A note on Hardy’s inequalities with boundary singularities

Mouhamed Moustapha Fall

Abstract. Let Ω be a smooth bounded domain in \( \mathbb{R}^N \) with \( N \geq 1 \). In this paper we study the Hardy-Poincaré inequalities with weight function singular at the boundary of Ω. In particular we give sufficient conditions so that the best constant is achieved.

Key Words: Hardy inequality, extremals, p-Laplacian.

1 Introduction

Let Ω be a domain in \( \mathbb{R}^N \), \( N \geq 1 \), with \( 0 \in \partial \Omega \) and \( p > 1 \) a real number. In this note, we are interested in finding minima to the following quotient

\[
\mu_{\lambda,p}(\Omega) := \inf_{u \in W_{0}^{1,p}(\Omega)} \frac{\int_{\Omega} |\nabla u|^p \, dx - \lambda \int_{\Omega} |u|^p \, dx}{\int_{\Omega} |x|^{-p}|u|^p \, dx},
\]

in terms of \( \lambda \in \mathbb{R} \) and \( \Omega \). If \( \lambda = 0 \), we have the \( \Omega \)-Hardy constant

\[
\mu_{0,p}(\Omega) = \inf_{u \in W_{0}^{1,p}(\Omega)} \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |x|^{-p}|u|^p \, dx}.
\]
which is the best constant in the Hardy inequality for maps supported by $\Omega$. The existence of extremals for $\mu_{3,2}(\Omega)$ was studied in [10] while for $\mu_{0,2}(\Omega)$, one can see for instance [6], [5], [21] and [19] for $\mu_{0,N}(\Omega)$.

Given a unit vector $\nu$ of $\mathbb{R}^N$, we consider the half-space $H := \{ x \in \mathbb{R}^N : x \cdot \nu \geq 0 \}$.

For $N = 1$, the following Hardy inequality is well known

$$ (p - 1) \int_0^\infty t^{-p} |u|^p dt \leq \int_0^\infty |u'|^p dt \quad \forall u \in W_0^{1,p}(0, \infty). $$

Moreover $\mu_{0,p}(H) = \left( \frac{p-1}{p} \right)^p$ is the $H$-Hardy constant and it is not achieved, see [15] for historical comments also.

For $N \geq 2$, it was recently proved by Nazarov [20] that the $H$-Hardy constant is not achieved and

$$ \mu_{0,p}(H) := \inf_{V \in W_0^{1,p}(\mathbb{S}_{+}^{N-1})} \frac{\int_{\mathbb{S}_{+}^{N-1}} \left( \frac{N-p}{p} \right)^2 |V|^2 + |\nabla_{\sigma} V|^2 \right)^{\frac{p}{2}} d\sigma}{\int_{\mathbb{S}_{+}^{N-1}} |V|^p d\sigma}, $$

where $\mathbb{S}_{+}^{N-1}$ is an $(N - 1)$-dimensional hemisphere. Notice that this problem always has a minimizer by the compact embedding $L^p(\mathbb{S}_{+}^{N-1}) \hookrightarrow W_0^{1,p}(\mathbb{S}_{+}^{N-1})$. The quantity $\mu_{0,p}(H)$ is explicitly known only in some special cases. Indeed, $\mu_{0,2}(H) = \frac{N^2}{4}$ while for $p = N$ then $\mu_{N,N}(H)$ is the first Dirichlet eigenvalue of the operator $-\text{div}(\nabla u |^{N-2} \nabla u)$ in $W_0^{1,N}(\mathbb{S}_{+}^{N-1})$ with the standard metric.

Problem (1.1) carries some similarities with the questions studied by Brezis and Marcus in [2], where the weight is the inverse-square of the distance from the boundary of $\Omega$ and $p = 2$. We also deal with this problem in the present paper for all $p > 1$ in Appendix A. We generalize here the existence result obtained by R.Musina and the author in [10] for any $p > 1$ and $N \geq 1$.

**Theorem 1.1** Let $p > 1$ and $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, with $0 \in \partial \Omega$. There exits $\lambda^*(p, \Omega) \in [-\infty, +\infty)$ such that

$$ \mu_{\lambda,p}(\Omega) < \mu_{0,p}(H), \quad \forall \lambda > \lambda^*(p, \Omega). $$

The infinimum in (1.1) is attained for any $\lambda > \lambda^*(p, \Omega)$. 

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The existence of \( \lambda^*(p, \Omega) \) comes from the fact that
\[
\sup_{\lambda \in \mathbb{R}} \mu_{\lambda, p}(\Omega) = \mu_{0, p}(H),
\]
see Lemma 2.2. Now observe that the mapping \( \lambda \mapsto \mu_{\lambda, p} \) is non-increasing. Moreover, for bounded domains \( \Omega \), letting \( \lambda_1 \) be the first Dirichlet eigenvalue of the \( p \)-Laplace operator \(-\text{div}(|\nabla u|^{p-2}\nabla u)\) in \( W^{1, p}_0(\Omega) \), it is plain that \( \mu_{\lambda_1, p}(\Omega) = 0 \). Then we define
\[
\lambda^*(p, \Omega) := \inf \{ \lambda \in \mathbb{R} : \mu_{\lambda, p}(\Omega) < \mu_{0, p}(H) \}
\]
so that \( \mu_{\lambda, p} < \mu_{0, p}(H) \) for all \( \lambda > \lambda^*(p, \Omega) \). In particular \( \lambda^*(p, \Omega) \leq \lambda_1 \). On the other hand there are various bounded smooth domains \( \Omega \) with \( 0 \in \partial \Omega \) such that \( \lambda^*(p, \Omega) \in [-\infty, 0) \), see Proposition 2.5 and Proposition 2.6. Furthermore if \( N = 1 \) then \( \mu_{0, p}(\mathbb{R} \setminus \{0\}) = \left( \frac{p-1}{p} \right)^p = \mu_{0, p}(H) \) thus \( \lambda^*(p, \Omega) \geq 0 \).

It is obvious that if \( \Omega \) is contained in a half-ball centered at the origin then \( \mu_{0, p}(\Omega) = \mu_{0, p}(H) \) thus \( \lambda^*(p, \Omega) \geq 0 \) and in addition
\[
\lambda^*(p, \Omega) = \inf_{u \in W^{1, p}_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^p \, dx - \mu_{0, p}(H) \int_{\Omega} |x|^{-p}|u|^p \, dx}{\int_{\Omega} |u|^p \, dx}.
\]

We have obtained the following result.

**Theorem 1.2** If \( \Omega \) is contained in a half-ball centered at the origin then there exists a constant \( c(N, p) > 0 \) such that
\[
(1.6) \quad \lambda^*(p, \Omega) \geq \frac{c(N, p)}{\text{diam}(\Omega)^p}.
\]

The constant \( c(N, p) \) appearing in (1.6) has the property that \( c(N, 2) \) is the first Dirichlet eigenvalue of \(-\Delta\) in the unit disc of \( \mathbb{R}^2 \). This type of estimates was first proved by Brezis-Vázquez in [3] when \( p = 2, N \geq 2 \) and later on, extended to the case \( 1 < p < N \) by Gazzola-Grunau-Mitidieri in [13] when dealing with \( \mu_{0, p}(\mathbb{R}^N \setminus \{0\}) := \left| \frac{N-p}{p} \right|^p \). More precisely they proved the existence of a positive constant \( C(N, p) \) such that for any open subset \( \Omega \) of \( \mathbb{R}^N \), there holds
\[
(1.7) \quad \int_{\Omega} |\nabla u|^p - \mu_{0, p}(\mathbb{R}^N \setminus \{0\}) \int_{\Omega} |x|^{-p}|u|^p \geq C(N, p) \left( \frac{\omega_N}{|\Omega|} \right)^{\frac{p}{N}} \int_{\Omega} |u|^p \quad \forall u \in W^{1, p}_0(\Omega),
\]
where $|\Omega|$ is the measure of $\Omega$ and $\omega_N$ the measure of the unit ball of $\mathbb{R}^N$. The constant $C(N, p)$ was explicitly given and $C(N, 2) = c(N, 2)$ as was obtained in [3].

The main ingredients to prove (1.7) is the Schwarz symmetrization and a "dimension reduction" via the transformation $x \mapsto \frac{x}{\omega}$, where $\omega(x) = |x|^\frac{N-p}{p}$ satisfies

$$\text{div}(|\nabla \omega|^{p-2} \nabla \omega) + \mu_{0,p}(\mathbb{R}^N \setminus \{0\}) |x|^{-p} \omega^{p-1} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$ 

For $p = 2$, the lower bound in (1.6) was obtained in [10] by a similar transformation and using the Poincaré inequality on $S^{N-1}$. However, in view of (1.4), such argument do not apply here when $p \neq 2$ and $p \neq N$. By analogy, to reduce the dimension, we will consider the mapping $x \mapsto \frac{x}{v}$, where $v(x) := |x|^{\frac{N-p}{p}} V \left( \frac{x}{|x|} \right)$ is a weak solution to the equation

$$\text{div}(|\nabla v|^{p-2} \nabla v) + \mu_{0,p}(H) |x|^{-p} |v|^{p-2} v = 0 \quad \text{in } D'(H)$$

whenever $V$ is a minimizer of (1.4). Then exploiting the strict convexity of the mapping $a \mapsto |a|^p$, estimate (1.6), for $p \geq 2$, follows immediately while the case $p \in (1, 2)$ carries further difficulties as it can be seen in Section 2.2.

The argument to prove the attainability of $\mu_{\lambda,p}(\Omega)$ is taken from de Valeriola-Willem [7]. It allows to show that, up to a subsequence, the gradient of the Palais-Smale sequences converges point-wise almost every where. Therefore an application of the Brezis-Lieb lemma with some simples arguments yields the existence of extremals.

### 2 Hardy inequality with one point singularity

Let $C$ be a proper cone in $\mathbb{R}^N$, $N \geq 2$ and put $\Sigma := C \cap S^{N-1}$. It was shown in [20] that the $C$-Hardy constant is not achieved and it is given by

$$\mu_{0,p}(C) = \inf_{V \in W^{1,p}_0(\Sigma)} \frac{\int_{\Sigma} \left( \frac{N-p}{p} \right)^2 |V|^2 + |\nabla_{\sigma} V|^2 \right)^{\frac{p}{2}} d\sigma}{\int_{\Sigma} |V|^p d\sigma}.$$
Letting $V \in W^{1,p}_0(\Sigma)$ be the positive minimizer to this quotient then the function

$$v(x) := |x|^{\frac{p-N}{p}} V \left( \frac{x}{|x|} \right)$$

satisfies

$$\int_C |\nabla v|^{p-2} \nabla v \cdot \nabla h = \mu_{0,p} (C) \int_C |x|^{-p} v^{p-1} h \quad \forall h \in C^1_c (C).$$

Notice that $\mu_{0,2} (C) = (\frac{N-2}{2})^2 + \lambda_1 (\Sigma)$, where $\lambda_1 (\Sigma)$ is the first Dirichlet eigenfunction of the Laplace operator on $\Sigma$ endowed with the standard metric on $S^{N-1}$. This was obtained in [21], [19] and [10].

### 2.1 Existence

In this Section we show that the condition $\mu_{\lambda,p} (\Omega) < \mu_{0,p} (H)$ is sufficient to guaranty the existence of a minimizer for $\mu_{\lambda,p} (\Omega)$.

We emphasize that throughout this section, $\Omega$ can be taken to be an open set satisfying the uniform sphere condition at $0 \in \partial \Omega$. Namely there are balls $B_+ \subset \Omega$ and $B_- \subset \mathbb{R}^N \setminus \Omega$ such that $\partial B_+ \cap \partial B_- = \{0\}$. This holds if $\partial \Omega$ is of class $C^2$ at 0, see [[16] 14.6 Appendix]. We start with the following approximate local Hardy inequality.

**Lemma 2.1** Let $\Omega$ be a smooth domain in $\mathbb{R}^N$, $N \geq 1$, with $0 \in \partial \Omega$ and let $p > 1$. Then for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that

$$\mu_{0,p} (\Omega \cap B_{r_\varepsilon} (0)) \geq \mu_{0,p} (H) - \varepsilon,$$

where $B_{r_\varepsilon} (0)$ is a ball of radius $r$ centered at 0.

**Proof.** If $N = 1$ then (2.4) is an immediate consequence of (1.3). From now on we can assume that $N \geq 2$. We denote by $N_{\partial \Omega}$ the unit normal vector-field on $\partial \Omega$. Up to a rotation, we can assume that $N_{\partial \Omega} (0) = E_N$, so that the tangent plane of $\partial \Omega$ at 0 coincides with $\mathbb{R}^{N-1} = \text{span} \{ E_1, \ldots, E_{N-1} \}$. Denote by $B^+_r = \{ y \in B_r (0) : y^N > 0 \}$. For $r > 0$ small, we introduce the following system of coordinates centered at 0 (see [9]) via the mapping $F : B^+_r \to \Omega$ given by

$$F(y) = \text{Exp}_0 (\tilde{y}) + y^N N_{\partial \Omega} (\text{Exp}_0 (\tilde{y})), $$

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where \( \tilde{y} = (y^1, \ldots, y^{N-1}) \) and \( \tilde{y} \mapsto \exp_0(\tilde{y}) \in \partial \Omega \) is the exponential mapping of \( \partial \Omega \) endowed with the metric induced by \( \mathbb{R}^N \). This coordinates induces a metric on \( \mathbb{R}^N \) given by 

\[
\dot{g}_{ij}(y) = \langle \partial_i F(y), \partial_j F(y) \rangle
\]

for \( i, j = 1, \ldots, N \). Let \( u \in C_c^\infty(F(B_r^+)) \) and put \( v(y) = u(F(y)) \) then

\[
\int_{F(B_r^+)} |\nabla u|^p \, dx = \int_{B_r^+} |\nabla v|^p \sqrt{|g|} \, dy, \quad \int_{F(B_r^+)} |x|^{-p} |u|^p \, dx = \int_{B_r^+} |F(y)|^{-p} |v|^p \sqrt{|g|} \, dy,
\]

with \(|g|\) stands for the determinant of the \( g \) while \( |\nabla v|^p_g = g(\nabla v, \nabla v)^\frac{p}{2} \). Since \(|F(y)| = |y| + O(|y|^2)\) and \( g_{ij}(y) = \delta_{ij} + O(|y|) \), we infer that

\[
\frac{\int_{B_r^+} |\nabla v|^p \sqrt{|g|} \, dy}{\int_{B_r^+} |F(y)|^{-p} |v|^p \sqrt{|g|} \, dy} \geq (1 - Cr) \frac{\int_{B_r^+} |\nabla v|^p \, dy}{\int_{B_r^+} |g|^{-p} |v|^p \, dy},
\]

for some constant \( C > 0 \) depending only on \( \Omega \) and \( p \). Furthermore since \( \mu_{0,p}(B_r^+) \geq \mu_{0,p}(H) \), using (2.5) we conclude that

\[
\mu_{0,p}(F(B_r^+)) \geq (1 - Cr) \mu_{0,p}(H).
\]

We are in position to prove (1.5) in the following

Lemma 2.2 Let \( \Omega \) be a smooth domain in \( \mathbb{R}^N \), \( N \geq 1 \), with \( 0 \in \partial \Omega \) and let \( p > 1 \). Then there exists \( \lambda^*(p, \Omega) \in [0, +\infty) \) such that

\[
\mu_{\lambda,p}(\Omega) < \mu_{0,p}(H) \quad \forall \lambda > \lambda^*(p, \Omega).
\]

Proof. We first show that

\[
\sup_{\lambda \in \mathbb{R}} \mu_{\lambda,p}(\Omega) = \mu_{0,p}(H).
\]

Step 1: We claim that \( \sup_{\lambda \in \mathbb{R}} \mu_{\lambda,p}(\Omega) \geq \mu_{0,p}(H) \). For \( r > 0 \) small, we let \( \psi \in C_c^\infty(B_r(0)) \) with \( 0 \leq \psi \leq 1, \psi \equiv 0 \) in \( \mathbb{R}^N \setminus B_{\frac{r}{2}}(0) \) and
ψ ≡ 1 in $B_{\frac{1}{4}}(0)$. For a fixed $\varepsilon > 0$ small, there holds
\[
\int_{\Omega} |x|^{-p}|u|^p = \int_{\Omega} |x|^{-p}|\psi u + (1 - \psi) u|^p \\
\leq (1 + \varepsilon) \int_{\Omega} |x|^{-p}|\psi u|^p + c(\varepsilon) \int_{\Omega} |x|^{-p}(1 - \psi)^p|u|^p \\
\leq (1 + \varepsilon) \int_{\Omega} |x|^{-p}|\psi u|^p + c(\varepsilon) \int_{\Omega} |u|^p.
\]
Now by (2.4)
\[
(\mu_{0,p}(H) - \varepsilon) \int_{\Omega} |x|^{-p}|\psi u|^p \leq \int_{\Omega} |\nabla(\psi u)|^p
\]
and hence
\[
(\mu_{0,p}(H) - \varepsilon) \int_{\Omega} |x|^{-p}|\psi u|^p \leq (1 + \varepsilon) \int_{\Omega} |\nabla(\psi u)|^p + c(\varepsilon) \int_{\Omega} |u|^p.
\]
Since $|\nabla(\psi u)|^p \leq (\psi |\nabla u| + |u||\nabla \psi|)^p$ we deduce that
\[
|\nabla(\psi u)|^p \leq (1 + \varepsilon)\psi |\nabla u|^p + c|u|^p |\nabla \psi|^p \leq (1 + \varepsilon)|\nabla u|^p + c|u|^p.
\]
Using (2.7), we conclude that
\[
(\mu_{0,p}(H)) \int_{\Omega} |x|^{-p}|\psi u|^p \leq (1 + \varepsilon) \int_{\Omega} |\nabla u|^p + c \int_{\Omega} |u|^p.
\]
This implies that $\mu_{0,p}(H) \leq \sup_{\lambda \in \mathbb{R}} \mu_{\lambda,p}(\Omega)$ and the claim follows.

**Step 2:** We claim that $\sup_{\lambda \in \mathbb{R}} \mu_{\lambda,p}(\Omega) \leq \mu_{0,p}(H)$.

Denote by $\nu$ the unit interior normal of $\partial \Omega$. For $\delta \geq 0$ we consider the cone
\[
C^\delta_+ := \{ x \in \mathbb{R}^N \mid x \cdot \nu > \delta |x| \}
\]
and put $\Sigma_\delta = C^\delta_+ \cap S^{N-1}$. For every $\eta > 0$, let $V \in C^\infty_c(\Sigma_0)$ such that
\[
\int_{\Sigma_0} \left( \left( \frac{N - p}{p} \right)^2 |V|^2 + |\nabla_\sigma V|^2 \right)^\frac{p}{2} d\sigma \\
\leq \mu_{0,p}(H) + \eta.
\]
On the other hand, there exists $\delta > 0$ small such that $\text{supp} V \subset \Sigma_\delta$. From this we conclude that
\[
(2.9) \quad \mu_{0,p}(H) \leq \mu_{0,p}(C^\delta_+) \leq \mu_{0,p}(H) + \eta.
\]
Since $\partial \Omega$ is smooth at 0, for every $\delta > 0$, there exists $r_\delta > 0$ such that $C^\delta \cap B_r(0) \subset \Omega$ for all $r \in (0, r_\delta)$. Clearly by scale invariance, $\mu_{0,p}(C^\delta \cap B_r(0)) = \mu_{0,p}(C^\delta)$. For $\varepsilon > 0$, we let $\phi \in W^{1,p}_0(C^\delta \cap B_r(0))$ such that

\[ \int_{C^\delta \cap B_r(0)} |\nabla \phi|^p \, dx \leq \mu_{0,p}(C^\delta) + \varepsilon. \]

From this we deduce that

\[ \mu_{\lambda,p}(\Omega) \leq \frac{\int_{C^\delta \cap B_r(0)} |\nabla \phi|^p \, dx - \lambda \int_{C^\delta \cap B_r(0)} |\phi|^p \, dx}{\int_{C^\delta \cap B_r(0)} |x|^{-p} |\phi|^p \, dx} \leq \mu_0(C^\delta) + \varepsilon + |\lambda| \frac{\int_{C^\delta \cap B_r(0)} |\phi|^p \, dx}{\int_{C^\delta \cap B_r(0)} |x|^{-p} |\phi|^p \, dx}. \]

Since $\int_{C^\delta \cap B_r(0)} |x|^{-p} |\phi|^p \, dx \geq r^{-p} \int_{C^\delta \cap B_r(0)} |\phi|^p \, dx$, we get

\[ \mu_{\lambda,p}(\Omega) \leq \mu_{0,p}(C^\delta) + \varepsilon + r^p|\lambda|. \]

The claim follows immediately by (2.9). Therefore (2.6) is proved.

Finally as the map $\lambda \mapsto \mu_{\lambda,p}(\Omega)$ is non increasing while $\mu_{\lambda_1,p}(\Omega) = 0 < \mu_{0,p}(H)$, we can set

\[ \lambda^*(p, \Omega) := \inf\{\lambda \in \mathbb{R} : \mu_{\lambda,p}(\Omega) < \mu_{0,p}(H)\} \]

so that $\lambda^*(p, \Omega) < \mu_{0,p}(H)$ for any $\lambda > \lambda^*(p, \Omega)$.

**Remark 2.3** Observe that the proof of Lemma 2.2 highlights that

\[ \lim_{r \to 0} \mu_{0,p}(\Omega \cap B_r(0)) = \mu_{0,p}(H) = \lim_{\lambda \to -\infty} \mu_{\lambda,p}(\Omega). \]

**Proof of Theorem 1.1**

Let $\lambda > \lambda^*(p, \Omega)$ so that $\mu_{\lambda,p}(\Omega) < \mu_{0,p}(H)$. We define the mappings $F, G$ :
\( W^{1,p}_0(\Omega) \to \mathbb{R} \) by

\[
F(u) = \int_{\Omega} |\nabla u|^p - \lambda \int_{\Omega} |u|^p
\]

and

\[
G(u) = \int_{\Omega} |x|^p |u|^p.
\]

By Ekeland variational principal, there is a minimizing sequence \( u_n \in W^{1,p}_0(\Omega) \) normalized so that

\[
G(u_n) = 1, \quad \forall n \in \mathbb{N}
\]

and with the properties that

\[
F(u_n) \to \mu_{\lambda,p}(\Omega),
\]

\[
J(u_n) = F'(u_n) - \mu_{\lambda,p}(\Omega) G'(u_n) \to 0 \text{ in } (W^{1,p}_0(\Omega))'.
\]

Up to a subsequence, we can assume that there exists \( u \in W^{1,p}_0(\Omega) \) such that

\[
\nabla u_n \rightharpoonup \nabla u \text{ in } L^p(\Omega),
\]

\( u_n \to u \) in \( L^p(\Omega) \) and \( u_n \to u \) a.e. in \( \Omega \). Moreover by (2.8), we may assume that

\[
|x|^{-1} u_n \rightharpoonup |x|^{-1} u \text{ in } L^p(\Omega).
\]

We set \( \theta_n = u_n - u \) and

\[
T(s) = \begin{cases} 
  s & \text{if } |s| \leq 1 \\
  \frac{s}{|s|} & \text{if } |s| > 1
\end{cases}
\]

It follows that for every \( r \geq 1 \)

\[
\int_{\Omega} |T(\theta_n)|^r \to 0.
\]

Moreover notice that

\[
\int_{\Omega} \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla T(\theta_n) = \langle J(u_n), T(\theta_n) \rangle + \mu_{\lambda,p}(\Omega) \int_{\Omega} |x|^{-p} |u|^{p-2} u_n T(\theta_n)
\]

\[
+ \lambda \int_{\Omega} |u_n|^{p-2} u_n T(\theta_n) - \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla T(\theta_n).
\]

Therefore by (2.10), (2.11) and (2.12) we infer that

\[
\int_{\Omega} \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla T(\theta_n) \to 0.
\]
Consequently by [7]-Theorem 1.1,

\[ \lim_{n \to \infty} \left( \int_\Omega |\nabla u_n|^p - \int_\Omega |\nabla \theta_n|^p \right) = \int_\Omega |\nabla u|^p. \]  

By Brezis-Lieb Lemma [4]

\[ 1 - \lim_{n \to \infty} \int_\Omega |x|^{-p}|\theta_n|^p = \int_\Omega |x|^{-p}|u|^p. \]

Fix \( \varepsilon > 0 \) small. By (2.8) and Rellich, there exists \( \lambda_\varepsilon \) such that

\[ (\mu_{0,p}(H) - \varepsilon) \int_\Omega |x|^{-p}|\theta_n|^p \leq \int_\Omega |\nabla \theta_n|^p - \lambda \varepsilon \int_\Omega |\theta_n|^p = \int_\Omega |\nabla \theta_n|^p + o(1). \]

Using this together with (2.13) and (2.14) we get

\[
\begin{align*}
\mu_{\lambda,p}(\Omega) \int_\Omega |x|^{-p}|u|^p &\leq \int_\Omega |\nabla u|^p - \lambda \int_\Omega |u|^p - \int_\Omega |\nabla u_n|^p - \lambda \int_\Omega |u_n|^p + o(1) \\
&\leq F(u_n) - (\mu_{0,p}(H) - \varepsilon) \int_\Omega |x|^{-p}|\theta_n|^p + o(1) \\
&\leq \mu_{\lambda,p}(\Omega) - (\mu_{0,p}(H) - \varepsilon) \left( 1 - \int_\Omega |x|^{-p}|u|^p \right) + o(1) \\
&\leq \mu_{\lambda,p}(\Omega) - \mu_{0,p}(H) + \varepsilon + (\mu_{0,p}(H) - \varepsilon) \int_\Omega |x|^{-p}|u|^p + o(1).
\end{align*}
\]

Send \( n \to \infty \) and then \( \varepsilon \to 0 \) to get

\[ (\mu_{\lambda,p}(\Omega) - \mu_{0,p}(H)) \int_\Omega |x|^{-p}|u|^p \leq \mu_{\lambda,p}(\Omega) - \mu_{0,p}(H). \]

Hence \( \int_\Omega |x|^{-p}|u|^p \geq 1 \) because \( \mu_{\lambda,p}(\Omega) - \mu_{0,p}(H) < 0 \) and the proof is complete.

As a consequence of the existence theorem, we have

**Corollary 2.4** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \), \( N \geq 2 \), with \( 0 \in \partial \Omega \). Then

\[ \mu_{0,p} \left( \mathbb{R}^N \setminus \{0\} \right) = \left| \frac{N - p}{p} \right| < \mu_{0,p}(\Omega) \leq \mu_{0,p}(H). \]

**Proof.** By (2.6) \( 0 < \mu_{0,p}(\Omega) \leq \mu_{0,p}(H) \). If the strict inequality holds, then there exists a positive minimizer \( u \in W^{1,p}_0(\Omega) \) for \( \mu_{0,p}(\Omega) \) by Theorem 1.1. But then \( \mu_{0,p} \left( \mathbb{R}^N \setminus \{0\} \right) < \mu_{0,p}(\Omega) \), because otherwise a null extension of \( u \) outside \( \Omega \) would achieve the Hardy constant in \( \mathbb{R}^N \setminus \{0\} \) which is not possible.
As mentioned earlier, we shall show that there are smooth bounded domains in $\mathbb{R}^N$ such that $\lambda^*(p, \Omega) \in [-\infty, 0)$. These domains might be taken to be convex or even flat at 0. For that we let $\nu \in S^{N-1}$ and $\delta, r, R > 0$. We consider the sector

\[(2.15) \quad C^\delta_{r,R} := \{ x \in \mathbb{R}^N \mid x \cdot \nu > -\delta|x|, r < |x| < R \}.\]

**Proposition 2.5** Let $N \geq 2$ and $p > 1$. Then for all $\delta \in (0, 1)$, there exist $r, R > 0$ such that if a domain $\Omega$ contains $C^\delta_{r,R}$ then $\mu_{0,p}(\Omega) < \mu_{0,p}(H)$.

**Proof.** Consider the cone

\[C^\delta := \{ x \in \mathbb{R}^N \mid x \cdot \nu > -\delta|x| \}\]

Notice that by Harnack inequality $\mu(C^\delta) < \mu(C^{\delta'})$ for any $0 \leq \delta' < \delta < 1$. Thus for any $\delta \in (0, 1)$, we can find $u \in C^\infty_c(C^\delta)$ such that

\[\int_{C^\delta} |\nabla u|^p < \mu_{0,p}(H).\]

Hence we choose $r, R > 0$ so that $\text{supp } u \subset C^\delta_{r,R}$. \qed

By Corollary 2.4, starting from exterior domains, one can also build various example of (possibly annular) domains for which $\lambda^*(p, \Omega) < 0$. The following argument is taken in [Ghoussoub-Kang [14] Proposition 2.4]. If $U \subset \mathbb{R}^N$, $N \geq 2$, is a smooth exterior domain (the complement of a smooth bounded domain) with $0 \in \partial U$ then by scale invariance $\mu_{0,p}(U) = \mu_{0,p}(\mathbb{R}^N \setminus \{0\})$. We let $B_r(0)$ a ball of radius $r$ centered at the 0 and define $\Omega_r := B_r(0) \cap U$ then clearly the map $r \mapsto \mu(\Omega_r)$ is decreasing with

\[(2.16) \quad \mu_{0,p}(\mathbb{R}^N \setminus \{0\}) = \inf_{r>0} \mu_{0,p}(\Omega_r) \quad \text{and} \quad \mu_{0,p}(H) = \sup_{r>0} \mu_{0,p}(\Omega_r).\]

We have the following result for which the proof is similar to the one given in [14] by Corollary 2.4 and Harnack inequality.

**Proposition 2.6** There exists $r_0 > 0$ such that the mapping $r \mapsto \mu_{0,p}(\Omega_r)$ is left-continuous and strictly decreasing on $(r_0, +\infty)$. In particular

\[\mu_{0,p}(\mathbb{R}^N \setminus \{0\}) < \mu_{0,p}(\Omega_r) < \mu_{0,p}(H), \quad \forall r \in (r_0, +\infty).\]
2.2 Remainder term

We know that for domains $\Omega$ contained in a half-ball $\lambda^*(p, \Omega) \geq 0$. Our aim in this section is to obtain positive lower bound for $\lambda^*(p, \Omega)$ by providing a remainder term for Hard’s inequality in these domains. In [13], Gazzola-Grunau-Mitidieri proved the following improved Hardy inequality for $1 < p < N$:

$$
(2.17) \quad \int_{\Omega} |\nabla u|^p - \mu_{0,p} (\mathbb{R}^N \setminus \{0\}) \int_{\Omega} |x|^{-p} |u|^p \geq C(N, p) \left( \frac{\omega_N}{|\Omega|} \right)^\frac{p}{N} \int_{\Omega} |u|^p,
$$

that holds for any bonded domain $\Omega$ of $\mathbb{R}^N$ and $u \in W^{1,p}_0(\Omega)$. Here the constant $C(N, p) > 0$ is explicitly given while $C(N, 2)$ is the first Dirichlet eigenvalue of $-\Delta$ of the unit disc in $\mathbb{R}^2$.

We shall show that such type of inequality holds in the case where the singularity is placed at the boundary of the domain. To this end, we will use the function $v(x) := |x|^{\frac{2}{p}} V \left( \frac{x}{|x|} \right)$ defined in (2.2) to “reduce the dimension”.

Throughout this section, we assume that $N \geq 2$ since the case $N = 1$ was already proved by Tibodolm [22] Theorem 1.1. Indeed, he showed that

$$
\int_0^1 |u'(r)|^p dr - \mu_{0,p}(H) \int_0^1 r^{-p} |u(r)|^p dr \geq (p-1)^2 \int_0^1 |u(r)|^p dr, \quad \forall u \in W^{1,p}_0(0, 1).
$$

We start with conic domains

$$
C_{\Sigma} = \{ x = r\sigma \in \mathbb{R}^N \mid r \in (0, 1), \sigma \in \Sigma \},
$$

where $\Sigma$ is a domain properly contained in $\mathbb{S}^{N-1}$ and having a Lipschitz boundary. We will denote by $V$ the positive minimizer of (2.1) in $\Sigma$ while $v(x) := |x|^{\frac{2}{p}} V \left( \frac{x}{|x|} \right)$ satisfies (2.3) in the infinite cone $\{ x = r\sigma \in \mathbb{R}^N \mid r \in (0, +\infty), \sigma \in \Sigma \}$. Finally we remember that by Harnack inequality $v \in L^{\infty}_{loc}(C_{\Sigma})$.

Recall the following inequalities (see [17] Lemma 4.2) which will be useful in the remaining of the paper. Let $p \in [2, \infty)$ then for any $a, b \in \mathbb{R}^N$

$$
(2.18) \quad |a + b|^p \geq |a|^p + \frac{1}{2p-1-1} |b|^p + p|a|^{p-2} a \cdot b.
$$

If $p \in (1, 2)$ then for any $a, b \in \mathbb{R}^N$

$$
(2.19) \quad |a + b|^p \geq |a|^p + c(p) \frac{|b|^2}{(|a| + |b|)^{2-p}} + p|a|^{p-2} a \cdot b.
$$

We first make the following observation.

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Lemma 2.7 Let \( u \in C^\infty_c(C_\Sigma) \), \( u \geq 0 \). Set \( \psi = \frac{u}{v} \) then

If \( p \geq 2 \)

\[ \left( 2.20 \right) \int_{C_\Sigma} |\nabla u|^p - \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} |x|^{-p}|u|^p \geq \frac{1}{2p-1 - 1} \int_{C_\Sigma} |v\nabla \psi|^p, \]

If \( 1 < p < 2 \)

\[ \left( 2.21 \right) \int_{C_\Sigma} |\nabla u|^p - \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} |x|^{-p}|u|^p \geq c(p) \int_{C_\Sigma} \frac{|v\nabla \psi|^2}{(|v\nabla \psi| + |\psi \nabla v|)^{2-p}}, \]

**Proof.** We prove only the case \( p \geq 2 \) as the case \( p \in (1, 2) \) goes similarly. Notice that \( \nabla u = v\nabla \psi + \psi \nabla v \) then we use the inequality (2.18) with \( a = v\nabla \psi \) and \( b = \psi \nabla v \) to get

\[ \int_{C_\Sigma} |\nabla u|^p \geq \int_{C_\Sigma} |\psi \nabla v|^p + p \int_{C_\Sigma} |\psi \nabla v|^{p-2} \psi \nabla v \cdot (v \nabla \psi) + \frac{1}{2p-1 - 1} \int_{C_\Sigma} |v\nabla \psi|^p. \]

It is plain that

\[ p|\psi \nabla v|^{p-2} \psi \nabla v \cdot (v \nabla \psi) = |\nabla v|^{p-2} \nabla v \cdot (v \nabla \psi) = |\nabla v|^{p-2} \nabla v \cdot \nabla (v \psi) - |\psi \nabla v|^p. \]

Inserting this in the first inequality and using (2.3) we deduce that

\[ \int_{C_\Sigma} |\nabla u|^p \geq \frac{1}{2p-1 - 1} \int_{C_\Sigma} |v\nabla \psi|^p + \int_{C_\Sigma} |\nabla v|^{p-2} \nabla v \cdot \nabla (v \psi) \]

\[ \geq \frac{1}{2p-1 - 1} \int_{C_\Sigma} |v\nabla \psi|^p + \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} |x|^{-p}|u|^p. \]

The improvement in the case \( p \geq 2 \) is an immediate consequence of the above lemma.

**Lemma 2.8** For all \( p \geq 2 \)

\[ \int_{C_\Sigma} |\nabla u|^p - \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} |x|^{-p}|u|^p \geq \frac{\Lambda_p}{2p-1 - 1} \int_{C_\Sigma} |u|^p, \quad \forall u \in C^\infty_c(C_\Sigma), \]

where \( \Lambda_p := \inf_{f \in C^1_c(0,1)} \frac{\int_0^1 |r^{p-1} f(r)|^p \, dr}{\int_0^1 r^{p-1} |f(r)|^p \, dr}. \)
Proof. Since $|\nabla|u|| \leq |\nabla u|$, we may assume that $u \geq 0$. We only need to estimate the right hand side in (2.20). We use polar coordinates $x \mapsto (x, \frac{x}{|x|}) = (r, \sigma)$ and denote by $\partial_r$ the radial direction. Then using (2.18),

$$\int_{C_S} |v \nabla \psi|^p = \int_\Sigma \int_0^1 r^{p-1} V^p \psi_r \partial_r + \nabla_\sigma \psi|^p \geq \int_\Sigma V^p \int_0^1 r^{p-1} |\psi|^p \geq \Lambda_p \int_\Sigma V^p \int_0^1 r^{p-1} |\psi|^p \geq \Lambda_p \int_\Sigma \int_0^1 u^{p-1} r^{-1} = \Lambda_p \int_{C_S} |u|^p.$$ 

The lemma readily follows from (2.20). \hfill \Box

It is easy to see that by integration by parts $\Lambda_p \geq 1$ while for integer $p \in \mathbb{N}$ then $\Lambda_p$ corresponds to the first Dirichlet eigenvalue of $-\Delta$ in the unit ball of $\mathbb{R}^p$.

We now turn to the case $p \in (1, 2)$ which carries more difficulties. We shall need the following intermediate result.

Lemma 2.9 Let $p \in (1, 2)$ and $u \in C_\infty^\infty(C_\Sigma)$, $u \geq 0$. Setting $\psi = \frac{u}{\tilde{v}}$ then there exists a constant $c = c(p, \Sigma) > 0$ such that

$$c \int_{C_\Sigma} r |\tilde{v} \nabla \psi|^p \leq \int_{C_\Sigma} r^{(2-p)/2} |v \nabla \psi|^p.$$ 

Proof. Let $\tilde{\psi} := r^{\frac{1}{p}} \tilde{v}$ and use $\tilde{v}^p v$ as a test function in the weak equation (2.3). Then by Hölder

$$\int_{C_\Sigma} |\tilde{\psi} \nabla \tilde{v}|^p \leq \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} \tilde{v}^{-p} \tilde{v}^p \tilde{v}^p \tilde{v}^p \int_{C_\Sigma} |\tilde{\psi} \nabla \tilde{v}|^p |\tilde{v} \nabla \tilde{\psi}|$$

$$\leq \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} \tilde{v}^{-p} \tilde{v}^p \tilde{v}^p + \left( \int_{C_\Sigma} |\tilde{\psi} \nabla \tilde{v}|^p \right)^{p-1} \left( \int_{C_\Sigma} |\tilde{v} \nabla \tilde{\psi}|^p \right)^{\frac{1}{p}}.$$ 

Therefore by Young’s inequality, for $\varepsilon > 0$ small there exists a constant $C_\varepsilon > 0$ depending on $p$ and $\Sigma$ such that

$$(1 - \varepsilon c(p)) \int_{C_\Sigma} |\tilde{\psi} \nabla \tilde{v}|^p \leq C_\varepsilon \int_{C_\Sigma} \tilde{v}^{-p} \tilde{v}^p \tilde{v}^p + C_\varepsilon \int_{C_\Sigma} |\tilde{v} \nabla \tilde{\psi}|^p.$$ 

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Recall that $\tilde{\psi} = r^{\frac{1}{p}} \psi$. Then since
\[ |\nabla \tilde{\psi}|^p \leq c(p) \left( r^{1-p} \psi^p + r |\nabla \psi|^p \right), \]
we conclude that there exists a constant $c = c(p, \Sigma)$ such that
\[ (2.22) \quad c \int_{C \Sigma} r^{1-p} \psi^p \leq \int_{C \Sigma} r^{1-p} \psi^p + \int_{C \Sigma} r^{(2-p)/p} |\nabla \psi|^p, \]
we have used the fact that $r \leq r^{(2-p)/p}$ for all $r \in (0, 1)$. To estimate the first term in the right hand side in (2.22) we will use the 2-dimensional Hardy inequality.

Through the polar coordinates $x \mapsto (r, \sigma)$
\[ \int_{C \Sigma} r^{1-p} \psi^p \leq \int_{\Sigma} V_p \int_0^1 r^{p-1} \left( \frac{\psi}{r} \right)^p r \]
\[ \leq \int_{\Sigma} V_p \int_0^1 \left( \frac{\psi}{r} \right)^p r \]
\[ \leq \left| \frac{p}{p-2} \right|^{-p} \int_{\Sigma} V_p \int_0^1 |\psi|^p r \]
\[ = \left| \frac{p}{p-2} \right|^{-p} \int_0^1 \int_{\Sigma} V_p r^{-(2-p)/p} |\nabla \psi|^p r \]
\[ = \left| \frac{p}{p-2} \right|^{-p} \int_{\Sigma} r^{N-p+1} |\nabla \psi|^p. \]
To conclude, we notice that $r^{N-p+1} = r^{N-\frac{2}{p}(2-p)/p} \leq r^{N-1}(2-p)/p$ as $p \in (1, 2)$ so that
\[ \int_{C \Sigma} r^{1-p} \psi^p \leq \left| \frac{p}{p-2} \right|^{-p} \int_{C \Sigma} r^{(2-p)/p} |\nabla \psi|^p. \]
Inserting this in (2.22) the lemma follows immediately.

We are now in position to prove the improved Hardy inequality for $p \in (1, 2)$.

\textbf{Lemma 2.10} \textit{Let $p \in (1, 2)$. Then there exists a constant $c = c(p, \Sigma) > 0$ such that}
\[ \int_{C \Sigma} |\nabla u|^p - \mu_0,C(\Sigma) \int_{C \Sigma} |x|^{-p}|u|^p \geq c \int_{C \Sigma} |u|^p, \quad \forall u \in C\infty_c (C \Sigma). \]

\textbf{Proof.} Here also we may assume that $u \geq 0$. We need to estimate the right hand
side of (2.21). Let $r = |x|$ then by Hölder and Lemma 2.9, we have

$$\int_{C_\Sigma} r^{\frac{2-p}{p}} |v \nabla \psi|^p = \int_{C_\Sigma} \frac{|v \nabla \psi|^2}{(|v \nabla \psi| + |\psi \nabla v|)^{(2-p)/2}} r^{\frac{2-p}{p}} (|v \nabla \psi| + |\psi \nabla v|)^{(2-p)/2}$$

$$\leq \left( \int_{C_\Sigma} \frac{|v \nabla \psi|^2}{(|v \nabla \psi| + |\psi \nabla v|)^{2-p}} \right)^{p/2} \left( \int_{C_\Sigma} r (|v \nabla \psi| + |\psi \nabla v|)^{(2-p)/2} \right)$$

$$\leq \left( \int_{C_\Sigma} \frac{|v \nabla \psi|^2}{(|v \nabla \psi| + |\psi \nabla v|)^{2-p}} \right)^{p/2} \times \left( 2^{p-1} \int_{C_\Sigma} r |v \nabla \psi|^p + 2^{p-1} \int_{C_\Sigma} r |\psi \nabla v|^p \right)^{(2-p)/2}$$

$$\leq c \left( \int_{C_\Sigma} \frac{|v \nabla \psi|^2}{(|v \nabla \psi| + |\psi \nabla v|)^{2-p}} \right)^{p/2} \left( \int_{C_\Sigma} r^{\frac{2-p}{p}} |v \nabla \psi|^p \right)^{(2-p)/2},$$

where $c$ a positive constant depending only on $p$ and $\Sigma$ and we have used once more the fact that $r \leq r^{(2-p)/p}$ for all $r \in (0, 1)$. Consequently by (2.21), we deduce that

$$(2.23) \quad \int_{C_\Sigma} |\nabla u|^p - \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} |x|^{-p} |u|^p \geq c \int_{C_\Sigma} r^{\frac{2-p}{p}} |v \nabla \psi|^p.$$

To proceed we estimate

$$\int_\Sigma \int_0^1 u^p r^{N-1} = \int_\Sigma V^p \int_0^1 r^{p-1} |\psi|^p \leq c(p) \int_\Sigma V^p \int_0^1 r |\psi_r|^p$$

$$\leq c(p) \int_\Sigma V^p \int_0^1 r^\frac{2}{2-p} |\psi_r|^p$$

$$\leq c(p) \int_{C_\Sigma} r^{\frac{2-p}{p}} |v \nabla \psi|^p.$$

The first inequality comes from the 2-dimensional embedding $W^{1,p}_0 \subset L^{\frac{2p}{2-p}} \subset L^{\frac{p}{2-p}}$, one can see [[13] page 2155] for the proof. Putting this in (2.23) we conclude that there exists a positive constant $c = c(p, \Sigma)$ such that

$$\int_{C_\Sigma} |\nabla u|^p - \mu_{0,p}(C_\Sigma) \int_{C_\Sigma} |x|^{-p} |u|^p \geq c \int_{C_\Sigma} |u|^p$$

which was the purpose of the lemma.
The main result in this section is contained in the following theorem.

**Theorem 2.11** Let $\Omega$ be a domain in $\mathbb{R}^N$ with $0 \in \partial \Omega$. If $\Omega$ is contained in a half-ball centered at $0$ then there exists a constant $c(N, p) > 0$ such that

$$\int_{\Omega} |\nabla u|^p - \mu_{0,p}(H) \int_{\Omega} |x|^{-p} |u|^p \geq \frac{c(N, p)}{diam(\Omega)^p} \int_{\Omega} |u|^p \quad \forall u \in W^{1,p}_0(\Omega).$$

**Proof.** Let $R = \text{diam}(\Omega)$ be the diameter of $\Omega$. Then $\Omega$ is contained in a half ball $B_R^+$ of radius $R$ centered at the origin. From Lemma 2.8 and Lemma 2.10 we infer that

$$\int_{B_R^+} |\nabla u|^p - \mu_{0,p}(H) \int_{B_R^+} |x|^{-p} |u|^p \geq \frac{c(N, p)}{R^p} \int_{B_R^+} |u|^p \quad \forall u \in C_\infty(\Omega)$$

by homogeneity. The theorem readily follows by density. $\square$

We do not know whether $\text{diam}(\Omega)$ might be replaced with $\omega_0(\Omega) \frac{\omega_N}{N}$ as in [13] at least when $\Omega$ is convex and $p \geq 2$. There might exists also “logarithmic” improvement as was recently obtained in [11] inside cones and $p = 2$. One can see also the work of Barbatis-Filippas-Tertikas in [1] for domains containing the origin or when $|x|$ is replaced by the distance to the boundary.

## A Hardy’s inequality

We denote by $d$ the distance function of $\Omega$:

$$d(x) := \inf\{|x - \sigma| : \sigma \in \partial \Omega\}.$$  

In this section, we study the problem of finding minima to the following quotient

$$(A.1) \quad \nu_{\lambda,p}(\Omega) := \inf_{u \in W^{1,p}_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^p \, dx - \lambda \int_{\Omega} |u|^p \, dx}{\int_{\Omega} d^{-p}|u|^p \, dx},$$

where $p > 1$ and $\lambda \in \mathbb{R}$ is a varying parameter. Existence of extremals to this problem was studied in [2] when $p = 2$ and in [18] with $\lambda = 0$. It is known (see for instance [18]) that $\nu_{0,p}(\Omega) \leq c_p$ for any smooth bounded domain $\Omega$ while for convex domain $\Omega$, the Hardy constant $\nu_{0,p}(\Omega)$ is not achieved and $\nu_{0,p}(\Omega) = \left(\frac{p-1}{p}\right)^p = c_p$. The main result in this section is contained in the following
Theorem A.1 Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ and $p > 1$, there exists $\tilde{\lambda}(p,\Omega) \in [-\infty, +\infty)$ such that
\[
(A.2) \quad \nu_{\lambda,p}(\Omega) < \left( \frac{p-1}{p} \right)^p, \quad \forall \lambda > \tilde{\lambda}(p,\Omega).
\]
The infimum in (A.1) is attained if $\lambda > \tilde{\lambda}(p,\Omega)$.

We start with the following result which is stronger than needed. It was proved in [2] for $p = 2$ and in [12] when $2 \leq p < N$ as the authors were dealing with Hardy-Sobolev inequalities.

Lemma A.2 Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ and $p \in (1, \infty)$. Then there exists $\beta = \beta(p,\Omega) > 0$ small such that
\[
(A.3) \quad \int_{\Omega_{\beta}} |\nabla u|^p \geq c_p \int_{\Omega_{\beta}} d^{-p}|u|^p \quad \forall u \in H^1_0(\Omega),
\]
where $\Omega_{\beta} := \{ x \in \Omega : d(x) < \beta \}$.

Proof. Since $|\nabla u| \leq |\nabla u|$, we may assume that $u \geq 0$. Let $u \in C^\infty_c(\Omega)$ and put $v = d^{\frac{1}{p-1}} u$. Using (2.18) and (2.19), we get
\[
(A.4) \quad |\nabla u|^p - c_p d^{-p}|u|^p \geq c(p) d^{-p-1} |\nabla v|^p + \left| \frac{p-1}{p} \right|^{p-1} \nabla d \cdot \nabla (v^p) \quad \text{if } p \geq 2,
\]
\[
(A.5) \quad |\nabla u|^p - c_p d^{-p}|u|^p \geq c(p) \left( \frac{1}{c_p |v| + d|\nabla v|} \right)^{p-2} + \left| \frac{p-1}{p} \right|^{p-1} \nabla d \cdot \nabla (v^p) \quad \text{if } p \in (1, 2).
\]
By integration by parts, we have
\[
\int_{\Omega_{\beta}} \nabla d \cdot \nabla (v^p) = -\int_{\Omega_{\beta}} \Delta d |v|^p + \int_{\partial\Omega_{\beta}} |v|^p \geq -c \int_{\Omega_{\beta}} |v|^p + \int_{\partial\Omega_{\beta}} |v|^p,
\]
for a positive constant depending only on $\Omega$. Multiply the identity $\text{div}(d\nabla d) = 1 + d\Delta d$ by $v$ in integrate by parts to get
\[
(1 + o(1)) \int_{\Omega_{\beta}} |v|^p = -p \int_{\Omega_{\beta}} d|v|^{p-1} \nabla d \cdot \nabla v + \int_{\partial\Omega_{\beta}} d|v|^p \leq c(p) \int_{\Omega_{\beta}} d|v|^{p-1} |\nabla v| + \int_{\partial\Omega_{\beta}} d|v|^p.
\]
By Hölder and Young’s inequalities

\[(A.6) \quad (1 + o(1) - c\varepsilon) \int_{\Omega_\beta} |v|^p \leq c_\varepsilon \int_{\Omega_\beta} d^p|\nabla v|^p + \int_{\partial\Omega_\beta} d|v|^p.\]

**Case** \(p \geq 2\). Using (A.6) we infer that

\[(1 + o(1) - c\varepsilon) \int_{\Omega_\beta} |v|^p \leq c_\varepsilon \beta \int_{\Omega_\beta} d^{p-1}|\nabla v|^p + \beta \int_{\partial\Omega_\beta} |v|^p.\]

It follows from (A.4) that for \(\varepsilon, \beta > 0\) small

\[\int_{\Omega_\beta} |\nabla u|^p - c_\varepsilon \int_{\Omega_\beta} d^{-p}|u|^p \geq c \left( \int_{\Omega_\beta} d^{p-1}|\nabla v|^p + \int_{\partial\Omega_\beta} |v|^p \right)\]

as desired.

**Case** \(p \in (1, 2)\). By Hölder and Young’s inequalities

\[\int_{\Omega_\beta} d^p|\nabla v|^p = \int_{\Omega_\beta} \frac{d^p|\nabla v|^p}{\left( \frac{1}{c_\varepsilon^p} |v| + d|\nabla v| \right)^{\frac{p(2-p)}{2}}} \leq c_\varepsilon \int_{\Omega_\beta} \frac{d^2|\nabla v|^2}{\left( \frac{1}{c_\varepsilon^p} |v| + d|\nabla v| \right)^{2-p}} + \varepsilon c \int_{\Omega_\beta} |v|^p + \varepsilon c \int_{\partial\Omega_\beta} d^p|\nabla v|^p\]

and thus

\[(1 - c\varepsilon) \int_{\Omega_\beta} d^p|\nabla v|^p \leq c_\varepsilon \int_{\Omega_\beta} \frac{d^2|\nabla v|^2}{\left( \frac{1}{c_\varepsilon^p} |v| + d|\nabla v| \right)^{2-p}} + \varepsilon \int_{\Omega_\beta} |v|^p.\]

Using this in (A.6) we obtain

\[(1 + o(1) - c\varepsilon) \int_{\Omega_\beta} |v|^p \leq c\beta \int_{\Omega_\beta} \frac{d|\nabla v|^2}{\left( \frac{1}{c_\varepsilon^p} |v| + d|\nabla v| \right)^{2-p}} + c\beta \int_{\partial\Omega_\beta} |v|^p.\]

By (A.5), we conclude that for \(\varepsilon, \beta > 0\) small

\[\int_{\Omega_\beta} |\nabla u|^p - c_\varepsilon \int_{\Omega_\beta} d^{-p}|u|^p \geq c \int_{\Omega_\beta} \frac{d|\nabla v|^2}{\left( \frac{1}{c_\varepsilon^p} |v| + d|\nabla v| \right)^{2-p}} + c \int_{\partial\Omega_\beta} |v|^p.\]

This ends the proof of the lemma.
**Lemma A.3** Let $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$. Then there exists $\tilde{\lambda}(p,\Omega) \in [-\infty, +\infty)$ such that

$$\nu_{\lambda,p}(\Omega) < c_p \quad \forall \lambda > \tilde{\lambda}(p,\Omega).$$

**Proof.** The proof will be carried out in 2 steps.

**Step 1:** We claim that $\sup_{\lambda \in \mathbb{R}} \nu_{\lambda,p}(\Omega) \geq c_p$.

For $\beta > 0$ we define $\Omega_\beta := \{ x \in \Omega : d(x) < \beta \}$.

Let $\psi \in C^\infty(\Omega_\beta)$ with $0 \leq \psi \leq 1$ in $\mathbb{R}^N \setminus \Omega_\beta$ and $\psi \equiv 1$ in $\Omega_{\beta / 4}$. For $\varepsilon > 0$ small, there holds

$$\int_\Omega d^{-p}|u|^p = \int_\Omega d^{-p}|\psi u + (1 - \psi)u|^p \leq (1 + \varepsilon) \int_\Omega d^{-p}|\psi u|^p + C \int_\Omega d^{-p}(1 - \psi)^p|u|^p \leq (1 + \varepsilon) \int_\Omega d^{-p}|\psi u|^p + C \int_\Omega |u|^p.$$

By (A.3), we infer that

$$c_p \int_\Omega d^{-p}|\psi u|^p \leq \int_\Omega |\nabla (\psi u)|^p$$

and hence

$$c_p \int_\Omega d^{-p}|u|^p \leq (1 + \varepsilon) \int_\Omega |\nabla (\psi u)|^p + C \int_\Omega |u|^p. \quad (A.7)$$

Since $|\nabla (\psi u)|^p \leq (\psi |\nabla u| + |u||\nabla \psi|)^p$ we deduce that

$$|\nabla (\psi u)|^p \leq (1 + \varepsilon) |\psi|^p |\nabla u|^p + C |u|^p |\nabla \psi|^p \leq (1 + \varepsilon) |\nabla u|^p + C |u|^p.$$

Using (A.7), we conclude that

$$c_p \int_\Omega d^{-p}|u|^p \leq (1 + \varepsilon)^2 \int_\Omega |\nabla u|^p + C(\varepsilon, \beta) \int_\Omega |u|^p.$$

This means that $c_p \leq \sup_{\lambda \in \mathbb{R}} \nu_{\lambda,p}(\Omega)$.

**Step 2:** We claim that $\sup_{\lambda \in \mathbb{R}} \nu_{\lambda,p}(\Omega) \leq c_p$. 

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Let $\beta > 0$ then by (1.3) and scale invariance we have $\mu_{0,p}(0,\beta) = c_p$. Hence for $\varepsilon > 0$ there exists a function $\phi \in W^{1,p}_0(0,\beta)$ such that

\begin{equation}
(A.8) \quad c_p + \varepsilon \geq \frac{\int_0^{\beta} |\phi'(s)|^p ds}{\int_0^{\beta} s^{-p}|\phi|^p ds}.
\end{equation}

Letting $u(x) = \phi(d(x))$, there exists a positive constant $C$ depending only on $\Omega$ such that

\[
\int_{\Omega} |\nabla u|^p = \int_{\Omega} |\phi'(s)|^p d\sigma_s \leq (1 + C\beta) |\partial \Omega| \int_0^{\beta} |\phi'(s)|^p ds.
\]

Furthermore

\[
\int_{\Omega} d^{-p}|u|^p = \int_{\Omega} s^{-p}|\phi(s)|^p d\sigma_s \geq (1 - C\beta) |\partial \Omega| \int_0^{\beta} |\phi(s)|^p ds.
\]

By (A.8) we conclude that

\[
\nu_{\lambda,p}(\Omega) \leq \frac{\int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} u^p dx}{\int_{\Omega} d^{-p}|u|^p dx} \leq (c_p + \varepsilon) \frac{1 + C\beta}{1 - C\beta} + |\lambda| \frac{\int_{\Omega} |u|^p dx}{\int_{\Omega} d^{-p}|u|^p dx}.
\]

Since $\int_{\Omega} d^{-p}|u|^2 dx \geq \beta^{-p} \int_{\Omega} |u|^p dx$, we get

\[
\nu_{\lambda,p}(\Omega) \leq (c_p + \varepsilon) \frac{1 + C\beta}{1 - C\beta} + \beta^p |\lambda|,
\]

sending $\beta$ to 0 we get the desired result.

Clearly the proof of Theorem A.1 goes similarly as the one of Theorem 1.1 and we skip it.

It was shown in [2] that $\tilde{\lambda}(2,\Omega) \in \mathbb{R}$ and that $\nu_{\lambda,p}(\Omega)$ is not achieved for any $\lambda \geq \tilde{\lambda}(2,\Omega)$. On the other hand by [8], there are domains for which $\tilde{\lambda}(2,\Omega) < 0$, see also [18].

We point out that if $\Omega$ is convex then by [22] there exists a constant $a(N,p) > 0$ (explicitly given) such that

\[
\tilde{\lambda}(p,\Omega) \geq \frac{a(N,p)}{|\Omega|^{\frac{1}{N}}}.
\]
We finish this section by showing that there are smooth bounded domains in \( \mathbb{R}^N \) such that \( \widetilde{\lambda}(p, \Omega) \in [-\infty, 0) \). We let \( U \subset \mathbb{R}^N, \ N \geq 2 \) with 0 \( \in \partial U \) be an exterior domain and set \( \Omega_r = B_r(0) \cap U \).

**Proposition A.4** Assume that \( p > \frac{N+1}{2} \) then there exists \( r > 0 \) such that \( \nu_{0,p}(\Omega_r) < \left( \frac{p-1}{p} \right)^p \).

**Proof.** Clearly \( \mu_{0,p}(\mathbb{R}^N \setminus \{0\}) = \left| \frac{N-p}{p} \right|^p < \left( \frac{p-1}{p} \right)^p \) provided \( p > \frac{N+1}{2} \). Let \( \varepsilon > 0 \) such that \( \left( \frac{p-1}{p} \right)^p > \left| \frac{N-p}{p} \right|^p + \varepsilon \) so by (2.16), there exits \( r > 0 \) such that

\[
\mu_{0,p}(\Omega_r) < \left| \frac{N-p}{p} \right|^p + \varepsilon < \left( \frac{p-1}{p} \right)^p.
\]

The conclusion readily follows since \( \nu_{0,p}(\Omega_r) \leq \mu_{0,p}(\Omega_r) \) because 0 \( \in \partial \Omega_r \).

**References**


