REMOVABLE SINGULARITIES
FOR THE EQUATION \( \text{div} \, v = 0 \)

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Abstract. A compact subset \( S \) of \( \mathbb{R}^N \) is removable for the equation \( \text{div} \, v = 0 \) if, by definition, every bounded Borel vectorfield whose distributional divergence vanishes outside \( S \) has a zero distributional divergence in the whole \( \mathbb{R}^N \). Here we establish that a compact subset of \( \mathbb{R}^N \) is removable for the equation \( \text{div} \, v = 0 \) if and only if its \((N-1)\)-dimensional Hausdorff measure vanishes.

1. Preliminaries and notations

Let \( N \geq 2 \) be an integer, and denote by \( \mathbb{R}^N \) the \( N \)-dimensional Euclidean space and by \( e_1, \ldots, e_N \) its usual basis. The usual inner product of \( x, y \in \mathbb{R}^N \) is written by \( x \cdot y \). The Euclidian norm of \( x \in \mathbb{R}^N \) is defined by \( |x| = \sqrt{x \cdot x} \), and we let \( B(x, r) \) (resp. \( B[x, r] \)) represent the open (resp. closed) ball in \( \mathbb{R}^N \) with center \( x \in \mathbb{R}^N \) and radius \( r > 0 \). Whenever \( S \subseteq \mathbb{R}^N \) is given, the notations \( \text{cl} \, S \), \( \text{int} \, S \), \( \partial S \) and \( d(S) \) stand for the closure, interior, boundary and diameter of \( S \) respectively, while \( |S| \) and \( \mathcal{H}^{N-1}(S) \) denote the Lebesgue measure and the \((N-1)\)-dimensional Hausdorff measure of \( S \), respectively. See [6] for details.

The set of all indefinitely differentiable functions with compact support in the open set \( \Omega \subseteq \mathbb{R}^N \) is denoted by \( \mathcal{D}(\Omega) \). The notation \( \mathcal{D}'(\Omega) \) stands for the space of all distributions in \( \Omega \). The support of \( T \in \mathcal{D}'(\Omega) \), denoted \( \text{supp} \, T \), is the set

\[
\mathbb{R}^N \setminus \bigcup \{ U \subseteq \Omega : U \text{ is open}, \langle T, \varphi \rangle = 0 \text{ whenever } \text{supp} \, \varphi \subseteq U \}.
\]

When \( \Omega \) is an open subset of \( \mathbb{R}^N \) and \( 1 \leq p < \infty \) is given, one defines \( L^p(\Omega) \) as the space of measurable functions \( u : \Omega \to \mathbb{R} \) for which \( |u|^p \) is Lebesgue-integrable on \( \Omega \). See [5] for details.

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Suppose $\Omega \subseteq \mathbb{R}^N$ is an open set. One says that $u \in L_{loc}^1(\Omega)$ is of bounded variation in $\Omega$ if and only if the extended real number

$$V(u, \Omega) = \sup \left\{ \int_{\Omega} u \, \text{div} \, w \, dx : \ w \in C^1_c(\Omega, \mathbb{R}^N), \|w\|_{\infty} \leq 1 \right\},$$

is finite, where $C^1_c(\Omega, \mathbb{R}^N)$ stands for the set of all vectorfields $w : \Omega \to \mathbb{R}^N$, of class $C^1$ in $\Omega$, with compact support in $\mathbb{R}^N$ and if we let $\|w\|_{\infty} = \max_{\Omega} |w|$ for $w \in C^1_c(\Omega, \mathbb{R}^N)$. The collection of all Lebesgue-integrable functions in $\Omega$ which have bounded variation in $\Omega$ is denoted $BV(\Omega)$:

$$BV(\Omega) = \left\{ u \in L^1(\Omega) : V(u, \Omega) < \infty \right\}.$$

The perimeter of a set $E \subseteq \mathbb{R}^N$ is the extended real number $P(E) = V(\chi_E, \mathbb{R}^N)$, where $\chi_E$ denotes the indicator function of $E$. In case $P(E)$ is finite, one says that $E$ is a set with finite perimeter in $\mathbb{R}^N$. The collection of all sets with finite perimeter in $\mathbb{R}^N$ will be denoted by $\mathcal{BV}$.

On the other hand, take a set $E \subseteq \mathbb{R}^N$ and a point $x \in \mathbb{R}^N$. One says $x \in \mathbb{R}^N$ is a dispersion point of the set $E \subseteq \mathbb{R}^N$ if the limit

$$\lim_{r \to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|},$$

vanishes, while $x$ is called a density point of $E$ is the limit in (1) has the value 1. The set of all density points of $E$ is called the essential interior of $E$ and denoted $\text{int}_e E$, while the essential closure of $E$ is denoted $\text{cl}_e E$ and defined by $\text{cl}_e E = \mathbb{R}^N \setminus \text{int}_e (\mathbb{R}^N \setminus E)$. Finally, the essential boundary of $E$, written by $\partial_e E$, satisfies $\partial_e E = \text{cl}_e E \setminus \text{int}_e E$. The inclusions $\text{int} E \subseteq \text{int}_e E \subseteq \text{cl}_e E \subseteq \text{cl} E$ are easy to check.

Whenever $E \subseteq \mathbb{R}^N$ is a bounded set with finite perimeter, there exists a vectorfield $\nu_E : \partial_e E \to \mathbb{R}^N$ (called the measure-theoretic exterior normal vector to $E$) satisfying $|\nu_E(x)| = 1$ for $\mathcal{H}^{N-1}$-a.e. $x \in \partial_e E$, and for which the Gauss-Green formula

$$\int_E \text{div} \, v \, dx = \int_{\partial_e E} v \cdot \nu_E \, d\mathcal{H}^{N-1},$$

holds whenever $v : \mathbb{R}^N \to \mathbb{R}^N$ is of class $C^1$ in $\mathbb{R}^N$. The theory of functions of bounded variation and sets of finite perimeter in $\mathbb{R}^N$ is developed, along with other subjects, in [6].

Rectifiable and purely unrectifiable sets are defined in [7].
2. The setting

The flux of a bounded Borel vectorfield $v : \mathbb{R}^N \to \mathbb{R}^N$ can be thought of as the set function

$$\mathcal{BV} \to \mathbb{R}, \quad E \mapsto \int_{\partial E} v \cdot \nu_E \, d\mathcal{H}^{N-1},$$

(2)

or as the distributional divergence of $v$,

$$\mathbf{div} v : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}, \quad \varphi \mapsto -\int_{\mathbb{R}^N} v \cdot \nabla \varphi \, dx.$$  

(3)

For our purposes, we shall use the second presentation in order to introduce removable sets.

2.1. Definition. Let $S \subseteq \mathbb{R}^N$ be a compact set. One says that $S$ is removable for the equation $\text{div} \, v = 0$ (or simply removable, in the sequel) if the following condition holds: for every bounded Borel vectorfield $v : \mathbb{R}^N \to \mathbb{R}^N$,

$$\text{supp}(\mathbf{div} \, v) \subseteq S \quad \text{implies} \quad \mathbf{div} \, v \equiv 0.$$  

In other words there is no bounded Borel vectorfield $v : \mathbb{R}^N \to \mathbb{R}^N$ whose nonzero distributional divergence vanishes outside a removable set. The main result of this note is the following.

2.2. Theorem. A compact set $S \subseteq \mathbb{R}^N$ is removable for the equation $\text{div} \, v = 0$ if and only if $\mathcal{H}^{N-1}(S) = 0$.

In order to prove the sufficient part of that result don’t need too much material. The following is contained in a work [4] of Th. De Pauw.

3. A sufficient condition for a compact set to be removable

Let

3.1. Lemma. Let $S \subseteq \mathbb{R}^N$ be a compact set satisfying $\mathcal{H}^{N-1}(S) = 0$. There exists a sequence $(\tilde{\chi}_n) \subseteq \mathcal{BV}(\mathbb{R}^N)$ satisfying the following conditions:

a) $\tilde{\chi}_n = 1$ in a neighbourhood of $S$, for each integer $n$;
b) $|\text{supp} \, \tilde{\chi}_n| \to 0$ as $n \to \infty$;
c) $V(\tilde{\chi}_n, \mathbb{R}^N) \to 0$ as $n \to \infty$. 

Proof. Let \( n \) be a nonnegative integer. Choose a finite collection of open cubes \( C_1^n, \ldots, C_m^n \) with \( d(C_j^n) \leq 1, 1 \leq j \leq m \), for which the following are satisfied:

\[
S \subseteq \bigcup_{j=1}^{m} C_j^n \quad \text{and} \quad \sum_{j=1}^{m} d(C_j^n)^{N-1} \leq \frac{1}{2N(n+1)}.
\]

Let \( U_n = \bigcup_{j=1}^{m} C_j^n \) and \( \tilde{\chi}_n = \chi_{U_n} \). One computes

\[
\text{V}(\tilde{\chi}_n, \mathbb{R}^N) = \mathcal{H}^{N-1}(\partial U_n) \leq \sum_{j=1}^{m} \mathcal{H}^{N-1}(\partial C_j^n),
\]

and

\[
\sum_{j=1}^{m} \mathcal{H}^{N-1}(\partial C_j^n) \leq 2N\sum_{j=1}^{m} d(C_j^n)^{N-1} \leq \frac{1}{n+1}.
\]

On the other hand, one has \(|\text{supp} \tilde{\chi}_n| = |U_n| \to 0 \text{ as } n \to \infty\). The result follows.

Using the regularization theorem for \( BV \) functions (see [10], section 5.3), one obtains the following corollary.

3.2. Corollary. Suppose \( S \subseteq \mathbb{R}^N \) compact and satisfies \( \mathcal{H}^{N-1}(S) = 0 \). There exists a sequence \((\chi_n) \subseteq \mathcal{D}(\mathbb{R}^N)\) satisfying the following conditions:

\( a) \ \chi_n = 1 \text{ in a neighbourhood of } S, \text{ for each } n; \)
\( b) \ |\text{supp} \chi_n| \to 0 \text{ as } n \to \infty; \)
\( c) \int_{\mathbb{R}^N} |\nabla \chi_n| \, dx \to 0 \text{ as } n \to \infty. \)

It is now easy to infer the removability of sets whose \((N-1)\)-dimensional Hausdorff measure vanishes.

3.3. Proposition. Suppose \( S \subseteq \mathbb{R}^N \) compact and satisfies \( \mathcal{H}^{N-1}(S) = 0 \). Then, \( S \) is removable for the equation \( \text{div} \ v = 0 \).

Proof. Let \( v : \mathbb{R}^N \to \mathbb{R}^N \) be a bounded Borel vectorfield whose distributional divergence is supported in \( S \). We have to show that \( \text{div} \ v \) is the zero distribution. For that purpose let \((\chi_n) \subseteq \mathcal{D}(\mathbb{R}^N)\) be a sequence associated to \( S \) by corollary 3.2. For \( \varphi \in \mathcal{D}(\mathbb{R}^N) \), write

\[
\langle \text{div} \ v, \varphi \rangle = \langle \text{div} \ v, \chi_n \varphi \rangle + \langle \text{div} \ v, (1 - \chi_n) \varphi \rangle,
\]

whenever \( n \) is a nonnegative integer. It is clear that one has \( \langle \text{div} \ v, (1 - \chi_n) \varphi \rangle = 0 \) by hypothesis and using the fact that \( (1 - \chi_n) \varphi \) is supported outside \( S \). On the other hand, the dominated convergence theorem guarantees that \( \lim_{n \to \infty} \langle \text{div} \ v, \chi_n \varphi \rangle = 0 \).
As we will see the condition $\mathcal{H}^{N-1}(S) = 0$ characterizes the removability property of compact subsets of $\mathbb{R}^N$.

4. Hausdorff measure of removable sets

4.1. A simple example. The co-area formula ([2], 3.40) yields the following result.

4.1. Proposition. Let $S \subseteq \mathbb{R}^N$ be a compact set, and $v : \mathbb{R}^N \to \mathbb{R}^N$ be a bounded Borel vectorfield which is continuous outside $S$. The following conditions are equivalent:

a) $\int_{\mathbb{R}^N} v \cdot \nabla \varphi \, dx = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$ satisfying $S \cap \text{supp} \, \varphi = \emptyset$;

b) $\int_{\partial E} v \cdot \nu_E \, d\mathcal{H}^{N-1} = 0$ for all bounded $E \in \mathcal{BV}$ with $S \cap \overline{E} = \emptyset$.

Moreover the conditions above imply that the following is satisfied:

c) $\int_{\partial E} v \cdot \nu_E \, d\mathcal{H}^{N-1} = 0$ for all bounded $E \in \mathcal{BV}$ for which $S \cap \partial E$ is closed and satisfies $\mathcal{H}^{N-1}(S \cap \partial E) = 0$.

The following example is selfexplanatory.

4.2. Example. Let $S = \mathbb{R}^{N-1} \times \{0\}$. We will show that $S$ is not removable for the equation $\text{div} \, v = 0$. Let $v : \mathbb{R}^N \to \mathbb{R}^N$ be the bounded Borel vectorfield defined by the formula $v(x) = \text{sgn}(x_N)e_N$. Define $E = [-1/2, 1/2]^N$. Of course, the distributional divergence of $v$ is supported in $S$. If $S$ were removable for the equation $\text{div} \, v = 0$, then the distributional divergence of $v$ would vanish identically in $\mathbb{R}^N$. This is not the case since

$$\int_{\partial E} v \cdot \nu_E \, d\mathcal{H}^{N-1} = 2.$$  

Similar construction can be made when $S$ is of the form $[a,b]^{N-1} \times \{0\}$ with real numbers $a < b$.

Adapting this construction one shows that a compact sumbanifold $M$ of $\mathbb{R}^N$ is removable for the equation $\text{div} \, v = 0$ if and only if $\mathcal{H}^{N-1}(M) = 0$.

4.2. Removability among rectifiable compact sets. The following result is due to contributions of Denjoy, Ahlfors & Beurling [1] and Coifman, McIntosh & Meyer [3].

4.3. Theorem. Suppose $S \subseteq \mathbb{R}^N$ is a compact $(N-1)$-rectifiable set with $\mathcal{H}^{N-1}(S) > 0$. Then there exists a Lipschitz function $u : \mathbb{R}^N \to \mathbb{R}$, harmonic outside $S$, but not harmonic in $\mathbb{R}^N$. 

Suppose $S \subseteq \mathbb{R}^N$ is a compact $(N - 1)$-rectifiable set satisfying $\mathcal{H}^{N-1}(S) > 0$. If we let $u$ be a Lipschitz map given by the previous result, and if we define $v = \nabla u$, one obtains a bounded vectorfield, smooth outside $S$, and whose distributional divergence doesn’t vanish in $\mathbb{R}^N$ while it does outside $S$. So we have the following.

4.4. Theorem. Suppose $S \subseteq \mathbb{R}^N$ is a compact $(N-1)$-rectifiable set with $\mathcal{H}^{N-1}(S) > 0$. Then $S$ is not removable for the equation $\text{div } v = 0$.

We could hope to have a proof of Theorem 4.4 which is inspired by the construction in Example 4.2 but, at this time, it seems to be a difficult and unsolved problem to construct “by hand” a vectorfield whose distributional divergence vanishes outside a $(N - 1)$-rectifiable compact subset of $\mathbb{R}^N$ with positive $(N - 1)$-dimensional Hausdorff measure, but not in the whole $\mathbb{R}^N$. Moreover, the proof of Theorem 4.3 relies on abstract Hahn-Banach-type duality arguments.

4.3. Removability among purely unrectifiable compact sets. The following result is due to Th. De Pauw, see [4].

4.5. Theorem. Suppose $S \subseteq \mathbb{R}^N$ is a compact, purely $(N - 1)$-unrectifiable set satisfying $\mathcal{H}^{N-1}(S) > 0$. Then $S$ is not removable for the equation $\text{div } v = 0$.

4.4. The general case. The following construction will allow us to prove that a compact set with positive $(N - 1)$-dimensional Hausdorff measure is not removable for the equation $\text{div } v = 0$.

Let $S \subseteq \mathbb{R}^N$ be compact and satisfy $\mathcal{H}^{N-1}(S) > 0$. Using ([7], 2.10.47) one can find a compact subset $S'$ of $S$ satisfying $0 < \mathcal{H}^{N-1}(S') < \infty$. Now let

$$\alpha = \sup \{ \mathcal{H}^{N-1}(R) : R \subseteq S' \text{ is (N - 1)-rectifiable} \} < \infty,$$

choose an increasing sequence $(R_n)$ of $(N - 1)$-rectifiable subsets of $S$ verifying $\mathcal{H}^{N-1}(R_n) \to \alpha$ as $n \to \infty$ and let $R = \bigcup_{n \in \mathbb{N}} R_n$. The set $R$ is $(N - 1)$-rectifiable, while $U = S' \setminus R$ is purely $(N - 1)$-unrectifiable. Either $\mathcal{H}^{N-1}(U) > 0$ or $\mathcal{H}^{N-1}(R) > 0$ as $S' = R \cup U$. In first case choose (using again [7], 2.10.47) a compact subset $K$ of $U$ with $\mathcal{H}^{N-1}(K) > 0$. Of course $K$ is purely $(N - 1)$-unrectifiable. In the second case a similar argument yields a compact $(N - 1)$-rectifiable subset $K$ of $S$ with $\mathcal{H}^{N-1}(K) > 0$. In both cases, $K$ is not removable for the equation $\text{div } v = 0$ using Theorems 4.4 and 4.5. As a subset of a removable one inherits its removability property, we can state the following result.

4.6. Theorem. Suppose $S \subseteq \mathbb{R}^N$ is a compact set with $\mathcal{H}^{N-1}(S) > 0$. Then $S$ is not removable for the equation $\text{div } v = 0$. 

Combining Proposition 3.3 and Theorem 4.6 yields us the following characterisation of removable sets for the equation \( \text{div} \, v = 0 \).

4.7. **Theorem.** A compact set \( S \subseteq \mathbb{R}^N \) is removable for the equation \( \text{div} \, v = 0 \) if and only if \( \mathcal{H}^{N-1}(S) = 0 \).

5. **Some generalization**

A simple observation shows that Theorem 4.4 remains true if the definition of a removable set is replaced by the following.

5.1. **Definition.** A compact set \( S \subseteq \mathbb{R}^N \) is said to be s-removable for the equation \( \text{div} \, v = 0 \) if the following condition holds: for every bounded Borel vectorfield \( v : \mathbb{R}^N \to \mathbb{R}^N \) of class \( C^\infty \) outside \( S \):

\[
\text{supp}(\text{div} \, v) \subseteq S \quad \text{implies} \quad \text{div} \, v \equiv 0.
\]

What about the purely unrectifiable case? In order to show Theorem 4.5 remains true for s-removability, let us recall a result of Th. De Pauw in [4].

5.2. **Lemma.** Suppose \( S \subseteq \mathbb{R}^N \) is a compact purely \((N-1)\)-unrectifiable set satisfying \( 0 < \mathcal{H}^{N-1}(S) < \infty \) and \( 1 \leq p < \infty \) is a real number. There exists a bounded Borel vectorfield \( w : \mathbb{R}^N \to \mathbb{R}^N \) satisfying the following conditions:

a) \( w \) is of class \( C^\infty \) outside \( S \),

b) \( \text{div} \, w \in L^p(\mathbb{R}^N) \),

c) there is a \( \varphi_* \in \mathcal{D}(\mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} \varphi_* \text{div} \, w \, dx \neq -\int_{\mathbb{R}^N} w \cdot \nabla \varphi_* \, dx \quad .
\] (4)

Let \( S \) satisfy the hypothesis of previous Lemma, choose a real number \( N < p < \infty \) and let \( w \) and \( \varphi_* \) be associated with \( S \) and \( p \) by the previous Lemma. Remark that calling \( u \) a solution of the Poisson equation \( \Delta u = \text{div} \, w \) and defining \( v = w - \nabla u \) one obtains a bounded Borel vectorfield of class \( C^\infty \) outside \( S \), continuous in \( \mathbb{R}^N \) (see [8], 4.2). As \( S \) is a set with finite \((N-1)\)-dimensional Hausdorff measure, the generalised integration by parts formula ([9], 2.10) guarantees the equalities

\[
\int_{\mathbb{R}^N} \varphi_* \text{div} \, w \, dx = \int_{\mathbb{R}^N} \varphi_* \text{div} \, \nabla u \, dx = -\int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi_* \, dx \quad ,
\]

are satisfied. Using (4), it yields

\[
\int_{\mathbb{R}^N} v \cdot \nabla \varphi_* \, dx \neq 0 \quad ,
\]
and so $\text{div} \, v$ is not the zero distribution. But one has $\langle \text{div} \, v, \varphi \rangle = 0$ whenever $\varphi$ is supported in $\mathbb{R}^N \setminus S$ using the integration by parts formula. With the previous computation, using the construction of section 4.4 and Proposition 3, one obtains the following generalisation of Theorem 4.7.

5.3. **Theorem.** A compact set $S \subseteq \mathbb{R}^N$ is s-removable for the equation $\text{div} \, v = 0$ if and only if $\mathcal{H}^{N-1}(S) = 0$.

**References**


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