Financial Institutions Externalities and Systemic Risk: Tales of Tails Symmetry

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Summary

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Financial Institutions Externalities and Systemic Risk: Tales of Tails Symmetry

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We investigate the contribution of banks to the systemic risk based on the asymmetry of the right and left tails of the loss distribution. The basic idea is that a bank should create as much risk as it undertakes. Any imbalance in the distribution of profit and losses is a sign of the bank’s failure to internalize its externalities or the social costs associated to its activities. In this paper we develop a theoretical background to show the importance of tail symmetry on the sustainability of the financial system. We also propose a mathematical definition and a measure of tail imbalance based on extreme value theory. This measure could help regulators and policy makers to have an insight on the contribution of each bank to the overall risk hidden in the financial system and waiting to burst in the context of a crisis.

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Banks and more generally financial institutions play a major role in the economy. The ability of the financial system to intermediate between those who are willing to lend and others in the need of borrowing is a key determinant of growth and economic welfare. In the absence of this system of intermediation it would be difficult for most companies to fulfil their need for investment and for individuals to invest in durable goods and consume non-durable
goods. Unfortunately, it was a global breakdown of the financial system that was required to witness enough attention from researchers to the issue of systemic stability.

Andrew Crockett, \(^1\) pointed out that some rational or even desirable decisions at the individual level may have an unwelcome collateral effect on the macro level (Crockett (2000)). This observation was his motivation to urge regulators years before the crisis of 2008 toward the need of balancing micro and macro-management approaches when regulating the financial system. In fact, he highlighted the importance of reinforcing the traditional Basel accords by marrying micro-risk management with macro-risk management.

The discipline of risk management in banks is oriented towards either the predication of market evolutions with negative impact on the equity value or the anticipation of forward events that may increase the cost of reimbursing debts. At the individual bank level, the risk managers’ main focus is to limit potential losses by controlling the level of risk undertaken by the bank’s traders and created by the different activities of the bank. Therefore, the risk management discipline is directly linked to losses which explains the misconception of risk as being losses. The regulatory institutions and the central banks main concern is the sustainability of the financial system. For that aim, they make incentives (or deterrences) toward the behavior of risk taking by banks. However, most investment decisions undertaken by the banks’ managers are either to make profit or to limit losses. The first apprehension is that decisions that may have the greatest impact on the financial system in time of distress are probably those seeking gains and more precisely large gains. In this paper, we propose a new risk management approach to regulators that aggregates information about gains and losses. Regulators should not forget that all banks are competing for gains and are too narrow in their interests. The main objective of banks is sustainable gains without explicit attention

\(^1\) General manager of the Bank of International Settlements from January 1994 until March 2003
to the viability of the financial system. The role of regulators should essentially be to act as the referee and make sure that the individual decision taken by any financial institution in order to increase its own welfare has limited negative effects or externalities on the overall system.

The objective of this paper is to propose a theoretical and practical approach for measuring the externalities generated by banks’ activities that may contribute to the embedded stress in the financial system and consequently increase its fragility. Several approaches have been published to fill up the gap on individual risk measures. A survey by Bisias et al. (2012) listed up to 30 systemic risk measures in the literature. The originality of our approach is that it accounts for all the internal decisions of the bank and then considers their net effect on the system. Notably, both sides of the profit and loss distribution will be examined in our analysis with a focus on the tails. The key concept is that outstanding positive returns should have their importance in assessing systemic risk as should extreme losses.

In this paper we suggest a new rule regarding the management of systemic risk by regulators: a financial activity with no negative externalities on the financial system should not alter the symmetry of the tails of the profit and losses distribution of the bank making benefits from that activity. Even though externalities connotes a powerful image of economic and financial plagues, it is hard to define it scientifically, by consequence the measurement of externalities is troublesome for regulator. Nevertheless, we consider that potential losses that an economic agent A can suffer after contracting a transaction with an agent B as being an externality generated by the later if at the moment of the signature A is not totally aware of the unfortunate event leading to losses while B is expecting that gain. For sake of parsimony, we always designate negative externalities by simply externalities.

Several arguments suggest that banks should have tails symmetry properties. First, it is crucial to draw the attention to the fact that the derivatives market has known a substantial
growth in the last few decades. In fact, the total value of the outstanding notional amounts on derivatives is larger than the world’s GDP by a scale of magnitude \(^2\). The main intuition behind the idea of looking into gains is that most derivatives are zero-sum games: for every winner there is a looser. Gains of one financial institution should be reflected in the losses part of one or several banks holding the opposite position on the same derivatives. In the presence of perfect symmetry of information \(^3\), market actors should have the same access of risk and more importantly the same expected extreme payoffs.

Based on this rule, the left and right tails of the profit and loss distribution of a bank should have the same tail fatness. Notice that we consider that the notion of symmetry, due to the zero-sum positions, is something that is only relevant to the tails. In fact, it is possible and even healthy for the financial market to have actors with different opinions and anticipations based on the same set of information. By contrast, the extreme events are by definition unpredictable and most important of all unseen in history. Hence, anticipation of extreme payoffs based on rare a market movement should be identical among all financial institutions.

In measuring systemic risk, it is also important to pay special attention to the run-up phase, in which systemic risk and bubbles are built up in the background and waiting to burst during a financial crisis. In this phase it is important to identify actors that are exposing the

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\(^2\) According to the World Bank the total GDP for 2010 is around 63 trillion of US$. Also according to the Bank of International Settlements, the total outstanding notional amount for the derivatives market is 601 trillion of US $ as of December 2010.

\(^3\) The symmetry of information implies also that both financial institutions have equivalent computational resources and human capital to process the available information and draw conclusions. Arora and Barak (2009) pointed out that computational complexity can increase the information asymmetry.
system to hidden risks to generate excessive positive returns. In this context, it is important not only to look at the correlations that may create a contagion of losses between financial institutions, it is also crucial to detect imbalances that are built up during the pre-crisis phase. Brunnermeier and Oehmke (In Press) pointed out the difficulties in accessing the intensity of the risk hidden in the system and waiting for a trigger to materialize in the financial system in the shape of a financial crisis. Our idea is that looking into gains should identify institutions creating invisible risks in the system. It is the extreme asymmetry between potential gains and possible losses that should trigger the concerns of regulators about the bank’s level of externalities. For example, Beltratti and Stulz (2012) found evidence supporting that large gain during the pre-crisis period is negatively correlated with their performance during the crisis. At first glance, this approach seems more comprehensive as it aggregates a wider set of information compared to loss based measures. In times of financial stability and growth, most investment decisions are observed on the profit side of the P&L distribution rather than on the loss side. In addition, losses are highly monitored and regulated by the financial authorities with different incentives to limit exposures to downside risk. Meanwhile, little attention is allocated to gains. The race to boost gains may create important pervert effects and the origin of potential systemic risk lies in the failure to internalize them.

So far, systemic risk is associated to the correlation between the left tails of both the bank and the financial market. However, it is possible that companies behind the financial system’s fragility are not the ones that suffer the most from the possible damages of the crisis. It is more likely that the abnormal profits that those companies yield in normal times shield them from critical decrease in capital value during turmoil. A prototypical example is the one presented by Bernard et al. (2013) about Ambac, a US company providing financial guarantee. If Ambac fails, many guarantees will become riskier and by consequence
its counterparts will see their risk increase dramatically leading to an overall increase in risk. However it is obvious that the activity of the company should increase the overall stability of the system. On the other hand, a company like Goldman Sachs made profit helping its clients take position in the housing sector via the creation of Collaterized Debt Obligation (CDO). These activities of were unseen in the loss side. However, it is obvious that they contributed to systemic risk.

From a pure conceptual point of view, every financial institution should pay for the risk generated by its investments. Expressed differently, no financial institution should benefit from the exposure to positive shocks without bearing the risk of losing important amount of money in the occurrence of some negative shocks. We develop this concept through the idea of tail symmetry. The symmetry perception should reflect both the notion of information symmetry and the internalization of the possible damaging impacts on the financial system. Said differently, the regulator should make sure that the bank internalizes all the negative effects on the system, i.e. its externalities. Moreover, our methodology has an ethical dimension. It seems unfitted that a bank $A$ hides its exposure to positive extreme gains from its counterpart $B$. In this situation, it is bank $B$ that ultimately pays the price of the risk taken by $A$. Moreover, in the spirit of Allen and Gale (1997) intermediaries such as banks ensure an intertemporal smoothing of nondiversifiable risks. The problem arises when banks fail in the risk transfer function and hide unsustainable risk in the near future. The concept of tail symmetry will help regulator to detect when banks are borrowing future gains at the expense of the future system stability.

The last argument in favor of tail symmetry is related to internal risk management within banks. In fact, banks should know that excessive gains are the results of a favorable outcome of an exposure to some risk factors. However, a negative downturn in the evolution of those
risk factors could have a disastrous impact on the value of the bank’s assets. For that reason, banks should also monitor their profits to identify exposures to risk factors that are only visible in the gains part.

The content of this paper will be organized as follows. First we will present the fast evolving literature dealing with systemic risk and we will try to point out where our approach stands out from the crowd. The second section is dedicated to the presentation of a theoretical framework to justify the idea of tail symmetry. Then we will introduce a toy model to test some theoretical claims via Monte Carlo simulation. In the third section, we will present how to benefit from the large literature of extreme value theory to compute risk measures based on tail symmetry. Finally, using publically available market data we will present the results of the new designed measure and study the evolution of the externalities measure before, within and after the financial crisis.

1. **State of the Art and contributions**

We think that this work is at the crossroad between the literature on measuring systemic risk and the one about detecting bubbles and banks externalities. In the year 2000, Crockettle (2000) pointed out that the literature on the measurement of systemic risk and banks externalities is at a very premature stage. He also urged researchers to dig up this question because of the potentially devastating effects of a systemic crisis. Nevertheless, the financial sector had to wait until the collapse of Lehman Brothers in 2008 and witness the unbearable cost of a global financial crisis to see the emergence of serious and practical work about the financial system stability.

While we all agree that strategies to mitigate the effect of a systemic crisis should be developed, no consensus has emerged on the definition of systemic risk. This divergence is even present among regulators across countries. The Bank of England considers a very broad
definition where any possible threat to the financial system is a systemic risk event. The FED governor Daniel Tarello defined the systemic risk with the spotlight on the financial system stability. In fact, he considered institutions to be systemically risky when any situation of distress of the bank may endanger the overall financial system. By contrast, the focus of the ECB is the impact on the real economy as it relates the systemic risk to the possible economic impact. According to it, the systemic risk is any event in the financial system that may affect the consumption, welfare and growth of the real economy (Hartmann et al. (2009)). The ECB also characterize its prospective of the systemic risk as being a "vertical" approach. Others developed a more specific definition and focused on threats that affect the public confidence on the financial system (Caballero (2010)). Despite the difference, those definitions share a retrospective vision of systemic risk by looking into its effects. We propose a prospect definition of systemic risk related to the pre-crisis situation as we identify extreme asymmetries as signs of the system's fragility.

For measuring systemic risk, one can distinguish two major approaches. The first is related to network analysis and focuses on identifying relationships between financial institutions to forecast contagious chains. The second is to access the interaction between a single institution and the financial market to detect capital shortages during crises.

This strand of literature based on networks borrows from the large one on epidemiology. The similarities between a crisis and the spread of an epidemiology is almost straightforward.

For a complete overview on networks for accessing systemic risk please refer to Cont et al. (2013), Elsinger et al. (2013) and Cont et al. (2013). It is also of interest the work of Barigozzi and Brownlees (2013) and Dungey et al. (2012) who designed econometric approaches to identify network interaction using publicly available data. Those approaches are best suited to examine spillover effects and identify clusters of banks that may fall together during
crises. It is important to cite the pioneer work of Eisenberg and Noe (2001) who designed a mechanism of clearing in case of default of one or several financial institutions.

This theoretical framework was the starting point to establish theoretically the importance of tails symmetry on the probability of observing extreme losses. In fact, we found that financial system where each of its banks has perfectly symmetric tails will bare exactly the same probability of making extreme losses. In other words, only exogenous shocks to the system that are endured by the whole economy are sources of systemic risk. In this configuration, regulators are able to force banks to trade only financial products with no externalities on the financial system.

The second main stream on measuring systemic risk is probably the answer of researchers to the question of the FED Governor on how to identify systemically important banks within the system and their contribution to the overall risk. The first work is by Adrian and Brunnermeier (2011) who proposed to evaluate the systemic risk as the total loss encountered by the system whenever a bank is in distress. They called their measure $\Delta CoVaR$. The idea is to evaluate the possible increase of the $VaR$ of the system when a bank is under stress. A similar approach by Acharya et al. (2011) defined the "Systemic Expected Shortfall (SES)" as the average return of the firm when the overall system is stressed. Econometric methods to evaluate the $SES$ were developed by Brownlees and Engle (2011). For a complete assessment of the statistical properties of both measures and a discussion about their differences, readers can refer to Bernard et al. (2013). Both techniques have the advantage of being applicable using market data. Those data have no confidential aspect and are updated on a high frequency basis.

While both measures introduced by Adrian and Brunnermeier (2011) and Acharya et al. (2011) capture aspect of systemic failure, both methods are focused on the time of distress.
Second building policies based on such measures can be hard to accept by bankers. It can be implied that both $\Delta CoVaR$ and $SES$ penalize banks for being successful. It fact, part of the systemic importance that a bank may suffer is because they were able to innovate and create a financial product that had some success among other banks. Our approach however, discriminates between transactions that create asymmetry and others that enable the bank to internalize completely its externalities. The main issue is that a systemic crisis has the annoying characteristic of being unique. In fact, the continuous innovation of financial engineering made the structure of each new crisis differs from the previous ones. Therefore, it is more important to pay attention to the build up phase to detect fragilities.

In this context it is also interesting to review the literature on bubbles, financial crises and the study of financial channels. In fact, what we try to propose could also be interpreted as a bubble detection technique based on extreme heterogeneous beliefs. For a complete historical overview of crises and the contagion channels during this crises please refer to Brunnermeier (2008) and Xiong (2013).

A great inspiration to our work is the article Engle (2011). While long term risk is rather ignored in the discipline of risk management, it turned to be an important factor in the recent financial crisis. In Engle (2011), skewness and asymmetry were associated to term long risk. Our work is a proposition to include the gain side of the Profit and Losses distribution in the design of a risk measure. Authors such as Valderrama et al. (2012) and Jondeau (2010) are the first to consider both gains and losses asymmetrically in the context of systemic risk. However, the main purpose of their works was to identify the response of financial institutions toward positive and negative shocks in order to detect the system instabilities. Ours, by contrast is introducing a new vision of externalities based on tail imbalances generated by information asymmetry. The difference in tail coefficient is from our point of view a failure for one bank to assume all the risk created by the bank’s position.
In addition, because we are modeling extreme events we tried to benefit from the rich literature on extreme value theory. In fact, based on this theory we were able to design an indicator for the bank externalities that can be computed and evaluated using publicly available market data. Thanks to this approach we were able to overcome a major challenge: the confidentiality of financial data. The same methodology could be applied to complete data about individual transactions. Of course, the use of detailed data will be more informative and regulators (such as the Office of financial Research in the US) are in the process of creating such databases. Nevertheless, the use of price data can give indications about the formation of a crisis and externalities.

We believe that the idea of tail symmetry shares a common ground with the concept of fragility and anti-fragility introduced in Taleb (2012) in a philosophical essay about randomness and developed in a more technical fashion in Taleb and Douady (2012) and Taleb et al. (2012). In short, fragility is the important nonlinear exposure to negative shocks that could be viewed as a concave loss function. By contrast anti-fragility could express the opposite behavior where rare positive events are tremendously beneficial without suffering from fragility problems. The similarity between our approach and Taleb’s idea could be highlighted in the chief ethical rule expressed in Taleb (2012): *Thou shalt not have anti-fragility at the expense of the fragility of others.*

2. Relation between Tails symmetry and extreme losses

2.1. A model of the banking system

Our model of the banking system is based on the theoretical model designed by Eisenberg and Noe (2001) using the formulation and notation of Rogers and Veraart (2012). This theoretical framework was tested by Elsinger et al. (2006) on the Austrian banking system and concluded that contagion should be the major concern of regulators. They also find out
that the costs to prevent contagions are surprisingly small. We will extend the definitions by adding the notion of a crisis to the model and use its framework to establish the theoretical importance of tails symmetry on the measuring of banks externalities.

Let us consider a set $\mathcal{N} = \{1..N\}$ of financial institutions. Notice that in the paper, without making any specific distinction a financial institution is also called a bank. Each bank $i \in \mathcal{N}$, has liabilities to other banks in the system. Those liabilities are defined by an $N \times N$ matrix.

**Definition 1.** Liabilities matrix: the liabilities matrix is given by $L \in \mathbb{R}^{N \times N}$, where the entry $L_{ij}$ is the total liabilities of bank $i$ toward bank $j$. Here we assume that $L_{ij} \geq 0 \, \forall i, j$. Also, $L_{ii} = 0$ because the bank could not have liabilities towards itself.

The model assumes the priority of debt claims and that all debts have the same seniority.

**Definition 2.** Total liabilities: The total nominal liabilities of one bank $i$ toward the financial system is denoted by $\bar{L}_i$. $\bar{L}_i$ is given by $\bar{L}_i = \sum_{j=1}^{N} L_{ij}$

**Definition 3.** Net asset: Let us denote by $e_i$ the net asset of the bank $i$ from sources outside the financial system. The corresponding vector of net asset is $e$

The net asset $e_i$ is always positive. In other words, we exude costs paid by the bank outside the financial system such as operational costs. Notice that such assumption is not restrictive because it is always possible to create a fictive institution inside the financial system that will have no obligation toward other banks and it will adsorb all the operational losses that are subjected by banks.
\( \mathbf{e} \) represents the link between banks and the real economy. The role of the financial system is to provide liquidity and mitigate risks via the inter-banking market represented by the liability matrix.

DEFINITION 4. Value of equity \( v_i \): is given by the total incomes less the value of liabilities paid to creditor

\[
v_i = \sum_{j=1}^{N} L_{ji} + e_i - \bar{L}_i
\]

DEFINITION 5. Financial System: given a liabilities matrix \( \mathbf{L} \) and a vector of net assets \( \mathbf{e} \). The couple \((\mathbf{L}, e)\) is a financial system.

This definition of the financial system suppose the knowledge of all the liabilities of banks toward other members of the system. In practice, such knowledge is hardly available even for regulator and central banks. Nevertheless, this framework is an excellent starting point to study interaction between banks and possible defaults. The famous Diamond and Dybvig (1983) model is well suited to study liquidity contagion and bank runs in a multi-period framework where bilateral clearing issues are trivial. Nevertheless, in the context of multi-lateral network with cyclical liabilities the solution of this problem is less obvious and the Eisenberg and Noe (2001) model is more adapted. In fact, Eisenberg and Noe (2001) proved that, under some mild condition, it is possible to find a vector of obligation \( \mathbf{L}^* \) which respects the limited liabilities of banks and the proportional sharing in case of defaults. In other terms, it is possible to find the expected payment to each bank in case of default of one or several financial institutions and of course the banks defaulting from the first series of collapse.
2.2. Definition of tails symmetry

Most of the literature studying the tails of distribution focus on one side of any event distribution. Usually more focus is attributed to the disastrous side of any given distribution because extreme positive events are usually welcomed. Therefore, practitioners feel little need to study both tails simultaneously only the probability of negative events is measured to mitigate their effects. This explains the absence of the idea of tails symmetry on the literature about heavy tailed distributions. Incorporating both tails in the analysis is to the best of our knowledge unique in the discipline of risk management and it is the main originality of our work.

**Definition 6.** A probability distribution $P$, having a zero mean, is said to be long tail symmetric if $\exists \kappa > 0$ where $\forall \mu > \kappa$ we have:

$$P(X > \mu) = P(X < -\mu) \quad (1)$$

$\kappa$ is called the tail symmetry threshold.

The idea of tail symmetry is different from the idea of skewness. While skewness is a measure of the symmetry of the entire distribution, the tail symmetry is only considering the extreme components. It is trivial that symmetrical distributions with zero skewness are also tail symmetric. Nevertheless, it is not possible to infer the value of skewness starting from the symmetry of tails. To illustrate the difference between both skewness and tail symmetry we will present the following example of distribution. Given one positive real number $\kappa$:

$$
\begin{align*}
    f_X(x) &= \alpha_1 x + \beta_1 \quad \text{for} \quad -\kappa \leq x < 0 \\
    f_X(x) &= \alpha_2 x^2 + \beta_2 \quad \text{for} \quad 0 \leq x < \kappa \\
    f_X(x) &= e^{-|x|} \quad \text{otherwise}
\end{align*}
$$
\(\alpha_1, \alpha_2, \beta_1\) and \(\beta_3\) are chosen such as \(f_X\) is a probability distribution. It can be shown that the choice of \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_3\) is always possible and that it is unique under the condition that \(f_X\) is continuous at all points.

![Figure 1](image_url)

The figure illustrates the probability distribution \(f_x\). It is visible that the distribution is skewed however tail symmetric.

The probability distribution \(f_x\) is skewed. However, the distribution is tail symmetric. In fact, \(\forall \mu > \kappa\), it is easy to verify that \(P(X > \mu) = P(X < -\mu)\). The idea of tail symmetry is only specific to the behaviour of the distribution far on the tails without paying attention to the symmetry of the distribution around the mean value. Therefore, the value of the skewness, usually dominated by asymmetry around the mean, is not a real indicator of the tails balance. The importance of introducing such definition of tail symmetry is to stress on the fact that we are only interested in tails as by definition systemic risk events are extreme events. Finally, it is important to notice that zero-skewed distributions can also have unbalanced tails. The skewness of the distribution was lately associated with long term risk in the paper Engle (2011). In his paper he argues that negative skewness is an indicator of
long term risk which was a major component in the 2008 financial crisis. Our paper tries to push this rationale to the limits and only focuses on extreme imbalances. While skewness is associated with the long term component of risk that could be mitigated, we think that tail imbalances are risk that are beyond mitigation and that the regulator should be well armed to face them in case they trigger a systemic crisis. A simple example that we can give about externalities is that of a chemical company that invest in high risk facilities that can increase dramatically the incomes of the company. Meanwhile, insurance companies are not pricing this risk and continue to impose low premiums for the chemical company. In other words, the later company will cash in gains and is not willing to pay for the potential losses.

2.3. Implication of tail symmetry in distress probability:

To incorporate the idea of crisis on our model we will begin by presenting the implication of a crisis on the banking system.

**Definition 7.** The condition of a systemic crisis is defined by an important negative shock to the net assets $e_i$ of all banks in the financial systems.

Rogers and Veraart (2012) argued that given an initially solvent financial system, only a substantial negative shock on the bank’s net asset from outside the financial system could lead to the situation of default of one or several financial institutions. This substantial negative shock is what defines a financial system crisis in the context of our paper. A change in the pay-off structure inside the financial system could not engender the crash of the system as a whole since value is always inside the system. The impact resulting from the decrease of the assets $e_i$ can be expressed in the following way: the bank $i$ will fail to pay its obligations toward other banks based on the expected revenue of assets outside the financial system. A straightforward simplification of the system that we will adapt in the context of crisis is to neglect the amount of the net assets $e_i$ relative to the liabilities of the banks toward the overall financial system. Notice that the model implicitly suppose that shocks that lead
to banking failure are originated from the real economy and not from inside the financial system. The inter-banking activities cannot create value. Banks raise funds to finance the lending and investing operation outside the financial system. Ex ante, they expect to have a positive equity balance after reimbursing debts. Ex-post when uncertainty is resolved and the state of nature turns out to be a situation of crisis, some banks will have a shortage of liquidity and all the money they collected will be provided to the clearing system. In this situation we also witness a destruction of value inside the financial system. Of course, we should assume that we have a perfect claim-enforcement technology.

**Theorem 1.** Given a financial system $(L,e)$ with at least two participants. Under the condition of systemic crisis $E$ as defined in definition (7) and tail symmetry of all financial institutions $i$, $i \in \mathcal{N} = \{1..n\}$ acting in a financial system $(L,e)$, and the assumption that the bank can only pay a fraction of its liabilities in case of default, we have:

$$\exists \psi > 0 \text{ where}$$

$$\forall i, j \in \mathcal{N} \text{ and } \forall \delta > \psi P(v_i \leq -\delta|E) = P(v_j \leq -\delta|E)$$

(2)

The importance of tail symmetry for the stability of the financial system is a direct implication of theorem 1. In fact, this theorem stipulates that given the tail symmetry of the financial system, all banks will have similar probabilities of realizing extreme losses. To give more intuition about the policy implication of this theorem, one should bear in mind that the situation of systemic crisis represents unpredictable events by both regulators and a set of financial institutions in the system. The banks failing to identify the crisis are innovation
followers who invest in derivatives designed by the competition and not fully understood by their risk managers. It is the leading banks who make a profit by exposing their portfolios to positive events while hiding part of the potential disastrous effects on their counterparts. The role of regulators is to make sure that banks can have homogeneous extreme beliefs. Said differently, the homogenization of extreme risk assessment across all banks will lead to tail symmetry. The threshold defining extreme losses($\psi$) in this context is also the maximum of tails symmetry thresholds $\kappa_i$ of all the financial system. In other words, if regulators can impose on banks a limit of asymmetry of profit and loss distribution the risk of experiencing events of extreme losses will be equivalent between all banks. It also implies that all banks had been successful at internalizing their externalities due to their financial activity in the financial system. Put differently, whenever a bank is changing its strategy to be exposed to some profitable event that same decision should make the bank vulnerable to other negative event that will balance both tails of the distribution. This idea can have a close link to the concept of no-arbitrage. In fact, the bank should not be potentially profiting from some unpredictable events in the market while other financial institutions are bearing the risk of extreme losses due to the same event.

The tails symmetry thresholds $\psi$ choice can have important implications on the financial system and that choice should reflect the regulator policy on the management of the financial system under his jurisdiction. In fact, we believe that setting $\psi$ to be very high raises two important issues. First, the regulator could be missing threatening financial products that have an effect below the symmetry threshold. In the long run those products cloud results in a major systemic failure. Moreover, the regulator can have hard time monitoring the tail symmetry of bank if the threshold is high simply because it will be hard to measure this effect due to the curse of black swans in the extremes.
By contrast, if the regulator chooses to set the tail symmetry threshold to be at a low levels, the banks should adjust their portfolios to meet the symmetry condition starting from relatively frequent and predictable outcomes. In fact, transactions in the financial markets are driven by the dispersion of beliefs between the market participants. Setting the threshold limit to a low level can lead to the homogenization of beliefs and by consequence limit the evolution of the financial market with all the economic implications that may arise from such restriction. A direct implication of the homogenisation of beliefs is to slow down financial innovation which is an important factor to sustain growth and support the fast changing business world. Moreover, such restriction can have a negative impact on competition between banks and by consequence increase the costs of financial services. At the end, this could lead to further income inequalities as argued by Beck et al. (2010).

In conclusion, the regulator need to strike a balance and choose the right level of tails symmetry threshold according to their policy about financial innovation and growth.

Presently, we believe that regulators are implicitly proposing an extreme threshold in the Basel III regulations. In fact, proposing that banks should predict 99% of the losses means that they consider that 1% are unpredictable. By consequence, regulator tolerate that banks are unable to predict 2% of the P&L distribution. According to theorem 1, central banks should also make sure that banks are tail symmetric starting from the 99% quantile.

2.4. System stability illustrated by simulations

To illustrate the relationship between tail symmetry and the stability of the financial system we will borrow the model of the banking system proposed by Ichiba and Fouque (2013). A similar model was also presented in Fouque and Sun (2013), the authors were able to illustrate the limitation of the diversification in the case of systemic events. In our case, we will use the model to show via Monte-Carlo simulations that the financial system can become unstable in the presence of tail asymmetry.
Let us consider the following financial system $Y := (Y_t := (Y_t^{(1)}, \cdots, Y_t^{(n)}), 0 \leq t < \infty)$ of $(N \geq 2)$ banks. $Y_t^{(i)}$ is the log-monetary reserve of bank $i, i \in [1, N]$ at time $t$.

It is important to highlight that we do not claim that the model is a full representation of the interactions between banks. In fact, the model is ill suited to study spillovers in the real financial system where each bank and each connection is unique in the system. Nevertheless, network analysis has shown that individuals tend to have similar behaviors in the situation of crisis and panics. For this reason, we believe it is safe to implicitly assume that banks have similar behaviors in our model and use this model to extract conclusions about extreme random behaviors of the financial system.

In the absence of interaction between banks, where no lending and borrowing is possible between market actors, $Y_t^{(i)}, i = 1 \cdots N$ are independent. Given these settings we assume that the banks in the system are only driven by a Brownian motion:

$$dY_t^{(i)} = \sigma dW_t^{(i)}$$ (3)

Where $(W_t^{(i)}, i \in [1, N])$ are independent standard Brownian motions that start at time $t = 0$ from $Y_0^{(i)} = y_0^{(i)}, i \in [1, N]$. Also, for the purpose of this study we choose to use a fixed and identical diffusion coefficient $\sigma$.

To model interaction between banks, it is important to distinguish between two channels of shocks transmission that are identified in the contagion literature. The first is through direct interbanking claims as defined by Allen and Gale (2000). By contrast, Diamond and Rajan (2005) argue that contagion is possible even in the absence of direct links. Because they refill their liquidity supplies throughout the interbanking liquidity market, the shrinkage of the later due to a situation of a stressed bank can have negative effects in other banks sharing the same market. Empirical evidence suggest that both types of contagion exist simultaneously, hence
in this section we show the externalities generated by tails asymmetry of both topology of the financial system. To focus our study on the effect of tails asymmetry, both structures of the financial system will be studied separately. This treatment, will also help to disentangle between fragility related to a specific type of structure. Figure (2) illustrates the two types of topology of the financial market that we will consider in this section.

First, we describe the modeling fashion when we only consider direct links between banks as the leading contagion channel as in the model of Allen and Gale (2000). In the latter, this topology of networks will be called direct networks. For that purpose, the interaction between banks is introduced throughout a drift term in the diffusion process. The drift \( (Y_t^{(i)} - Y_t^{(j)}) \) is proportional to the rate at which the bank \( i \) borrows from bank \( j \). The dynamic of interactions is in line with the previous modeling of the banking system. In fact, the link between banks in the inter-banking market is the driver of contagions and interaction between actors in the system in both situations. This model is suited to study systemic risk events considering that the failure of banks within a crisis context is usually coupled with a dramatic fall in its monetary reserve with a failure to get liquidities from the inter-banking money market.

Here the dependant model is then:

\[
dY_t^{(i)} = \frac{1}{N} \sum_{j=1}^{N} \alpha_{ij}(Y_t^{(i)} - Y_t^{(j)})dt + \sigma dW_t^{(i)}, \ i \in [1, N] 
\] (4)
Where $\alpha_{ij}$ is a Bernoulli random variable with a parameter $p$. In fact $p = 0$ represents the independent system where all banks are independent from each other. By contrast, $p = 1$ is a complete network in the sense of Allen and Gale (2000). In the latter configuration, each bank in the system is having link with all other banks throughout the lending/borrowing mechanism.

\[ P = 0.25 \quad \quad P = 0.75 \]

**Figure 3** Graphical representation of the links between banks in a direct network for $p = 0.25$ and $p = 0.75$

Because the diffusion process of all banks is identical in the model introduced in equation (4), the banking system presents the characteristics of tail symmetric banks. To study the effect of tail asymmetry on the fragility of the system, we will compare the symmetric system generated with equation 4 to a modified version where we introduce tail asymmetry to one of the financial institutions.
The model will be then:

\[
\begin{align*}
    dY^{(i)}_t &= \frac{1}{\sum_{j=1}^{N} \alpha_{ij}} \sum_{j=1}^{N} \alpha_{ij} (Y^{(i)}_t - Y^{(j)}_t)dt + \sigma dW^{(i)}_t, \quad i \in [2, N] \\
    dY^{(1)}_t &= \frac{1}{\sum_{j=1}^{N} \alpha_{1j}} \sum_{j=1}^{N} \alpha_{1j} (Y^{(1)}_t - Y^{(j)}_t)dt + \sigma dW^{(1)}_t + J(t)dq(t)
\end{align*}
\]

(5)

Where \( J(t) > 0 \), is the jump intensity. \( dq(t) \) is a Poisson counter process variable with intensity \( \lambda \) such as \( \mathbb{P}[dq(t) = 1] = \lambda dt \). The jump process is identical to the jump component introduced by Merton (1976) in the price dynamic.

The realization of \( q(t) = 1 \) represents a rare event that leads to excessive gains for bank 1 in the system. Of course, the jump introduced in the diffusion process of bank 1 will induce a distortion of the right tail of the gains distribution.

In both financial systems defined by the process (4) and (5), we consider that the bank is defaulting if the value of its log monetary reserve falls below a certain threshold \( \eta \).

\[\text{Figure 4} \quad \text{One realization of the trajectories of the coupled diffusion (5) with } \alpha = 1. \text{ The solid line represents the default level } \eta = -0.7\]
Figure (5) illustrates the impact of introducing a jump component to the diffusion process of bank 1. The left figure clearly shows that the quantiles of the right tail are higher than those of the normal distribution. However, such difference is less relevant in the case of the right plot where no jump component is introduced in the diffusion. Of course, the positive jump has very little impact on the left tail.

![Q-Q plot for the coupled diffusion introduced in equation (5).](image)

**Figure 5**  Q-Q plot for the coupled diffusion introduced in equation (5). The right plot is for the bank with a jump component $J(t) = 0.02 Y_{[t-1]}^{(1)}$. The left plot represents the quantiles of a bank with no jump component on the diffusion.

We will use Monte-Carlo simulations to compare the coupled diffusion (4) with the diffusion presented by the system (5). The aim of the simulation is to study the effect of introducing possible rare but extreme gains to one bank on the overall stability of the system. The latter will be proxied by the distribution of the number of failing banks in the system according to our default criteria. This distribution will be called the default distribution for the sake of parsimony.
For the simplicity of our simulation, we assume the following parameters: a common \( \sigma = 1 \), \( N = 20 \), and we used the Euler scheme with time-step \( \Delta = 10^{-3} \), up to time \( T \). Finally, we assume \( y_0^i = 10, i = 1..N \) and that \( \eta \)

![Graph showing the probability distribution of the number of defaulting banks in the system with \( p = 0.8 \) and jump component \( J(t) = 0.02 Y_{t-1}^{(1)} \).

We can see from the illustration of default distribution for both diffusion with and without positive jump probability, that the asymmetry of the tail of one bank seems to weaken the system. In fact, the loss distribution corresponding to diffusion (5) has a higher right skewness compared to the system where all banks are tails symmetric. Tail asymmetry have also an impact on the expected value of failing banks which is equal to 6.946 in the presence of jumps and is only 5.343 in the other configuration \(^4\).

\(^4\)The difference is <1% significant following a t-test and a Wilcoxon test
Figure 7 illustrates the variation of the expected number of the failing bank $E_F$ in each configuration modeled by the diffusion process 4 and 5. Two important conclusions could be drawn from this figure. First, the financial system with a toxic bank in its premises is riskier compared to a system described by equation (4) in the whole range of $p$. Second, and most importantly we can notice that the evolution of $E_F$ is characterized by two different regimes. $E_F$ starting from a low value of corresponding to the independent system $p = 0$ will continue to increase until reaching its maximum at $p$ around 20%. Then, $E_F$ will decrease however with a slower rate until $p = 1$ for the complete network. The change of monotonicity is due to the trade-off between the beneficial effect of links in the financial network as highlighted by Allen and Gale (2000) and the role of links as a channel of financial contagion.

To check the robustness of these results, we will consider the case of a banking system where the major source of contagion is the inter-banking liquidity market following the model of Diamond and Rajan (2005). The proposed dynamic of liquidity reserves $Y^i$, $i \in [1,N]$
\[
\begin{aligned}
}& dY_t^{(i)} = \frac{\alpha}{1+Y_t^{(i)}} \left( Y_t^{(i)} - \frac{1}{N} \sum_{j=1}^{N} Y_t^{(j)} \right) dt + \sigma dW_t^{(i)}, \quad i \in [2, N] \\
& dY_t^{(1)} = \frac{\alpha}{1+Y_t^{(1)}} \left( Y_t^{(1)} - \frac{1}{N} \sum_{j=1}^{N} Y_t^{(i)} \right) dt + \sigma dW_t^{(1)} + J(t) dq(t)
\end{aligned}
\]

Where we assume that the mean reversion rate \( \alpha > 0 \). \( \alpha \) is a parameter that expresses the level of dependence between banks. In fact, \( \alpha = 0 \) represents the independent system, reader interested in discussions about the impact of parameter \( \alpha \) can refer to Fouque and Sun (2013). Notice that the larger the parameter \( \alpha \) the more stability is observed on the system, but the impact of a systemic event will be more destructive to the system in that case.

Two important features are considered in the design of the interaction between banks in equation (6). \( Y_t^{(i)} \) is driven by the rate of lending and borrowing between the banks and the average available liquidity in the market \( \sum_{j=1}^{N} Y_t^{(j)} \). The bank’s liquidity target is again the average available liquidity in the inter-banking market. Notice that, the banks will perceive positive interests whenever its liquidity is higher than the average. It will also borrow to reach its target and pays interest whenever it has low liquidity provisions. Moreover, average interest paid by banks rates is decreasing whenever the banks have higher available liquidity \( Y_t^{(i)} \). The rational behind this choice is that the liquidity market is organized in successive rounds. Each round will see the exchange of a fixed amount of liquidity and rates. The interest rates are decreasing between rounds. Banks with higher cash available (needs) should participate in more rounds to satisfy their needs. Therefore, the average interest rates will be lower for banks engaging in several rounds of liquidity exchanges.

The crucial fact of those simulations is that regardless of the nature of the banking interaction channels, introducing an asymmetry of the tail’s distributions results on an increase of the overall instability of the system and leads to greater chances of systemic crisis. We
Figure 8 On the left, we show the Probability distributions of the number of defaulting banks in the system with $p = 0.8$ and jump component $J(t) = 0.02Y_{t-1}^{(1)}$.

do not claim that those simulations are some sort of proof that asymmetries are the origin of fragility in the system. Nevertheless, we can conclude that the asymmetry amplifies the negative effects of regular contagion channels.

3. Measure of tail asymmetry and capital provisions

3.1. Tail index

The empirical study of this research will focus on identifying evidence of externalities based on the study of financial institutions stock price. At this level it is convenient to call for the large literature about tails and extreme events developed in extreme value theory (EVT).

In the approach of EVT, we focus on the tails regardless of their behavior around the mean value. This approach have the great advantage of the possibility to characterize the tails with the need of only one parameter for a wide range of distributions. The main
idea is that a very large class of probability distributions could be approached in the tails by a probability distribution called Generalized Extreme Value distribution (GEV). The cumulative distribution function of the GEV distribution could be expressed as:

$$F_\xi(x) = \begin{cases} 
  \exp\{-1 + \xi x\}^{-\frac{1}{\xi}} & \text{for } \xi \neq 0 \\
  \exp\{-e^{-x}\} & \text{for } \xi = 0 
\end{cases} \quad (7)$$

The particular case of $\xi > 0$ characterizes the heavy tailed distribution to which we acquire a special interest. In fact, authors such as Guillaume and Dacorogna (1997), Longin (1996) and Loretan and Phillips (1992) found empirical evidence that series of return in the stock market or the foreign exchange market usually present heavy tails. It is safe then to assume that series of returns in finance are heavy tailed and we will focus our study on the particular case of heavy tails.

Thanks to Gnedenko (1970), we can write a simpler formulation of the heavy tailed distribution $F$.

$$\overline{F}(x) = 1 - F(x) = x^{-\frac{1}{\xi}}L(x) \quad (8)$$

Where $L$ is a slow varying function.  

The importance of the formulation in equation (8), is that it is easier to understand the signification of the parameters $\xi$. Distributions with a large value of $\xi$ have fatter tails and by consequence the occurrence of extreme events is more frequent in that case. A series of distribution such as the Student-t, Pareto are in the class of heavy tailed distribution. Figure

---

5 The function $L$ is said to be slow varying if

$$\lim_{x \to \infty} \frac{L(tx)}{L(t)} = 1, \forall t > 0$$
9 illustrates the evolution of $F(x)$ for different values of $\xi$. Negative values indicate short tail distribution, which means that the maximum values are capped. $\xi = 0$ indicates distributions with exponentially decaying tails such as the Normal distribution. Finally, $\xi > 0$ is for the class of heavy tailed distribution. The greater the value of $\xi$ the slower the tail decay and by consequence the greater the probability of extremes occurring.

A widely used technique to estimate the tail factor is a semi-parametric approach that uses a Hill type estimator. The strategy of the estimator is defined in Embrechts et al. (1997). The idea is to choose a threshold point $x_0$ such that all observations exceeding that point are considered to be from a Pareto distribution. The set of observations exceeding the threshold will be used to form a maximum likelihood estimator for the tail parameter $\alpha = \xi^{-1}$ that we need to estimate.

We can set $x_0$ to be the $p$ quantile of gains and losses observations $(X_j)_{1 \leq j \leq n}$. We choose to denote it by $VaR_p$ to be in line with the risk management notations. Then assuming that all observations exceeding the $VaR_p$ belong to the tail that can be approached by a Pareto like distribution, we can estimate the tail factor $\hat{\alpha}_{p,n}^H$ as being:
Where $\text{Exceeds}$ is the set of observations that exceed the $\text{VaR}_p$. It is common to choose the risk horizon for losses to be 10 days and the probability level to be 99% due to the regulatory requirement of the BIS. However, in the context of the choice of the optimal threshold to compute the tail factor, it is recommended to use a Hill-plot to visualize the region where the risk factor is robust for the choice of threshold. The convergence properties of the Hill estimator are very thoroughly studied in the literature about heavy tailed distributions. The consistency of the estimator is established under some technical conditions. For further discussion about the Hill estimator please refer to Resnick and Stărică (1995), Resnick and Stărică (1998) and Embrechts et al. (1997).

### 3.2. Tail imbalance factor

In heavy tail theory, the modeling of extremes is usually focused on one side of the distribution. However, the theoretical foundation allows for the evaluation of the tail index for both tails. Our proposed heuristic is based on the balance between the tail index in both tail of the distribution. Assuming that both tails of the profit and losses distribution have fat tails, following equation (8) we have that:

\[ \exists \xi_g, \xi_l > 0, \text{ and two slowly varying function } L_1, L_2 \text{ such that} \]

\[
\begin{cases}
\bar{F}(x) = x^{-\frac{1}{\xi_g}} L_1(x) \\
F(-x) = (-x)^{-\frac{1}{\xi_l}} L_2(-x)
\end{cases}
\]
ξ_\text{g} and ξ_\text{l} are called respectively the gains tail factor and the losses tail factor.

In this context we will define the tail imbalance factor \( \lambda \) to be:

\[
\lambda = \frac{(1 + \xi_\text{g})}{(1 + \xi_\text{l})} - 1
\]  \hspace{1cm} (11)

Where \( \alpha_\text{g} = \frac{1}{\xi_\text{g}} \) and \( \alpha_\text{l} = \frac{1}{\xi_\text{l}} \)

The tail imbalance factor is a measure of the asymmetry of the tails as it expresses the ratio of both tails factors. Different intuitions are behind the choice of the formulation of the tail imbalance factor. It is easy to see that the fist order Taylor expansion of \( \lambda \) is actually the difference between \( \xi_\text{l} \) and \( \xi_\text{g} \). Second using Karamata’s theorem \(^6\), this formulation leads to equal conditional expectations for values exceeding thresholds. In other words, the difference between conditional gains and conditional losses will decrease when the imbalance measured by \( \lambda \) is fading. Distributions with symmetric tails will yield approximately the same tail factors \( \xi_\text{g} \) and \( \xi_\text{l} \) for both tails. In those conditions, the tail factor will be close to zero. Distributions with tail index \( \xi = 0 \) have exponentially decaying tails such as the normal distribution. Hence we consider the zero point as the reference for no detectable externalities.

\(^6\) Embrechts et al. (1997)
While the main purpose of this paper is to measure externalities in the financial sector, the measurement procedure could be applied to any series of returns. It is possible to apply this technique to other sectors of the economy. The idea is that the concept of externalities in the financial sector is very similar in principle to pollutions and green house gas emissions in industry. The analogy is based on the idea that both pollution and bank’s externalities are side effects that are not fully assumed by the emitting entity. The rational behind this empirical test is that we can observe the behaviour of $\lambda$ for sector that their externalities are identified and measured.

We evaluated the value of $\lambda$ for different sectors of the economy. In fact, our analysis is based on the return of index funds that tries to track some sector indices globally. The funds are managed by BlackRock and their return are published on a daily basis.

<table>
<thead>
<tr>
<th>Sector</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td></td>
</tr>
<tr>
<td>Global Energy</td>
<td>0.01</td>
</tr>
<tr>
<td>Global Clean Energy</td>
<td>-0.03</td>
</tr>
<tr>
<td>Global Nuclear Energy</td>
<td>0.04</td>
</tr>
<tr>
<td>MSCI Global Energy Producers</td>
<td>-0.04</td>
</tr>
<tr>
<td>MSCI Emerging Markets Energy Capped</td>
<td>0.02</td>
</tr>
<tr>
<td>Financials</td>
<td></td>
</tr>
<tr>
<td>MSCI Europe Financials</td>
<td>0.02</td>
</tr>
<tr>
<td>Global Financials</td>
<td>0.09</td>
</tr>
<tr>
<td>MSCI Emerging Markets Financials</td>
<td>-0.01</td>
</tr>
<tr>
<td>MSCI Far East Financials</td>
<td>-0.09</td>
</tr>
<tr>
<td>Materials</td>
<td></td>
</tr>
<tr>
<td>Global Timber &amp; Forestry</td>
<td>0.07</td>
</tr>
<tr>
<td>MSCI Global Metals &amp; Mining Producers</td>
<td>0.13</td>
</tr>
<tr>
<td>MSCI Global Agriculture Producers</td>
<td>-0.06</td>
</tr>
</tbody>
</table>

Table (1) reports the value of $\lambda$ for some indices in the energy, financial and raw material sectors. Positive values indicate sectors with important externalities, which means that these
industries failed to limit the impact of their activity on the environment and the society in general. Looking closer at the results for the energy sector we can notice that the clean energy index have negative $\lambda$ as opposed to the positive value of the nuclear sector. This result indicates that the value of $\lambda$ is coherent with our expectations regarding both nuclear and clean energy. A quick overview of recent history should indicate that the nuclear energy sector can have important negative externalities. For the raw material sector, mining and metals producers have positive $\lambda$ which is in line with the highly polluting dimension of those industries. With reference to the financial sector, Far east and emerging market banks seems to have limited externalities compared to global banks. To conclude, this table seems to confirm that $\lambda$ can be used to detect externalities. In fact, we were able to point out sectors in which their polluting nature is identified and are easily measured. This confirms the intuition that we can use this approach to detect externalities for the banking sector.

4. Measuring the bank externalities based on asymmetry

Risk measures such as the VaR or Expected Shortfall (ES) are very popular in the financial industry because they are easy to grasp and then derive policy implication based on them. The reason is that they are measured in monetary units and represent real potential losses. While the tail imbalance factor can detect externalities based on its sign, the absolute value is hard to interpret. Moreover, this measure does not take into consideration the size of the financial institution and by consequence can only compare banks of equivalent sizes. To remediate to such shortages, we will introduce a new measure of banks externalities that we call Value of Externalities (VoE). The proposed measure is inspired from the estimator of the VaR in the case of extreme value theory and based on the tail factor (see for example Embrechts et al. (1997)). Again in the context of extreme value theory, it is possible to have an estimate of the Value at Risk based on the empirical distribution of losses and the Hill
estimate of the risk factor. First let $X^{(1)} \geq X^{(2)} \geq \cdots \geq X^{(n)}$ be the order statistics of a historical sample of losses of size $n$. Assuming $u = X^{(k)}$ a very high threshold and $\frac{k}{n}$ the probability associated with $u$ (form the empirical distribution) A proposed estimate of the VaR is:

$$\hat{\text{VaR}}_q(X) = X^{(k+1)} \left( \frac{n}{k} (1 - q) \right)^{-\xi}$$

(12)

This estimator assumes a Pareto type shape of the tail. In equation 12 the historical quantile $X^{(k+1)}$ is corrected to take into account the tail fatnesses. The correction factor is greater than one for positive value of $\xi$. This formula will be our inspiration for the design of the $VoE$. Consequently, the estimator of the Value of Externalities will be defined as follows:

$$\hat{\text{VoE}}_q(X) = |X^{(k+1)} - X^{(n-k+1)}| \left( \frac{n}{k} (1 - q) \right)^{-\hat{\lambda}}$$

(13)

Where $\hat{\lambda}$ is:

$$\hat{\lambda} = \frac{\max(1 + \xi_g, 1 + \xi_l)}{\min(1 + \xi_g, 1 + \xi_l)} - 1$$

(14)

We must keep in mind that the $VoE$ is a not an alternative to losses tails only measures, such as the $\text{VaR}$, but rather complementary. In fact, $VoE$ is not designed to detect tail fatness in both sides. It is the difference that has an impact on the value of externalities. Under banking regulation, financial institutions should already monitor their losses, hence the losses tail should be thinned to avoid costly capital provision. The value of the $VoE$ should be zero for tail symmetric distribution which means limited externalities as well. One straight forward implication of the $VoE$, is that it may prevent from the risk hiding in the tails that may be an effect of regulations and the traders compensations system. In fact, the
compensation system for trades usually provides asymmetric incentives because traders will receive high bonuses in case of highly risky strategies that will pay-off. In the same time, equivalent amount cannot be drawn back from them simply because their salaries cannot go below zero. Such behaviour can create highly asymmetric distribution encouraged by the regulation that tends to ignore risk far in the losses tail which is the case of regulation based on measures such as the \( \text{VaR} \). Eventually, an important tail asymmetry could be a strong signal for regulators and policy makers that the banks’ portfolio is hiding a large amount of risk that may endanger the system once revealed.

Notice that in equation (13) we used \( \hat{\lambda} \) to characterize asymmetries instead of using \( \lambda \). First of all, \( \hat{\lambda} \) a slightly modified version of \( \lambda \) that measures the absolute tail asymmetry without any specific attention to the direction of imbalance (whether right tail is heavier or the opposite). The rational behind this choice is that as opposed to individual risk measures we think that systemic risk indicators should consider on its equation the system adaptation and mutation due to the regulation based on the indicator itself. The best way to satisfy such features by the \( \text{VoE} \) is that a bank cannot decrease its own externalities without decreasing the total externalities hidden in the system. This measure should not tolerate risk transfer between banks. In other words, the increase of a single institution’s welfare should be coupled with an increase of the welfare of the system as a whole.

4.1. Data description

We construct a sample of publically listed banks and financial institutions in general included in the NYSE financial index. The sample includes the major US financial institutions traded in the NYSE. About 180 financial institutions are included in the sample. Prices and market capitalizations are downloaded from yahoo finance for the period from 2000 to 2014.

Table (2) gives a quick overview on the symmetry structure of the returns distributions for different periods from 2000 to 2013. The 1 % \( \text{Diff} \) variable which is the difference
Table 2  Summary statistics of the data of 10-days log returns for the period beginning in 2000.

1%Diff is the difference between the 99% quantile and the 1% quantile: a first indicator of tail symmetry. AnnualizedVol is the annualized volatility. Skweness is the skewness of the distribution of the 10 days log returns

<table>
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<tr>
<th></th>
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<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>1% Diff</td>
<td>Vol</td>
<td>Skweness</td>
</tr>
<tr>
<td>10% Quantile</td>
<td>-3.55989791</td>
<td>26.91474917</td>
<td>-0.567011721</td>
</tr>
<tr>
<td>Median</td>
<td>1.361919287</td>
<td>35.04526628</td>
<td>0.03611241</td>
</tr>
<tr>
<td>90% Quantile</td>
<td>7.389700809</td>
<td>54.72670706</td>
<td>0.584560235</td>
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<td></td>
<td></td>
<td></td>
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<td>2007-2009</td>
<td>1% Diff</td>
<td>Vol</td>
<td>Skweness</td>
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<tr>
<td>10% Quantile</td>
<td>-19.49117547</td>
<td>37.96785892</td>
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<tr>
<td>Median</td>
<td>-5.442209874</td>
<td>62.1110698</td>
<td>-0.435953561</td>
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<tr>
<td>90% Quantile</td>
<td>1.224938577</td>
<td>108.1860531</td>
<td>0.134818111</td>
</tr>
</tbody>
</table>

Figure 11  In the left, we show a Q-Q plot for the returns of JPMorgan for the period 2002 – 2006. The right plot is the returns of the S&P500 index for the same period.

between the 99% quantile and the 1%quantile indicates a sign a tail asymmetry in the data. Nevertheless, the median value is close to zero, which suggests that the market has some aspect of tail symmetry on average. This idea is also visible in the Q – Q plot for the return of the S&P 500 index returns displayed in figure 11. The table also exhibits the changes
that occurred to the symmetry structure between different periods. The indicative measure of asymmetry presented suggests that the embedded stress in the system due to imbalance between losses and gains is reduced in time of high volatility.

Although table (2) and figure (11) suggests that the overall system may have some characteristics of balanced distribution with symmetric tails, it also strengthens the feeling that some banks may have strong tail asymmetry that can be a source of fragility and embedded stress in the financial system. This asymmetry is visible in the QQ plot of the returns of *JP Morgan* for example.

### 4.2. Results

Results on the value of the *VoE* are summarized in table (3). More specifically, we reported the banks that made it to the top 10 of the ranking for contribution to the system fragility throughout externalities according the metric of *VoE*. We only report the top 10 of each period. For instance, institutions like AIG are not included in 2006 but are part of the reported banks in 2007. The results are reported in term of relative contribution to the overall embedded stress in terms of percentage of the total *VoE* of the institutions in our sample. Because we only show the top 10 the reported values does not sum up to 100%.

Obviously, any analysis based on return cannot discriminate effect based on the size of the form, we corrected to *VoE* to the size by simply multiplying by the relative size of each bank in the *NYSE* financial index.

At this stage it is worth making some observations about figures in table (3. The most important finding is that the top 10 banks are the of origin more of than 50% of the externalities existing in the system and that contributes to the overall system fragility. What can be even more striking is that the top 5 institutions captures almost the third of the total *VoE*. It is also important to notice that financial institutions that are listed in our ranking
Table 3  Table represents the top 10 US financial institutions for the periods of 2006 to 2013 according to the VoE metric. SVoE is the percentage of the VoE compared to the total VoE of the financial institutions in our sample. Note that $VoE$ is corrected by the relative size of each institution in the system

<table>
<thead>
<tr>
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<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
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<tbody>
<tr>
<td>Ticker</td>
<td>SVoE</td>
<td>Ticker</td>
<td>SVoE</td>
<td>Ticker</td>
</tr>
<tr>
<td>MS</td>
<td>3.30</td>
<td>WFC</td>
<td>3.09</td>
<td>BBT</td>
</tr>
<tr>
<td>MET</td>
<td>3.32</td>
<td>AIG</td>
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<tr>
<td>BEN</td>
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<td>3.94</td>
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are also the most important banks in terms of market capitalization and activities. One possible explanation of the externalities' concentration is the implicit too big to fail guarantee (TBTF). Recall that, we consider that what give the bank the opportunity to make gains and not fully paid by it is an externality. This concept applies also to the TBTF that is a guarantee fully supported by the tax payers and not directly by the financial institution. We also notice that this concentration is more relevant starting from 2008 as it was apparent to the market after the fall of Lehman Brothers that the government would not allow for a second failure and that intensified the implicit guarantee for the TBTF banks.
Moreover, it is interesting to follow the evolution of the ranking of some of the biggest commercial banks in the US, for instance Bank of America(BoA) and Citi bank. Before 2007, BoA was considered to be a conservative institution relative to other big bank in the US. However, at the end of 2008 the bank was ranked top 2 relative to externalities which corresponds to the acquisition of Merrill Lynch and Wachovia. Citi bank and AIG were both ranked in the top 3 at the end of 2008. Both financial institution were the first to feel the heat of the financial crisis because of their deep implication in the mortgage business and are also one of the first to benefit from the bailout program. Again, according to our definition those bailout could be seen as externalities which in turn can explain those important ranking according to our measure.

![Graphs showing the evolution of the tail imbalance factor λ for six major US banks.](image)

**Figure 12** Those figures show the evolution of the tail imbalance factor $\lambda$ for the period of 2003-2014 for six major US banks. The value are updated monthly with a fixed timespan of 4 years. The value for the year 2006 mean that we include all the returns up to 12-2006.

Figures (12) and (13) show the time series evolution of both $\hat{\lambda}$ and $VoE$ between 2003 and 2014. The important conclusion is that all banks feature a spike in their externalities
Figure 13 Those figures show the evolution of the VoE for the period of 2003-2014 for six major US banks. The value are updated monthly with a fixed timespan of 3 years. The VoE is computed for $q = 99\%$. The value for the year 2006 mean that we include all the returns up to 12-2006.

around mid 2008. Precisely in 2007 we see that the value of externalities of all banks started to increase sharply which could be translated by an important externalities and systemic fragility. These spikes fall right after 2008 which mean that the hidden stress has materialized into a full scale systemic crisis. According to these figures, in 2007 the regulator should have been alarmed by the declining health of the financial system. Of course retrospectively we know that the market already had indication about the situation of the financial system starting from 2007. But at that time, opinions were mitigated about the scale of the hidden stress in the system. Using an indicator such as the VoE could have shown that banks were reaching unforeseen level of externalities and that the system would soon or later collapse to this stress.

At this stage it is important to draw the attention to the fact that the methodology that we propose should not be interpreted as an alternative to the measures of systemic risk.
proposed by Acharya et al. (2011) and Adrian and Brunnermeier (2011). In fact, the last two approaches could be seen as measures of simultaneous failure of the system and a financial institution and by consequence focus on the system in time of a crisis. The main insight that both measures propose to regulators is the identification of financial institutions that require special attention in time of distress either because they can be identified as SIFI or that their survival is at stake.

Our, by contrast, propose a measure of externalities of banks based on the principle of tail asymmetry that can be the source of system fragilities. The constant gathering of those fragilities may lead to a systemic crisis. It is the evolution of the measures that we propose that can enlighten regulators on financial institutions that adventure at activities with high externalities and are pushing the system toward its collapse. Regulators should react when they observe the increase of the level of asymmetry in time of economic stability as this can be seen as a premonition of future turmoil. Moreover, this indicator of symmetry can encourage banks to re-examine their relationships with their counter-parties that are highly exposed to positive shocks at the expanse of the system fragility. This peer evaluation by banks to other banks' activities will undoubtedly contribute to the overall stability of the system.

Nevertheless, the approaches of Acharya et al. (2011) and Adrian and Brunnermeier (2011) do have some common grounds. We also rely on publicly available market data to be able to compute and evaluate the system stability based on our measures. While we share all the advantages of using market data, we also suffer from the curse of all the critics that could be addressed to methodologies using stock prices.

However, it is important to notice that the key concept of tail symmetry could also be applied to bottom-up approaches such as the Basel methodology to compute VaR. In theory,
banks using their internal risk assessment models to evaluate their risk measures can also rely on the same model and simulations to deliver information about their right tails. In addition, the perfect conditions would be regulators who have access to all transactions of the financial institutions and then able to reconstruct their portfolios. Then using the available information and unified pricing model compute metrics about tail imbalances and react accordingly. With the use of unified pricing models, tail imbalance could be easily unjustified and even reprehensible by regulators. This ideal situation may seem out of reach in the present conditions. Nevertheless with the development strategy of the OFR to construct a complete data warehouse recording all the transactions in the banking system, and the power given by the Dodd–Frank (2010) to this office, the application of a monitoring system of tails imbalance fairness seems possible. Meanwhile, relying on market data appear to be the best alternative.

5. The post financial crisis fines

Two years after the peak of the financial crisis, the US market has experienced a major shift, itself a result of a change in Obama’s administrations’ policy. In fact, after the biggest bailout procedure in the history of the banking industry, Wall Street banks and the major foreign banks operating in the US have paid out more than $100 billion on fines. To grasp the magnitude of those fines, it is important to highlight that the Supervisory Capital Assessment Program (SCAP) conducted stress tests in 2009 on the 19 largest US banks. They concluded that the banks needed to raise $74.6 billion if the economy were to get worse. In addition, these fines to the banking industry set a record in the history of legal settlements in the US and are only exceeded by the historical litigation of the tobacco industry however on a period of 25 years ending in 2025. The fines reflected that the willingness of the Obama administrations to persuade the general public that the bankers would not get off lightly for
their role in the ignition of the financial crisis. Besides the political motivation of those fines, they are much in the spirit of our analysis of externalities in this paper. The fines covers the banking practices in different business areas ranging from lending to fraudulently issuing mortgage-backed securities and market manipulation.

Of course measuring *ex-ante* the externalities that resulted from the banks’ activities was generally considered to be a difficult task, the fines that resulted from a thorough analysis of the collateral damage of those externalities could be seen as an *ex-post* credible measure. The data on fines and penalties billed to the financial industries are collected and updated regularly by the Financial Times on their website. The data is collected between 05-2007 and 05-2014. The 4 biggest US banks (JP Morgan & Chase, Wells Fargo, Citigroup, Bank of America) total as much as 57.1 billion $ of fines. Foreign based banks with activities in the US such as HSBC and Deutsche Bank have a bill of over 15.5 billion $.

Using fines to proxy externalities in the financial industry is also one of the originality of this work. Nevertheless, the idea of linking fines to social welfare and externalities in industrious such as coal or tobacco industries goes back to 1920 in Pigou (2006). Similar proxies are unused in the other disciplines like the transportation industry where fines are

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7 see http://blogs.ft.com/ftdata/2014/03/28/bank-fines-data/
considered as the cost paid by consumers and the industry to cover for the social costs of environmental externalities (see for example Hultkrantz et al. (2012)).

The question in this section is to what extent our newly designed measure captures the level of externalities measured \textit{ex-post} via the fines imposed on banks. More precisely, we will also compare the \textit{SVoE} to other measures of systemic risk such as the \textit{MES} of Acharya et al. (2011) and \textit{SRISK} by Brownlees and Engle (2011). For that purpose, table 4 provides on OLS regression analysis that explains the fines with regressors \textit{VoE}, \textit{MES} and \textit{SRISK} The \textit{VoE} is based on the \textit{VoE} computed following the methodology explained in the previous section and scaled by the share of the domestic activities of the US banks. The domestic activity is evaluated from the ratio of foreign assets divided by the total assets of the banks. The data for US banks are extracted from the FR Y-9C report \footnote{reports are available at http://www.ffiec.gov/}. For non US banks, we used the geographical distribution of revenue to proxy the activities in the US. Data were extracted from the annual report of each bank available on their websites. \textit{MES} and \textit{SRISK} were extracted from the V-Lab website \footnote{http://vlab.stern.nyu.edu/} where those measures are regularly maintained and published for a large panel of banks. The extraction was made as of 31-12-2007 and return for the years of 2004-2007 were used to compute the value of \textit{VoE}. The choice of time frame is due to the fact that we want to access the ability of each of those measures to forecast \textit{ex-ante} the observed externalities of the banks after the crisis.

The striking result from Table 4 is that \textit{VoE} is highly significant in regression (3) and (4). For example in regression (3) where only \textit{VoE} is included as an explanatory variable the \textit{t-value} is 5.929 with an \textit{adjusted} $-R^2$ equal to 44.29\%. It is also important to notice that the variable \textit{SRISK} had little significance in the regression (2) with a \textit{t-value} of 2.925.
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Table 4  The dependant variable is the amount of fines. Model (1) to (3) are OLS regression with a single variable in the model. Model (4) is an OLS regression with $MES$, $SRISK$ and $SVoE$ as regressors. $MES$, $SRISK$ are computed as of Dec-2007 and $VoE$ is evaluated as of dec-2007 with a 4 years window.

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<tr>
<td>SRISK</td>
<td>0.9881*</td>
<td></td>
<td>5.601</td>
<td></td>
</tr>
<tr>
<td>VoE</td>
<td></td>
<td>6.035***</td>
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<td>5.429***</td>
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Adjusted $R^2$ -1.952% 12.91% 44.29% 43.06%

and an adjusted $-R^2$ equal to 12.91%. Nevertheless, this variable lost all its significance when regressed with the $VoE$. Also, note that the adjusted $-R^2$ had a small drop between regression (3) and (4) which indicated that $MES$ and $SRISK$ had a small marginal added-value to predict externalities compared to $VoE$. The important point is that the $VoE$ seems to better capture the externalities measured via fines and penalties than $MES$ and $SRISK$. Of course, this is not a fair horse-race. The $MES$ and $SRISK$ were specially designed to capture the shortage of capital in the financial institutions during the periods of turmoil. They do not take into consideration the externalities or the social costs associated to the bankers activities on the system. Nevertheless, $MES$ do a good job forecasting the capital shortage and seems to predict for example the results of the stress test that were performed on banks in 2009 by the SCAP. The bottom line of this analysis is that the $VoE$ designed to capture externalities should not be interpreted as an alternative to systemic risk measures such as the $MES$ and $SRISK$ and the policy implication of both approach are different.
While high $SRISK$ indicates that the banks have a capital shortage that could require a government intervention during a crisis situation, a high value of the $VoE$ suggests that the banks have important externalities and have an important marginal contribution to the system’s fragility.

Conclusions

The systemic crisis of 2008 initiated a series of research to identify important financial institution that the system cannot survive their failure. Nevertheless, those research had little focus on the pre-crisis era where risk were building up in the background to burst in the form of a crisis. This paper introduces the idea that banks should have symmetric tail in order to limit the systemic risk created by their activities. While we consider that skewness is acceptable in financial markets because it is the result of different expectations, tails asymmetry is not tolerated because if extreme losses are unpredictable so should extreme gains. The failure to pay attention to the imbalance between potential gains and potential losses in prosperous times allowed banks to increase their externalities. In fact, we translate the difference between tail asymmetry into a measure of banks externalities that could result into a systemic crisis. It was shown using both a theoretical model on banks interaction and Monte Carlo simulations that the asymmetry of tails of banks can lead to a riskier financial system. In addition, this paper proposes an estimation procedure to overcame the data availability problem. In fact, we were able to propose a measure of externalities based on publicly available price data. The measure proposed is building on extreme value theory and the hill estimator of tail factor. We also performed an \textit{ex-post} test of the $SVoE$ measure considering the fines and penalties paid by the banking industry after the crisis of 2008. We show that the $SVoE$ can have greater explanatory power than measures of systemic risk such as $SRISK$.
The results of this paper can have direct policy implications. As tail asymmetry is proven to be potentially harmful for the long run survivorship of the financial system and the sustainability of the financial services, it is suggested that regulators should regularly monitor both right and left tails of the profit and losses distribution of banks. It was suggested that long period of financial prosperity and growth can give premonition of future crisis. This paper provides the theoretical and empirical evidences that support such a claim.
Proof of theorem 1  The proof will be done via induction on the size \( N \) on the financial system. Without any loss of generality we will suppose that the Profit and Losses distribution have zero means for all financial institutions.

Part 1 : Case of size 2  In the context of two banks financial system the value of equity \( v_i \), \( i = 1, 2 \) of each bank is:

\[
v_1 = L_{21} - L_{12} + e_1 \quad \text{and} \quad v_2 = L_{12} - L_{21} + e_2
\]

Starting from \( \mathbb{P}(v_1 \leq -\delta | E) = \mathbb{P}(L_{21} - L_{12} + e_1 \leq -\delta | E) \) and given the assumption of a systemic financial crisis, we have that \( v_1 = L_{21} - L_{12} + e_1 \approx L_{21} - L_{12} \) and \( v_2 = L_{12} - L_{21} + e_2 \approx L_{12} - L_{21} \). Then we can conclude that \( \mathbb{P}(v_1 \leq -\delta | E) = \mathbb{P}(L_{21} - L_{12} \leq -\delta | E) = \mathbb{P}(L_{12} - L_{21} \geq \delta | E) \). Finally this leads to:

\[
\mathbb{P}(v_1 \leq -\delta | E) = \mathbb{P}(v_2 \leq \delta | E)
\]

On the other hand, we have that the Profit and losses distribution of bank 2 is tail symmetric. In other words, we can find \( \psi > 0 \) where \( \forall \alpha > \psi \), we have:

\[
\mathbb{P}(v_2 \leq -\alpha | E) = \mathbb{P}(v_2 \geq \alpha | E)
\]

With the combination of the results on crisis conditions and tail symmetry we have that:

\[
\delta > \psi \quad \mathbb{P}(v_1 \leq -\alpha | E) = \mathbb{P}(v_2 \leq -\alpha | E)
\]

Part 2 : Case of size 3  In the context of three banks in system the value of equity \( v_i \), \( i = 1, 2, 3 \) and of each bank is:

\[
v_i = \left( \sum_{j=1}^{3} (1 - \delta_{ij})L_{ji} - \sum_{j=1}^{3} (1 - \delta_{ij})L_{ij} \right) + e_1
\]

Where \( \delta_{ij} \) is the kronecker factor. We can start from:

\[
\mathbb{P}(v_1 \leq -\delta | E) = \mathbb{P}((L_{12} + L_{32}) - (L_{21} + L_{22}) - (L_{32} - L_{23}) + e_1 \geq \delta | E)
\]

Given the assumption of a systemic financial crisis, we have that

\[
v_1 = \left( \sum_{j=1}^{3} (1 - \delta_{ij})L_{ji} - \sum_{j=1}^{3} (1 - \delta_{ij})L_{ij} \right) + e_1 \approx ((L_{12} + L_{32}) - (L_{21} + L_{22}) - (L_{32} - L_{23}))
\]

Then we can write that \( \mathbb{P}(v_1 \leq -\delta | E) = \mathbb{P}(v_2 \geq \delta + (L_{32} - L_{23}) | E) \). Regarding that \( \delta \) is rather on the tail and is a significant of a big loss then we can reasonably assume that \( \delta + (L_{32} - L_{23}) \approx \delta \) again, with the
combination of the results on crisis conditions and tail symmetry for bank (2) we have that as in part (1) we can conclude that

$$\delta > \psi \quad \mathbb{P}(v_1 \leq -\alpha|E) = \mathbb{P}(v_2 \leq -\alpha|E)$$

**Part 3 : Induction**

Next we claim by mathematical induction that this Theorem 1 is true for all financial system with size \( \leq k - 1 \), and let us prove that this equality is valid for a banking system with size \( k \). First, let us consider a financial system \( S = (L, e) \) with \( k \) banks acting in the system. Let us imagine that bank \( C \) is the fusion of bank 1 and 2. Without any loss of generality, the bank should be chosen as a solvent bank. The case of the merger of two solvent banks will have no impact on the solvency and cash-flows on other banks in the system because every bank will continue to pay its obligation toward the bank. However, if the bank 2 is insolvent the merger of the two banks 1 and 2 will only be a bank \( C \) where expected payments from the bank \( C \) are the expected payments of bank 2 and the obligation of any financial institution with regards to \( C \) are the obligation toward the second bank plus the net obligation toward the failing bank. Because the bank, can only pay a fraction of its obligation in case of default. In other word, a default will always have its costs on the system. Such merger is always possible and bank will have incentive to perform it as presented in Rogers and Veraart (2012).

After the fusion of banks 1 and 2 into a single bank \( C \), we have a new financial system \( S' = (L', e) \). With the merger of two banks, the new financial system is of size \( k - 1 \).

According to the induction hypothesis, we have that:

$$\exists \psi_1 > 0 \text{ where}$$

$$\forall i,j \text{ a bank } S \text{ and } \forall \delta > \psi_1 \quad \mathbb{P}(v_i \leq -\delta|E) = \mathbb{P}(v_j \leq -\delta|E)$$

(15)

So far, we have established the desired equality for \( k - 2 \) bank in the initial system \( S \). Choosing a different financial system \( S'' \), where this time we will merge the bank 1 with a different bank will make the relationship applicable for the bank 2. This is true because the system \( S \) and \( S'' \) will have banks in common. Finally, to prove the extreme losses property for the financial system including the non defaulting bank 1. We will
distinguish the case where we have a second non defaulting bank from the case where all banks beside bank 1 are defaulting. In the first case, to demonstrate that equation 15 holds even if we are including the bank 1 we can simply merge the second non defaulting bank with any of the banks except bank 1. Then the equality become evident. Finally, the case were all banks besides bank 1 will default is also simple. Notice that the equation 15 holds for the merger of bank 1 and 2. But the bank 2 is defaulting without inducing any costs on the system. Therefore, the value of the bank $C$ is equal to the value of the bank 1. Thus the equation 15 holds for the bank 1 as well.

Finally, via mathematical induction we can establish Theorem 1
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