g) \) is a proper subset of \( V \); since \( V \) is either unbounded or \( \partial V \cap \partial U \neq \emptyset \). As \( V \) is open, \( V \cap \{g \neq 0\} \) is also a proper subset of \( V \), a contradiction. \( \square \)

**Lemma 3.5.** Let \( \{g_i\} \) be a sequence in \( BV_c(U) \) such that

\[
\sup \left\{ \int_U v \cdot d(Dg_i) : v \in B \text{ and } i = 1, 2, \ldots \right\} < \infty
\]

for each bounded set \( B \subset C(U; \mathbb{R}^m) \). Then \( \{g_i\} \) is compactly supported.

**Proof.** There are open sets \( U_i \) such that \( K_i := \text{cl} U_i \) is contained in \( U_{i+1} \), and \( U = \bigcup_{i=1}^\infty U_i \). If the sequence \( \{\text{supp } Dg_i\} \) is not compactly supported, then we can construct inductively subsequences of \( \{U_i\} \) and \( \{g_i\} \) still denoted by \( \{U_i\} \) and \( \{g_i\} \), so that \( \text{supp } Dg_i \) meets the open set \( U_{i+1} - K_i \). Consequently, there are \( v_i \in C(U; \mathbb{R}^m) \) supported in \( U_{i+1} - K_i \) such that \( |v|_\infty \leq 1 \) and \( a_i := \int_{U_i} v_i \cdot d(Dg_i) \) is different from zero. Let \( b_i = \max\{|a_1|^{-1}, \ldots, |a_i|^{-1}\} \). The bounded set

\[
B := \{ v \in C(U; \mathbb{R}^m) : |v|_{\infty, K_{i+1}} \leq ib_i \text{ for } i = 1, 2, \ldots \}
\]

contains \( w_i := (ib_i)v_i \), \( i = 1, 2, \ldots \). As \( \int_{U_i} w_i \cdot Dg_i \geq i \), we have a contradiction. Thus there is a compact set \( K \subset U \) containing the support of each \( Dg_i \). An application of Lemma 3.3 and Observation 3.4 completes the argument. \( \square \)

**Proposition 3.6.** If \( \Gamma^* \) is the adjoint map of

\[
\Gamma : C(U; \mathbb{R}^m) \to CH_s(U),
\]

then \( \Gamma^*[CH_s(U)^*] \) is sequentially closed in the strong topology of \( C(U; \mathbb{R}^m)^* \).

**Proof.** To simplify the notation, let \( \mathcal{C} := C(U; \mathbb{R}^m) \) and \( \mathcal{S} := CH_s(U) \). Observe

\[
\langle \Gamma^*(S), v \rangle = \langle S, \Gamma(v) \rangle = \langle \Phi(g), \Gamma(v) \rangle = \langle \Gamma(v), g \rangle = -\int_U v \cdot d(Dg) \tag{3.2}
\]

for \( S \in \mathcal{S}^* \) and \( g := \Phi^{-1}(S) \). Select a sequence \( \{S_i\} \in \mathcal{S}^* \) so that \( \{\Gamma^*(S_i)\} \) converges strongly to a \( T \in \mathcal{C}^* \), and note that \( \{\Gamma^*(S_i)\} \) is uniformly bounded on each bounded subset of \( \mathcal{C} \). Applying (3.2) to \( g_i := \Phi^{-1}(S_i) \), Lemma 3.5 implies that the sequence \( \{g_i\} \) in \( BV_c(U) \) is compactly supported. The set

\[
B := \{ v \in \mathcal{C} : |v|_\infty \leq 1 \}
\]

is a bounded subset of \( \mathcal{C} \). Letting \( \|R\| := \sup_{v \in B} \langle R, v \rangle \) for \( R \in \mathcal{C}^* \), we have

\[
\|R\| \leq \sup_{v \in \mathcal{C}} \{ \langle R, v \rangle : v \in \mathcal{C} \text{ and } |v|_{\infty, K} \leq 1 \} < \infty
\]

for any compact set \( K \subset U \). Since \( \lim \|\Gamma^*(S_i) - T\| = 0 \), there is a \( c > 0 \) such that

\[
\|g_i\| = \sup \left\{ \int_U v \cdot d(Dg_i) : v \in C^1(U; \mathbb{R}^m) \text{ and } |v|_\infty \leq 1 \right\} \leq \sup \left\{ \|\Gamma^*(S_i), v\| : v \in B \right\} = \|\Gamma^*(S_i)\| \leq c
\]
for \( i = 1, 2, \ldots \). By Poincaré inequality, there is a constant \( \kappa > 0 \), depending only on the dimension \( m \), such that \( |g_i|_m \leq \kappa \|g_i\| \); in particular \( g_i \in L_{\text{w*-compact}}(U) \). Since \( L_{\text{w*-compact}}(U) \) is the dual space of \( L^m(U) \), and since

\[
V := \left\{ h \in L^m(U) : |h|_m \leq \frac{1}{\kappa c} \right\}
\]

is a neighborhood of zero in \( L^m(U) \), the Banach-Alaoglu theorem \([15, \text{Section } 3.15]\) shows that

\[
\mathcal{K} := \left\{ f \in L_{\text{w*- compact}}(U) : \left| \int_U f(x)h(x) \, dx \right| \leq 1 \text{ for each } h \in V \right\}
\]

is w*-compact subset of \( L_{\text{w*- compact}}(U) \). By the Hölder and Poincaré inequalities,

\[
\left| \int_U g_i(x)h(x) \, dx \right| \leq |g_i|_m |h|_m \leq \kappa \|g_i\| \cdot |h|_m \leq \kappa c|h|_m \leq 1
\]

for each \( g_i \) and each \( h \in V \). Thus the sequence \( \{g_i\} \) has a w*-cluster point \( g \in \mathcal{K} \). As \( \{g_i\} \) is compactly supported, \( \text{supp } g \) is a compact subset of \( U \); in particular \( g \in L^1(U) \). Equality (3.2) implies

\[
\lim \langle \Gamma^*(S_i), v \rangle = \lim \int_U g_i(x) \text{div } v(x) \, dx = \int_U g(x) \text{div } v(x) \, dx \quad (3.3)
\]

for each \( v \in C^1_c(U; \mathbb{R}^m) \); the last equality holds, since \( \int_U g \text{div } v \) is the cluster point of a convergent sequence \( \left\{ \int_U g_i \text{div } v \right\} \). Thus for \( v \in C^1_c(U; \mathbb{R}^m) \) with \( |v|_\infty \leq 1 \),

\[
\int_U g(x) \text{div } v(x) \, dx \leq \sup \|g_i\| \leq c.
\]

We infer \( g \in BV_c(U) \), and let \( S := \Phi(g) \). By equalities (3.3) and (3.2),

\[
\langle T, v \rangle = \lim \langle \Gamma^*(S_i), v \rangle = - \lim \int_U v \cdot d(Dg_i)
\]

\[
= - \int_U v \cdot d(Dg) = \langle \Gamma^*(S), v \rangle
\]

for each \( v \in C^1_c(U; \mathbb{R}^m) \). As \( C^1_c(U; \mathbb{R}^m) \) is a dense subspace of \( \mathcal{E} \), we see that \( T = \Gamma^*(S) \) belongs to \( \Gamma^*(\mathcal{S}^*) \). \( \square \)

**Theorem 3.7.** \( \mathcal{F}(U) = CH_s(U) \).

**Proof.** According to the Closed Range Theorem \([6, \text{Theorem } 8.6.13]\), the following claims are equivalent:

(a) \( \Gamma^* \left[ CH_s(U)^* \right] \) is strongly closed in \( C(U; \mathbb{R}^m)^* \);
(b) \( \Gamma^* \left[ CH_s(U)^* \right] \) is w*-closed in \( C(U; \mathbb{R}^m)^* \);
(c) \( \Gamma \left[ C(U; \mathbb{R}^m) \right] \) is closed in \( CH_s(U) \).

However, a careful look at the proof of implication (a) \( \Rightarrow \) (b) presented in [6] reveals that the assumption “strongly closed” can be relaxed to “strongly sequentially closed”. In view of this and Proposition 3.6, the space \( \mathcal{F}(U) = \Gamma \left[ C(U; \mathbb{R}^m) \right] \) is closed in \( CH_s(U) \). As \( \mathcal{F}(U) \) is dense in \( CH_s(U) \) by Lemma 3.1, the theorem follows. \( \square \)
Theorem 3.8. Let $F \in CH_s(U)$. For each $\varepsilon > 0$ and each amiable set $K \subset U$, there is a $v \in C(U; \mathbb{R}^m)$ such that $\Gamma(v) = F$ and
\[
\|F\|_{s,K} \leq \|v\|_{\infty,K} \leq (1 + \varepsilon)\|F\|_{s,K}.
\]

Proof. The first inequality, which holds for any compact set $K \subset U$, is obvious. Choose an $\varepsilon > 0$ and an amiable set $K \subset U$. We simplify the notation by letting $|v| := |v|_{\infty,K}$ for each $v \in C(U; \mathbb{R}^m)$, and $\|F\| := \|F\|_{s,K}$. To avoid a triviality, assume that $\|F\| > 0$. It suffices to show that the nonempty convex sets
\[
A := \{v \in C(U; \mathbb{R}^m) : |v| < (1 + \varepsilon)\|F\|\},
\]
\[
B := \{v \in C(U; \mathbb{R}^m) : \Gamma(v) = F\}
\]
have a nonempty intersection. Proceeding toward a contradiction suppose that $A \cap B = \emptyset$. As $A$ is open, it follows from the Hahn-Banach theorem that there are $T \in \left[C(U; \mathbb{R}^m)^*\right]$ and $\gamma \in \mathbb{R}$ such that
\[
\langle T, v \rangle < \gamma \leq \langle T, w \rangle
\]
for each $v \in A$ and each $w \in B$ [15, Theorem 3.4, (a)]. Note $\gamma > 0$, because $v = 0$ belongs to $A$. For the reminder of the proof, select a $w \in B$. If $u \in \Gamma^{-1}(0)$, then $w + tu$ belongs to $B$ for each $t \in \mathbb{R}$. Hence $t\langle T, u \rangle \geq \gamma - \langle T, w \rangle$ for each $t \in \mathbb{R}$, and consequently $\langle T, u \rangle = 0$. Therefore $\Gamma^{-1}(0) \subset T^{-1}(0)$. Since $\Gamma$ is surjective, and hence open by the Open Mapping Theorem [15, Corollary 2.12, (a)], there is an $S \in \left[CH_s(U)^*\right]$ with $T = S \circ \Gamma$. The function $g := F^{-1}(S)$ belongs to $BV_c(U)$, and
\[
\int_U v \cdot d(Dg) = \langle \Gamma(v), g \rangle = \langle S, \Gamma(v) \rangle = \langle T, v \rangle
\]
for each $v \in C(U; \mathbb{R}^m)$. If $v \in C(U; \mathbb{R}^m)$ and $\{v \neq 0\} \cap K = \emptyset$, then $|v| = 0$. Thus both $tv$ and $-tv$ belong to $A$ for each $t \in \mathbb{R}$, and inequality (3.4) implies $Tv = 0$. By equality (3.5), the support of $Dg$ is contained in $K$, and by Observation 3.4, so is the support of $g$. Choose a positive $\eta < \varepsilon$ and a $u \in C^1_c(\mathbb{R}^m; \mathbb{R}^m)$ with $|u|_{\infty} \leq 1$. Clearly $v := -(1 + \eta)\|F\|u$ belongs to $A$, and by (3.5) and (3.4),
\[
\int_U g(x) \text{div} u(x) \, dx = - \int_U u \cdot d(Dg) = \frac{1}{(1 + \eta)\|F\|} \int_U v \cdot d(Dg)
\]
\[
= \frac{1}{(1 + \eta)\|F\|} \langle T, v \rangle < \frac{\gamma}{(1 + \eta)\|F\|}.
\]
We infer $\|g\| \leq \gamma / [(1 + \eta)\|F\|]$. As the support of $g$ is contained in $K$, a contradiction follows from (3.4) and (3.5):
\[
\gamma \leq \langle T, w \rangle = \langle \Gamma(w), g \rangle = \langle F, g \rangle \leq \|g\| \cdot \|F\| \leq \frac{\gamma}{1 + \eta} < \gamma.
\]

Let $K \subset \mathbb{R}^m$ be a compact set, and let $BV(K)$ be the linear space of all functions $g \in BV(\mathbb{R}^m)$ with $\{g \neq 0\} \subset K$. A linear functional $F : BV(K) \to \mathbb{R}$ is called an $s$-charge in $K$ if given $\varepsilon > 0$, there is a $\theta > 0$ such that
\[
|F(g)| \leq \theta |g_1| + \varepsilon \|g\|
\]
for each \( g \in BV(K) \). The linear space of all s-charges in \( K \), denoted by \( CH_s(K) \), is equipped with the Banach norm

\[
\| F \|_s := \sup \{ F(g) : g \in BV(K) \text{ and } \| g \| \leq 1 \}
\]

for \( F \in CH_s(K) \). Given \( K \subset U \), the restriction map \( \rho_s : F \mapsto F \restriction_{BV(K)} : CH_s(U) \to CH_s(K) \) is linear and continuous. If \( \Omega \) is a bounded Lipschitz domain, then \( CH_s(\overline{\Omega}) \) is linearly homeomorphic to \( \{ F \in CH_s(\mathbb{R}^m) : F = F \restriction_{\Omega} \} \) topologized as a subspace of \( CH_s(\mathbb{R}^m) \).

As the definitions of s-charges in an open set \( U \) and a in compact set \( K \) are similar, most of the properties established for s-charges in \( U \) hold also for s-charges in \( K \), and the corresponding proofs are analogous. Since \( CH_s(K) \) is a Banach space, proving properties of s-charges in \( K \) is often less technical.

Let \( K \subset \mathbb{R}^m \) be a compact set. If \( v \in C(K; \mathbb{R}^m) \), then the functional

\[
F_v : g \mapsto \int_K v \cdot Dg : BV(K) \to \mathbb{R}
\]

is an s-charge in \( K \), still called the flux of \( v \). Topologizing \( C(K; \mathbb{R}^m) \) by the Banach norm \( |v|_\infty \), we have a continuous linear surjection

\[
\Gamma_K : v \mapsto F_v : C(K; \mathbb{R}^m) \to CH_s(K)
\]

(cf. Theorem 3.7), and the following diagram commutes

\[
\begin{array}{ccc}
C(\mathbb{R}^m; \mathbb{R}^m) & \xrightarrow{\rho} & C(K; \mathbb{R}^m) \\
\uparrow{\Gamma} & & \uparrow{\Gamma_K} \\
CH_s(\mathbb{R}^m) & \xrightarrow{\rho_s} & CH_s(K)
\end{array}
\]

As the restriction map \( \rho : v \mapsto v \restriction_{K} \) is surjective, so is \( \rho_s \); in particular

\[
CH_s(K) = \{ F \restriction_{BV(K)} : F \in CH_s(\mathbb{R}^m) \}.
\]

However, note that for an \( F \in CH_s(\mathbb{R}^m) \), the inclusion

\[
\{ v \restriction_{K} : v \in \Gamma^{-1}(F) \} \subset \Gamma_K^{-1}[F \restriction_{BV(K)}]
\]

may be proper. The next proposition, whose proof is analogous to that of Theorem 3.8, holds for any compact set \( K \subset \mathbb{R}^m \).

**Proposition 3.9.** Let \( F \) be an s-charge in a compact set \( K \), and let \( \varepsilon > 0 \). There is a \( v \in C(K; \mathbb{R}^m) \) such that \( F = \Gamma_K(v) \) and

\[
\| F \|_s \leq |v|_\infty \leq (1 + \varepsilon) \| F \|_s.
\]

4. Rotation invariant charges

In this section we consider \( \Gamma \) restricted to a map from the space of all rotation invariant vector fields to the space of all rotation invariant s-charges, and construct a continuous right inverse of \( \Gamma \).

Working with the standard orthonormal base in \( \mathbb{R}^m \), we view the orthogonal group \( O(m) \) as the multiplicative group of orthonormal matrices, and employ the usual matrix multiplication. Vectors and one-forms are viewed as one-column and
one-row matrices, respectively. In particular, \( x \in \mathbb{R}^m \) is a one-column matrix, and the gradient \( \nabla \varphi \) of a \( \varphi \in C^1(\mathbb{R}^m) \) is a one-row matrix; in this interpretation, \( x \cdot \nabla \varphi(x) = [\nabla \varphi(x)]^T x \). The Haar probability on \( O(m) \) is denoted by \( \theta \).

Throughout this section, select a positive \( R \leq \infty \), and let
\[
U := \{ x \in \mathbb{R}^m : |x| < R \} \quad \text{and} \quad U_0 := \{ x \in \mathbb{R}^m : 0 < |x| < R \}.
\]
The group \( O(m) \) acts linearly and continuously on the spaces \( BV_c(U) \), \( CH_s(U) \), and \( C(U; \mathbb{R}^m) \) by the following rules:
\[
\langle A \cdot g, x \rangle := \langle g, Ax \rangle, \quad \langle A \cdot F, g \rangle := \langle F, A \cdot g \rangle, \quad \langle A \cdot v, x \rangle := A^{-1}(v, Ax)
\]
for every \( A \in O(m) \), \( g \in BV_c(U) \), \( F \in CH_s(U) \), \( v \in C(U; \mathbb{R}^m) \), and \( x \in U \). Let
\[
CH^\text{inv}_s(U) := \{ F \in CH_s(U) : A \cdot F = F \},
\]
\[
C^\text{inv}(U; \mathbb{R}^m) := \{ v \in C(U; \mathbb{R}^m) : A \cdot v = v \},
\]
and give these spaces the subspace topology. If \( v \in C^\text{inv}(U; \mathbb{R}^m) \) then \( v(0) = 0 \), since \( A^{-1}v(0) = v(0) \) for each \( A \in O(m) \). Observe
\[
\langle \Gamma(A \cdot v), \varphi \rangle = -\int_U \nabla \varphi(x) [A(v, A^{-1}x)] \, dx = -\int_U [\nabla \varphi(Ay) A] v(y) \, dy = -\int_U \nabla (A \cdot \varphi)(y) v(y) \, dy = \langle \Gamma(v), A \cdot \varphi \rangle = \langle A \cdot \Gamma(v), \varphi \rangle
\] (4.1)
for every \( A \in O(m) \), \( v \in C(U; \mathbb{R}^m) \), and \( \varphi \in \mathcal{D}(U) \). Thus \( \Gamma(A \cdot v) = A \cdot \Gamma(v) \), and it follows that \( \Gamma \) maps \( C^\text{inv}(U; \mathbb{R}^2) \) into \( CH^\text{inv}_s(U) \).

**Observation 4.1.** The map \( \Gamma : C^\text{inv}(U; \mathbb{R}^2) \rightarrow CH^\text{inv}_s(U) \) is surjective.

**Proof.** If \( F \in CH^\text{inv}_s(U) \) then by Theorem 3.7, there is a \( v \in C(U; \mathbb{R}^2) \) such that \( \Gamma(v) = F \). Defining a \( w := \int_{O(m)} A \cdot v \, d\theta(A) \), we have \( \Gamma(w) = F \). Indeed for each \( \varphi \in \mathcal{D}(U) \), Fubini’s theorem and (4.1) yield
\[
\langle \Gamma(w), \varphi \rangle = \int_U \left[ \int_{O(m)} (A \cdot v)(x) \, d\theta(A) \right] \cdot \nabla \varphi(x) \, dx = \int_{O(m)} \left[ \int_U (A \cdot v)(x) \cdot \nabla \varphi(x) \, dx \right] d\theta(A) = \int_{O(m)} \langle A \cdot \Gamma(v), \varphi \rangle \, d\theta(A) = \int_{O(m)} \langle A \cdot F, \varphi \rangle \, d\theta(A) = \int_{O(m)} \langle F, \varphi \rangle \, d\theta(A) = \langle F, \varphi \rangle.
\]

View the sphere \( S_r := \partial B(r) \) as a Riemannian submanifold of \( \mathbb{R}^m \), and denote by \( T_x(S_r) \) its tangent space at \( x \in S_r \). The measure \( \mathcal{H}^{m-1}(B(r)) \) defines an \( O(m) \)
invariant probability in \( S_r \), denoted by \( \sigma_r \). For \( x \in S_r \) and \( \varphi \in \mathcal{D}(S_r) \), let
\[
|\nabla \varphi|(x) := \sup\{|X\varphi| : X \in T_xS_r \text{ and } |X| = 1\},
\]
\[
\|\varphi\| := \int_{S_r} |\nabla \varphi|(x) \, d\sigma_r(x).
\]
With this notation at hand, we can introduce charges and s-charges in \( S_r \) by the obvious modification of Definition 2.3. Observation 2.4 readily translates to charges and s-charges in \( S_r \), and a charge in \( S_r \) is determined by its values on BV sets in \( S_r \) (cf. Remark 2.8). A charge \( G \) in \( S_r \) is called \textit{invariant} if
\[
\langle G, A \cdot \varphi \rangle = \langle G, \varphi \rangle
\]
for each \( A \in O(m) \) and each \( \varphi \in \mathcal{D}(S_r) \).

\textbf{Proposition 4.2.} If \( G \) is an invariant charge in \( S_r \), then
\[
\langle G, g \rangle = G(S_r) \int_{S_r} g(x) \, d\sigma_r(x)
\]
for each \( g \in BV(S_r) \).

\textit{Proof.} In view of Observation 2.4, it suffices to prove the proposition when \( g \) is a test function. Choose a \( \varphi \in \mathcal{D}(S_r) \), and for each \( x \in S_r \), let
\[
f(x) := \int_{O(m)} \varphi(Ax) \, d\theta(A).
\]
Since \( f(Bx) = f(x) \) for each \( B \in O(m) \), and since \( O(m) \) acts transitively on \( S_r \), the function \( f \) equals a constant \( c \). By Fubini’s theorem
\[
c = \int_{S_r} f(x) \, d\sigma_r(x) = \int_{O(m)} \left[ \int_{S_r} \varphi(Ax) \, d\sigma_r(x) \right] \, d\theta(A)
\]
\[
= \int_{O(m)} \left[ \int_{S_r} \varphi(x) \, d\sigma_r(x) \right] \, d\theta(A) = \int_{S_r} \varphi(x) \, d\sigma_r(x),
\]
and hence
\[
G(f) = G(c\chi_{S_r}) = cG(S_r) = G(S_r) \int_{S_r} \varphi(x) \, d\sigma_r(x).
\]
We complete the proof by showing that \( G(\varphi) = G(f) \). To this end, consider collections \( P := \{(E_1, A_1), \ldots, (E_p, A_p)\} \) such that \( E_1, \ldots, E_p \) are disjoint Borel subsets of \( O(m) \) whose union is \( O(m) \), and \( A_i \in E_i \) for \( i = 1, \ldots, p \). Given such a collection \( P \), define a test function \( f_P := \sum_{i=1}^p (A_i \cdot \varphi) \theta(E_i) \), and observe
\[
|f_P|_{\infty} \leq \sum_{i=1}^p |A_i \cdot \varphi|_{\infty} \theta(E_i) \leq |\varphi|_{\infty} \sum_{i=1}^p \theta(E_i) = |\varphi|_{\infty},
\]
\[
\|f_P\| \leq \sum_{i=1}^p \|A_i \cdot \varphi\| \theta(E_i) \leq \|\varphi\| \sum_{i=1}^p \theta(E_i) = \|\varphi\|.
\] (4.2)
The first inequality is obvious, and since \( |\nabla (A_i \cdot \varphi)|(x) = |\nabla \varphi|(A_ix) \) for each \( x \in S_r \) and \( i = 1, \ldots, p \), the second one follows. Now the function \( (A, x) \mapsto \varphi(Ax) \) is uniformly continuous on \( O(m) \times S_r \). Thus making the diameter of each \( E_i \) sufficiently small, \( f_P \) approximates \( f \) uniformly with an arbitrary precision; in particular, \( f_P \) can be arbitrarily close to \( f \) in the \( L^1 \) norm of \( L^1(S_r, \sigma_r) \). In view
of Remark 2.7, this and inequalities (4.2) imply that $G(f_P)$ can be arbitrarily close to $G(f)$. Since

$$G(f_P) = \sum_{i=1}^{p} \theta(E_i)G(A_i \cdot \phi) = G(\phi) \sum_{i=1}^{p} \theta(E_i) = G(\phi).$$

for each $P$, we obtain $G(f) = G(\phi)$. \hfill \Box

**Proposition 4.3.** The map $\Gamma : C^{inv}(U; \mathbb{R}^m) \rightarrow CH_s^{inv}(U)$ has a linear right inverse $\Upsilon : CH_s^{inv}(U) \rightarrow C^{inv}(U; \mathbb{R}^m)$ defined by the formula

$$\langle \Upsilon(F), x \rangle = \begin{cases} \frac{[F, B(x)]}{\|B(x)\|} \cdot \frac{x}{|x|} & \text{if } x \in U_0, \\ 0 & \text{if } x = 0, \end{cases} \quad (4.3)$$

for each $F \in CH_s^{inv}(U)$. The equality $|\Upsilon(F)|_{\infty, B[r]} = \|F\|_{s, B[r]}$ holds for each positive $r < R$; in particular $\Upsilon$ is continuous.

**Proof.** Clearly $\Upsilon$ is a linear map. Select an $F \in CH_s^{inv}(U)$, and note the vector field $v := \Upsilon(F)$ belongs to $C^{inv}(U; \mathbb{R}^m)$ by Proposition 2.6. We show that $F = F_v$. For $0 < r < R$ and a BV set $E$ in $S_r$, the cone

$$C_E := \{sx : x \in E \text{ and } 0 \leq s \leq r\}$$

is a bounded BV subset of $U$. In view of Remark 2.8, we can define an invariant charge $G_r$ in $S_r$ by letting $G_r(E) := F(C_E)$ for each BV set $E$ in $S_r$. By Proposition 4.2,

$$G_r(E) = G_r(S_r) \frac{\mathcal{H}(E)}{\|B(r)\|}$$

for every BV set $E$ in $S_r$. Given $0 < t < r < R$ and a BV set $E_t$ in $S_r$, the set $E_t := C_{E_t} \cap S_t$ is a BV set in $S_t$. If $C := C_{E_t} - C_{E_t}$, then

$$F(C) = G_r(S_r) \frac{\mathcal{H}(E_t)}{\|B(r)\|} - G_t(S_t) \frac{\mathcal{H}(E_t)}{\|B(t)\|}$$

$$= \int_{\partial C, C_{E_t}} v \cdot \nu_{C_{E_t}} d\Sigma - \int_{\partial C, C_{E_t}} v \cdot \nu_{C_{E_t}} d\Sigma = F_v(C).$$

Cover $U_0$ by charts $(J_1, \phi_1), \ldots, (J_n, \phi_n)$ where $J_i$ are open subintervals of $\mathbb{R}^m$ and $\phi_i : J_i \rightarrow U_0$ are defined by means of the spherical coordinates. If $K$ is a compact subinterval of $J_i$, we call $\phi_i(K)$ a “rectangle” in $U_0$. As the above calculation shows that $F(I) = F_v(I)$ for each “rectangle” $I$, we infer from Remark 2.8 that $F(E) = F_v(E)$ for each bounded BV set $E$ with $\text{cl} \ E \subset U_0$. If $E$ is a bounded BV subset of $U$, then

$$F(E) = \lim_{r \rightarrow 0} F\left[ E - B(r) \right] = \lim_{r \rightarrow 0} F_v\left[ E - B(r) \right] = F_v(E)$$

and Remark 2.8 implies $F = F_v$. Since $|v(x)| \leq \|F\|_{s, B[r]}$ for each $x \in B[r]$, we have $|\Upsilon(F)|_{\infty, B[r]} = |v|_{\infty, B[r]} \leq \|F\|_{s, B[r]}$. The reverse inequality has been established prior to Lemma 3.1:

$$\|F\|_{s, B[r]} = |\langle \Gamma, \Upsilon(F) \rangle|_{\infty, B[r]} \leq |\Upsilon(F)|_{\infty, B[r]}.$$

Noting that each compact subset of $U$ is contained in $B[r]$ for some $r < R$ completes the argument. \hfill \Box
Remark 4.4. We present a different proof of Proposition 4.3, which is available in dimension $m = 2$, but may not generalize to higher dimensions.

For $x = (\xi_1, \xi_2)$ in $\mathbb{R}^2$, let $\bar{x} = (-\xi_2, \xi_1)$. Given $v \in C^{\text{inv}}(U; \mathbb{R}^m)$, there are continuous functions $a_1, a_2$ defined on $[0, R)$ such that $a_1(0) = a_2(0) = 0$, and $v(x) = a_1(|x|)x + a_2(|x|)\bar{x}$ for each $x \in U$. Define vector fields $\pi_i v \in C^{\text{inv}}(U; \mathbb{R}^2)$, $i = 1, 2$, by

$$\pi_1 v(x) := a_1(|x|)x \quad \text{and} \quad \pi_2 v(x) := a_2(|x|)\bar{x}$$

for every $x \in U$. Interpreting derivatives in the distributional sense, observe that $\text{div } \pi_2 v = 0$, and that $\text{div } \pi_1 v = 0$ implies $ta'(t) + 2a_1(t) = 0$ for $0 < t < R$. In $(0, R)$, the continuous distributional solutions of the last equation are the same as the classical solutions $a_1(t) = ct^{-2}$ where $c \in \mathbb{R}$. As $a_1$ is bounded in the neighborhood of zero, $\text{div } \pi_1 v = 0$ implies $\pi_1 v = 0$. Now

$$\langle \Gamma(v), \varphi \rangle = -\int_U \pi_1 v(x) \cdot \nabla \varphi(x) \, dx - \int_U \pi_2 v(x) \cdot \nabla \varphi(x) \, dx$$

$$= \int_U \varphi(x) \text{div } \pi_1 v(x) \, dx + \int_U \varphi(x) \text{div } \pi_2 v(x) \, dx$$

$$= \int_U \varphi(x) \text{div } \pi_1 v(x) \, dx$$

for each $\varphi \in \mathcal{D}(U)$, and we conclude that $\Gamma(v) = 0$ implies $\pi_1 v = 0$.

Choose an $F \in CH^{\text{inv}}_s(U)$, and use Observation 4.1 to find a $v \in C^{\text{inv}}(U; \mathbb{R}^2)$ with $\Gamma(v) = F$. By the previous paragraph, $\pi_1 v \in C^{\text{inv}}(U; \mathbb{R}^2)$ does not depend on the choice of $v$. Thus letting $\Upsilon(F) = \pi_1 v$ for any $v \in C^{\text{inv}}(U; \mathbb{R}^2)$ with $\Gamma(v) = F$, we have defined a right inverse $\Upsilon$ of $\Gamma : C^{\text{inv}}(U; \mathbb{R}^2) \to CH^{\text{inv}}_s(U)$. Since

$$\langle F, B(r) \rangle = \int_{\partial B(r)} \pi_1 v \cdot \nu_{B(r)} \, d\mathcal{H} = ra_1(r) \|B(r)\|$$

for $0 < r < R$, the vector field $\Upsilon(F)$ is defined by formula (4.3).

5. Charges

Under the name “continuous additive functions”, charges were introduced in [12] as a common generalization of ac-charges and fluxing distributions. They facilitate a definition of a multidimensional Riemann type integral that provides a Gauss-Green theorem for any differentiable vector field (cf. Section 6 below). In this section, we show that the common generalization given by charges is minimal: the space $CH(U)$ of all charges in $U$ is the smallest linear space containing both $CH_{\text{ac}}(U)$ and $CH_s(U)$. The idea of the proof is similar to that of Theorem 3.7.

We give $CH(U)$ a Fréchet topology defined by the seminorms

$$\|F\|_K := \sup \{ F(g) : g \in BV(U), \; \{g \neq 0\} \subset K, \; \text{and} \; \|g\|_\infty + \|g\| \leq 1 \}$$

where $F \in CH(U)$, and $K \subset U$ is a compact set. Since $\|F\|_K \leq \|F\|_{s,K}$ for each s-charge $F$, the inclusion map $CH_s(U) \hookrightarrow CH(U)$ is continuous. However, $CH_s(U)$ is not topologized as a subspace of $CH(U)$.

The product topology in $L^1_{\text{loc}}(U) \times C(U; \mathbb{R}^m)$ is defined by the seminorms

$$\|(f, v)\|_K := \max \{ \|f\|_{1,K}, \|v\|_{\infty,K} \}.$$
A linear map $\Theta : L^1_{\text{loc}}(U) \times C(U;\mathbb{R}^m) \to CH(U)$, defined by the formula
\[
\langle \Theta(f, v), g \rangle := \Lambda(f) + \Gamma(v) = \int_U f g \, d\mathcal{L}^m - \int_U v \cdot d(Dg)
\]
for $(f, v) \in L^1_{\text{loc}}(U) \times C(U;\mathbb{R}^m)$ and $g \in BV^\infty_c(U)$, is continuous. Indeed,
\[
\left| \langle \Theta(f, v), g \rangle \right| \leq |g|_{1,K} \cdot |f|_{1,K} + \|g\| \cdot \|v\|_{1,K} \leq (|g|_{\infty} + \|g\|) \left| \langle f, v \rangle \right|_K
\]
whenever $K \subset U$ is compact and $g \neq 0 \in K$, and hence $\|\Theta(f, v)\|_K \leq \left| \langle f, v \rangle \right|_K$.

**Proposition 5.1.** If $\Theta^*$ is the adjoint map of $\Theta$, then $\Theta^*[CH(U)^*]$ is sequentially closed in the strong topology of $[L^1_{\text{loc}}(U) \times C(U;\mathbb{R}^m)]^*$.

**Proof.** To simplify the notation, let $\mathcal{C} := C(U;\mathbb{R}^m)$, and write $BV^\infty_c$, $L^1_{\text{loc}}$, and $CH$ instead of $BV^\infty_c(U)$, $L^1_{\text{loc}}(U)$, and $CH(U)$, respectively. By [13, Theorem 4.3.5], there is a linear bijection $\Psi : BV^\infty_c \to CH^*$ such that
\[
\langle \Psi(g), F \rangle = \langle F, g \rangle.
\]
for each $g \in BV^\infty_c$ and each $F \in CH$. Observe that
\[
\langle \Theta^*(S), (f, v) \rangle = \langle S, \Theta(f, v) \rangle = \langle \Psi(g), \Theta(f, v) \rangle = \langle \Psi(g), F \rangle = \langle F, g \rangle,
\]
for each $g \in BV^\infty_c$ and $F \in CH$. Let $g := \Psi^{-1}(S)$, and write $\Phi^{-1}(S)$. Select a sequence $\{S_i\}$ in $CH^*$ so that $\{\Theta^*(S_i)\}$ converges strongly to a $T$ in $L^1_{\text{loc}} \times \mathcal{C}^*$, and note that $\{\Theta^*(S_i)\}$ is uniformly bounded on each bounded subset of $L^1_{\text{loc}} \times \mathcal{C}$. Applying (5.1) to $g_i = \Psi^{-1}(S_i)$ and $(0, v)$, Lemma 3.5 implies that the sequence $\{g_i\}$ in $BV^\infty_c$ is supported in a compact set $C \subset U$. The set
\[
B := \{(f, v) \in L^1_{\text{loc}} \times \mathcal{C} : |f|_1 \leq 1 \text{ and } |v|_{\infty} \leq 1\}
\]
is a bounded subset of $L^1_{\text{loc}} \times \mathcal{C}$. Letting $\|R\| := \sup \{\langle R, (f, v) \rangle : (f, v) \in B \}$ for $R \in (L^1_{\text{loc}} \times \mathcal{C})^*$, we have
\[
\|R\| \leq \sup \left\{ \left| \langle R, (f, v) \rangle \right| : (f, v) \in L^1_{\text{loc}} \times \mathcal{C} \text{ and } \left| \langle f, v \rangle \right|_K \leq 1 \right\} < \infty
\]
for any compact set $K \subset U$. Since $\lim \|\Theta^*(S_i) - T\| = 0$, there is a $c > 0$ such that $\|\Theta^*(S_i)\| \leq c$ for $i = 1, 2, \ldots$. From
\[
|g_i|_{\infty} = \sup \left\{ \int_U f(x)g_i(x) \, dx : f \in L^1(U) \text{ and } |f|_1 \leq 1 \right\},
\]
\[
\|g_i\| = \sup \left\{ \int_U v \cdot d(Dg_i) : v \in C^1_c(U;\mathbb{R}^m) \text{ and } |v|_{\infty} \leq 1 \right\},
\]
and equality (5.1), we obtain
\[
|g_i|_{\infty} + \|g_i\| \leq \sup \left\{ \|\Theta^*(S_i)(f, v)\| : (f, v) \in B \right\} = \|\Theta^*(S)\| \leq c.\]
Now $L^\infty(U)$ is the dual of $L^1(U)$, and $\mathcal{V} := \{ h \in L^1(U) : |h|_1 \leq 1/c \}$ is a neighborhood of zero in $L^1(U)$. According to the Banach-Alaoglu theorem,
\[
\mathcal{K} := \left\{ f \in L^\infty(U) : \left| \int_U f(x)h(x) \, dx \right| \leq 1 \text{ for each } h \in \mathcal{V} \right\}
\]
Remark 5.3. From [13, Proposition 4.2.2] and Lemma 3.1, we see that both spaces are closed w*-compact subset of $L^\infty(U)$. Every $g_i$ belongs to $BV^\infty \subset L^\infty(U)$, and
\[ \left| \int_{\mathbb{R}^m} g_i(x)h(x) \, dx \right| \leq |g_i|_\infty \cdot |h|_1 \leq 1 \]
for each $h \in \mathcal{V}$. Thus the sequence $\{g_i\}$ has a w*-cluster point $g \in \mathcal{X}$. As $\{g_i\}$ is supported in $C$, the support of $g$ is a subset of $C$. The sequence $\{\Theta^*(S_i)\}$ converges strongly to $T$, and a fortiori, it w*-converges to $T$. Equality (5.1) implies
\begin{align*}
\lim \langle \Theta^*(S_i), (f, 0) \rangle &= \lim \int_U f(x)g_i(x) \, dx = \int_U f(x)g(x) \, dx, \\
\lim \langle \Theta^*(S_i), (0, v) \rangle &= \lim \int_U g_i(x) \text{div} v(x) \, dx = \int_U g(x) \text{div} v(x) \, dx
\end{align*}
for each $(f, v) \in L^1(U) \times C^1_c(U; \mathbb{R}^m)$; the last equalities hold, since the right hand sides are cluster points of convergent sequences $\{\int_U f g_i\}$ and $\{\int_U g_i \text{div} v\}$. For each $v \in C^1_c(U; \mathbb{R}^m)$ with $|v|_\infty \leq 1$, the second equality in (5.2) implies
\[ \int_U g(x) \text{div} v(x) \, dx = \lim \int_U g_i(x) \text{div} v(x) \, dx \leq \sup \|g_i\| \leq c. \]
We infer $g \in BV^\infty$, and let $S := \Psi(g)$. By equalities (5.2) and (5.1),
\[ \langle T, (f, v) \rangle = \lim \langle \Theta^*(S_i), (f, v) \rangle \]
\[ = \lim \langle \Theta^*(S_i), (f, 0) \rangle + \lim \langle \Theta^*(S_i), (0, v) \rangle \]
\[ = \lim \int_U f(x)g_i(x) \, dx - \lim \int_U v \cdot d(Dg_i) \]
\[ = \int_U f(x)g(x) \, dx - \int_U v \cdot d(Dg) = \langle \Theta^*(S), (f, v) \rangle \]
for each $(f, v)$ in $L^1(U) \times C^1_c(U; \mathbb{R}^m)$. As $L^1(U) \times C^1_c(U; \mathbb{R}^m)$ is a dense subspace of $L^1_\text{loc} \times \mathcal{E}$, we see that $T = \Theta^*(S)$ belongs to $\Theta^*(CH^*)$. \hfill \qed

Theorem 5.2. Each charge is the sum of an ac-charge and an s-charge.

Proof. As in the proof of Theorem 3.6, we deduce from Proposition 5.1 and the Closed Range Theorem that $\Theta[L^1_\text{loc}(U) \times C(U; \mathbb{R}^m)]$ is a closed subspace of $CH(U)$. By [13, Proposition 4.2.2], the space $CH_{ac}(U) = \Theta[L^1_\text{loc}(U) \times \{0\}]$ is dense in $CH(U)$. Consequently
\[ CH_{ac}(U) + \mathcal{F}(U) = \Theta[L^1_\text{loc}(U) \times C(U; \mathbb{R}^m)] = CH(U), \]
and the theorem follows from Theorem 3.7. \hfill \qed

Remark 5.3. From [13, Proposition 4.2.2] and Lemma 3.1, we see that both spaces $CH_{ac}(U)$ and $CH_s(U)$ are dense in $CH(U)$.

6. The Gauss-Green theorem

According to Definition 2.1, the distributional divergence of $v \in C(U; \mathbb{R}^m)$ is defined as the flux $F_v$ of $v$. In this framework the Gauss-Green theorem is a mere tautology, which gains its usual meaning when the distribution $F_v$ is given by a function $f \in L^1_\text{loc}(U)$ [11, Proposition 4.1]. This is a well-known case: the flux $F_v$ is an ac-charge whose density $f$ is obtained by derivating $F_v$ with respect to a suitable derivation basis. However, one may wish to look at a more general situation when
$F_v$ is not an ac-charge, but still has a density $f$ obtained by derivation. Then $f$ is not in $L^1_{loc}(U)$, and two questions arise.

(i) When is $F_v$ determined uniquely by its density $f$?
(ii) If $F_v$ is determined uniquely by its density $f$, then how can $F_v$ be recovered from $f$?

Answers to these questions lead to extensions of the classical Gauss-Green theorem — a topic to which we devote the remainder of our paper.

For a bounded BV set $A$ contained in $U$, let

$$r(A) := \begin{cases} \frac{|A|}{d(A)} & \text{if } |A| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a sequence $\{A_i\}$ of bounded BV sets contained in $U$ tends to $x \in U$ if $x$ belongs to each $A_i$, $\lim d(A_i) = 0$, and $\inf r(A_i) > 0$. A charge $F$ in $U$ is derivable at $x \in U$ whenever a finite limit

$$DF(x) := \lim_{i} \frac{F(A_i)}{|A_i|}$$

exists for each sequence $\{A_i\}$ of bounded BV sets contained in $U$ that tends to $x$. The number $DF(x)$, called the derivative of $F$ at $x$, does not depend on a particular sequence $\{A_i\}$. If $v \in C(U; \mathbb{R}^m)$ is differentiable at $x \in U$, the it is easy to verify that the flux $F_v$ of $v$ is derivable at $x$ and $DF_v(x) = div v(x)$.

Denote by $CH_{DF}(U)$ the linear space of all charges in $U$ that are derivable at almost all $x \in U$, and by $L^0(U)$ the space of all measurable functions defined on $U$. According to Luzin’s theorem for charges [10], the map

$$D_U : F \mapsto DF : CH_{DF}(U) \to L^0(U),$$

is surjective, and we call it the derivation in $U$. By our choice of derivation basis, the derivation $D_U$ is a natural transformation of the functors $CH_D : U \mapsto CH_D(U)$ and $L^0 : U \mapsto L^0(U)$ defined on the category $\text{Lip}_{loc}$ of open subsets of $\mathbb{R}^m$ and proper local lipomorphisms [13, Section 4.6]. The map $D_U$ has a nontrivial kernel $CH_{sing}(U) := D_U^{-1}(0)$, whose elements are called singular charges in $U$. Consequently, $D_U$ has no natural right inverse. Notwithstanding, we may find a functorial subspace $X(U)$ of $CH_D(U)$ so that $X(U) \cap CH_{sing}(U) = \{0\}$, in which case the restriction $D_U \upharpoonright X(U)$ is a bijection from $X(U)$ onto $J_X(U) := D_U(X)$. We denote the inverse map

$$(D_U \upharpoonright X)^{-1} : J_X(U) \to X(U)$$

by $I_{X,U}$, and call it the integration in $U$ induced by $X$. Clearly, the integration $I_{X,U}$ is a natural transformation of the functors $X : U \to X(U)$ and $J_X : U \to J_X(U)$ defined on $\text{Lip}_{loc}$.

The following are classical examples of the procedure we described.

(1) Letting $X(U) := CH_{ac}(U)$, we obtain $J_X(U) = L^1_{loc}(U)$ and $I_{X,U}$ is the Lebesgue integration in $U$.

(2) Let $X(U) := CH_{BD}(U)$ be the linear space of all charges in $U$ that are derivable at each $x \in U$. Then $X(U) \cap CH_{sing}(U) = \{0\}$ by [13, Section 2.6], and
the resulting integration $I_{X,U}$ generalizes the *Newton integral* of elementary calculus.

Since neither of the spaces $CH_{ac}(U)$ and $CH_{DD}(U)$ contains the other, it is inviting to look for a functorial space $X(U) \subset CH_D(U)$ such that

$$CH_{ac}(U) + CH_{DD}(U) \subset X(U) \quad \text{and} \quad X(U) \cap CH_{sing}(U) = \{0\}.$$ 

While such a space $X(U)$ is by no means unique, practical considerations limit the choices. We seek an $X(U)$ that is large and well behaved — a delicate balancing act still open for investigation. Below we describe a particular definition of $X(U)$ that proved useful in applications.

A *gage* on a set $E \subset \mathbb{R}^m$ is a nonnegative function $\delta$ defined on $E$ such that the measure $\mathcal{H}\lfloor \{\delta = 0\}$ is $\sigma$-finite (see Remark 6.6 below for the motivation). Given $F \in CH(U)$ and $E \subset U$, let

$$V_\star F(E) := \sup_{\eta > 0} \inf_\delta \sup_{i=1}^p |F(A_i)|$$

where $\delta$ is a gage on $E$ and the supremum is taken over all collections

$$\{(A_1, x_1), \ldots, (A_p, x_p)\}$$

such that $A_1, \ldots, A_p$ are disjoint BV sets in $U$, and $x_i \in A_i$, $d(A_i) < \delta(x_i)$, and $r(A_i) > \eta$ for $i = 1, \ldots, p$.

It is not difficult to prove that $V_\star F : E \mapsto V_\star F(E)$ is a Borel regular measure in $U$ [13, Proposition 3.5.1]. It follows from [13, Proposition 3.5.3] that $V_\star F$ restricted to BV subsets of a compact interval $J \subset U$ is the least additive function larger than or equal to $|F \lfloor J|$. In particular $|F(J)| \leq V_\star F(J)$ for each compact interval $J \subset U$. An easy argument reveals that $F$ is an ac-charge if and only if $V_\star F$ is absolutely continuous and locally finite [13, Proposition 3.6.1]. This fact suggests the following definition.

**Definition 6.1.** An $F \in CH(U)$ is called an *ac-charge* if the measure $V_\star F$ is absolutely continuous.

Denoting by $CH_\star(U)$ the linear space of all ac-charges, it is immediate that $CH_{ac}(U) \subset CH_\star(U)$; in fact, it follows from Theorems 5.2 and 3.7 that

$$CH_\star(U) = CH_{ac}(U) + \mathcal{F}(U) \cap CH_\star(U).$$

A direct verification of the inclusion $CH_{DD}(U) \subset CH_\star(U)$ is straightforward [13, Theorem 3.6.7]. Establishing the functoriality of $CH_\star : U \mapsto CH_\star(U)$ on the category $\textbf{Lip}_{loc}$ is not difficult, but requires some work [13, Section 4.6]. On the other hand, proving the next fundamental theorem is hard. We refer the interested reader to [13, Sections 3.5 and 3.6].

**Theorem 6.2.** $CH_\star(U) \subset CH_D(U)$ and

$$V_\star F(E) = \int_E |DF(x)| \, dx$$

for each $F \in CH_\star(U)$ and each measurable set $E \subset U$. 


If $F \in CH_*(U) \cap CH_{sing}(U)$, then Theorem 6.2 yields $|F(J)| \leq V_c F(J) = 0$ for each compact interval $J \subset U$. From this and Remark 2.8, we obtain the following essential corollary.

**Corollary 6.3.** $CH_*(U) \cap CH_{sing}(U) = \{0\}$.

The next theorem, proved in [13, Section 4.5], is important for applications [13, Sections 5.2 and 5.3]. It indicates a good behavior of the space $CH_*(U)$.

**Theorem 6.4.** Let $F \in CH_*(U)$ and $g \in BV^{\infty}(U)$. Then $F \downarrow g \in CH_*(U)$ and

$$D(F \downarrow g)(x) = DF(x)g(x)$$

for almost all $x \in U$.

A vector field $v : U \to \mathbb{R}^m$ is called *pointwise Lipschitz* in a set $E \subset U$ if

$$\lim_{y \to x} \sup_{y \neq x} \frac{|v(y) - v(x)|}{|y - x|} < \infty$$

for each $x \in E$. By Stepanoff’s theorem [9, Theorem 3.1.9], a vector field $v$ that is pointwise Lipschitz in $E \subset U$ is differentiable at almost all $x \in E$; in particular, the classical $\text{div} \, v$ is defined almost everywhere in $E$. Now we can generalize the classical Gauss-Green theorem.

**Theorem 6.5.** Let $E \subset U$ be such that the measure $\mathcal{H}^1 E$ is $\sigma$-finite, and let $v \in C(U; \mathbb{R}^m)$ be pointwise Lipschitz in $U - E$. Then within $CH_*(U)$, the flux $F_v$ of $v$ is uniquely determined by the classical $\text{div} \, v$. If $\text{div} \, v$ belongs to $L^1_{loc}(U)$, then

$$F_v(A) = \int_A \text{div} \, v(x) \, dx$$

for each bounded $BV$ set $A$ with $\overline{A} \subset \subset U$.

**Proof.** Since $v$ is pointwise Lipschitz almost everywhere in $U$, Stepanoff’s theorem implies $DF_v(x) = \text{div} \, v(x)$ for almost all $x \in U$. However, more is true. Utilizing that $v$ is pointwise Lipschitz in $U - E$ and that the measure $\mathcal{H}^1 E$ is $\sigma$-finite, it is easy to find gages on negligible sets which demonstrate the absolute continuity of the measure $V_c F_v$. Consequently $F_v \in CH_*(U)$, and the first claim follows from Corollary 6.3. If $\text{div} \, v$ belongs to $L^1_{loc}(U)$, then the charge $G : A \mapsto \int_A \text{div} \, v(x) \, dx$ belongs to $CH_{ac}(U)$, and hence to $CH_*(U)$. By the classical derivability result,

$$DG(x) = \text{div} \, v(x) = DF_v(x)$$

for almost all $x \in U$, and another application of Corollary 6.3 completes the argument.

**Remark 6.6.** The simplicity of the previous proof is due to an application of Corollary 6.3. Of course, if $\text{div} \, v$ does not belong to $L^1_{loc}(U)$, we must address question (ii) concerning the recovery of $F_v$ from $\text{div} \, v$. The answer is affirmative: each $F \in CH_*(U)$ can be recovered from $DF$ by means of an averaging
process akin to the generalized Riemann integral of Henstock and Kurzweil [13, Section 5.5].

References


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