Adaptive estimation procedure with applications

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Outline

Problems and Solutions

Problems

- Modeling joint dependence is critical for financial time series
- Model volatility of price change of financial assets
- Different approaches are used to model correlations, i.e. MV-GARCH:
  - VEC model (Bollerslev, Engle and Wooldridge, 1988)
  - BEKK model (Baba, Engle, Kraft and Kroner, 1991)
  - DCC model (Engle, 2002) and etc.

but

- “Stocks tend to crash together, but not boom together”
- Need to model asymmetric behavior of the assets
- $\Rightarrow$ use Copula

What are Copulas?

Copulas allow us to model the dependence relationships among r.v. independently of their marginal distributions [1]

Definition:

Function $C : [0,1]^2 \longrightarrow [0,1]$ such that

$$F(x,y) = C\{F_1(x), F_2(y)\}$$

is a copula

Example: Clayton copula

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

where $\theta \in (0,\infty)$ - dependence parameter

[1] For more details see Embrechts, Lindskog and McNeil (2001)

Modeling of marginal distributions

Copula models for marginal distributions

Lets define log-differences of 2 different assets as x_t and y_t

Assume that x_t and y_t follow typical univariate $GARCH(1,1)$

$$\begin{aligned} x_t|\mathcal{F}_t &\sim N(0, h_t) \\ h_t &= \omega_x + \alpha_x x_{t-1}^2 + \beta_x h_{t-1} \\ y_t|\mathcal{F}_t &\sim N(0, v_t) \\ v_t &= \omega_y + \alpha_y y_{t-1}^2 + \beta_y v_{t-1} \end{aligned}$$

After deGARCHing

$$\epsilon_t = \frac{x_t}{\sqrt{h_t}}, \eta_t = \frac{y_t}{\sqrt{v_t}}$$

Then ϵ_t and η_t are cleaned from GARCH effect $\Longrightarrow$ use Copula

$$F_{t-1}(\epsilon_t, \eta_t) = C_\theta\left(F_x(\epsilon_t), F_y(\eta_t)\right)$$

Local Change Point procedure

- **Approach proposed by** D.Mercurio and V.Spokoiny

  "Estimation of time dependent volatility via LCP analysis with applications to VaR" (2004)

- **Idea**: application to copula: $\exists$ intervals of time homogeneity where dependence parameter of a copula is constant

Notations

- I - interval candidate and $\mathcal{T}_I \in I$

$$H_0 : \forall \tau \in \mathcal{T}_I, \theta_t = \theta$$

$$H_1 : \exists \tau \in \mathcal{T}_I : \quad \begin{array}{lll} \theta_t = \theta_1 & \text{for} & t \in J \\ \theta_t = \theta_2 & \text{for} & t \in J^c \end{array}$$

Estimation

Canonical Maximum Likelihood

Canonical Maximum Likelihood (CLM)

maximize pseudo *log-likelihood* function

$$l(\theta) = \sum_{t=1}^{T} \log c \left\{ \widehat{F}_{X_1}(x_{1,t}), \ldots, \widehat{F}_{X_d}(x_{d,t}); \theta \right\}$$

with *empirical* marginal distributions

$$\widehat{F}_{X_j}(x) = \frac{1}{T+1} \sum_{t=1}^{T} \mathbb{I}\{X_{j,t} \leq x\}$$

$$\widehat{\vartheta}_{CML} = \arg\max_{\theta} l(\theta)$$

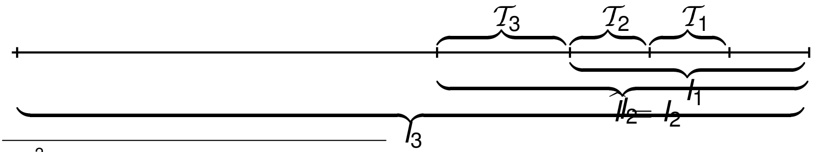
Local Change Point (LCP) procedure [2]

Adaptive choice of the *intervals of homogeneity*

Start with $k = 1$

1. test $H_{0,k}$ within I_k on $\mathcal{T}_k$
2. if $H_{0,k}$ is not rejected, take I_{k+1} and repeat until homogeneity is rejected
3. if $H_{0,k}$ is rejected within I_k, then $\widehat{I} = I_{k-1}$
4. if the largest possible interval is reached, then $\widehat{I} = [0, t_0]$

θ is estimated from observations in $\widehat{I}$, i.e. $\widehat{\theta}_t = \widetilde{\theta}_{\widehat{I}}$



[2] For more details see Giacomini, Härdle, Ignatieva and Spokoiny (2006)

Parametric time variation in the conditional copula

- **Approach proposed by** Andrew J.Patton
  “Modeling asymmetric exchange rate dependence” (2006)

- **Idea**: copula dependence parameter is time-varying and follows a parametric model

$$
\begin{aligned}
C(u, v) &= C_{\theta_t}(u_t, v_t) \\
\theta_t &\sim ARMA(p, q) - type
\end{aligned}
$$

Parametric time variation in the conditional copula
Choice of a copula

- *the Joe-Clayton copula*

$$
\begin{aligned}
C(u,v|\tau^U,\tau^L) &= 1-\left(1-\left\{[1-(1-u)^{\kappa}]^{-\gamma}+[1-(1-v)^{\kappa}]^{-\gamma}-1\right\}^{-1/\gamma}\right)\\
\kappa &= \frac{1}{log_2(2-\tau^U)}, \gamma = -\frac{1}{log_2(\tau^L)}\\
\tau_t^i &= \Lambda\left(\omega_i+\beta_i\tau_{t-1}^i+\alpha_i\frac{1}{10}\sum_{j=1}^{10}|u_{t-j}-v_{t-j}|\right), i=U,L
\end{aligned}
$$

where $\Lambda(x) \equiv (1+e^{-x})^{-1}$ logistic transformation to keep τ^U and τ^L in $(0,1)$ at all times

Parametric modeling of dependence parameter
Choice of a copula

- **Here**: *Clayton copula*

$$
\begin{aligned}
C(u, v|\theta) &= (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \\
\theta_t &= \overline{\Lambda}\left(\omega_\theta + \beta_\theta \theta_{t-1} + \alpha_\theta \frac{1}{10}\sum_{j=1}^{10} |u_{t-j} - v_{t-j}|\right) \qquad (1)
\end{aligned}
$$

where $\overline{\Lambda}(x) \equiv |x|$

The log-likelihood is given by

$$
\begin{aligned}
l(\theta) &= \log(1+\theta) - (\theta+1)\log(uv) \\
&\quad -(\frac{1}{\theta}+2)\log(u^{-\theta} + v^{-\theta} - 1)
\end{aligned}
$$

then

$$\hat{\theta} = \arg\max_{\theta} l(\theta)$$

given that (1) holds

Local Change Point (LCP) procedure

Simulation set

Conducted simulations

Test sensitivity of LCP to changes in

- angle of growth and reduction of θ_t
- height of increase of θ_t
- dimension of data $d = 2, 6$ and 10

$$\theta_t = \begin{cases} 0.1 & \text{if} \quad t \in [1, 100] \\ 0.1 + \frac{1}{\Delta t_{up}} \Delta(t - 100) & \text{if} \quad t \in [101, 100 + \Delta t_{up}] \\ \vartheta & \text{if} \quad t \in [101 + \Delta t_{up}, 200 + \Delta t_{up}] \\ \vartheta - \frac{1}{\Delta t_{dn}} \Delta(t - \Delta t_{up} - 200) & \text{if} \quad t \in [201 + \Delta t_{up}, 200 + \Delta t_{up} + \Delta t_{dn}] \\ 0.1 & \text{if} \quad t \in [201 + \Delta t_{up} + \Delta t_{dn}, 300 + \Delta t_{up} + \Delta t_{dn}] \end{cases}$$

Local Change Point (LCP) procedure
Simulation set: example

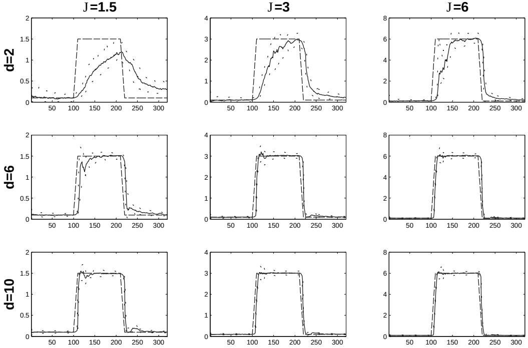


Figure: For Δt_{up}=Δt_{dn}=10. From top down: 2-, 6- and 10-dim. From left to right: jump in θ from 0.1 to 1.5, 3 and 6

Parametric time variation approach
Simulations

Simulations in progress

Test parametric approach to changes in

- structural changes (sudden jumps in θ)
- smooth functional behavior of θ, i.e. $\theta_t = sin(t)$
- transformation function $\Lambda(x)$
- number of lags p, q in $\theta \sim ARMA(p, q)$

Parametric time variation approach

Simulation example: Jump from 2 to 4, Clayton copula, 100 paths

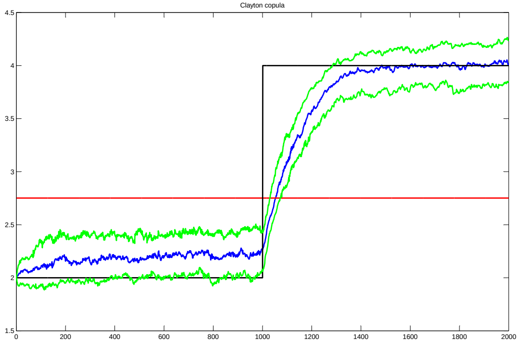


Figure: True θ_t (black), time-invariant θ (red), median of time-varying θ_t (blue), 25% and 75% quantiles.

Parametric time variation approach

Simulation example: $\theta_t = sin(t)$, Clayton copula, 100 paths

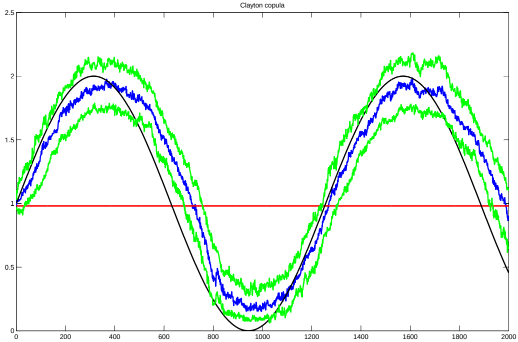


Figure: True θ_t (black), time-invariant θ (red), median of time-varying θ_t (blue), 25% and 75% quantiles.

Data set
Dow Jones Industrial Average and NASDAQ composite

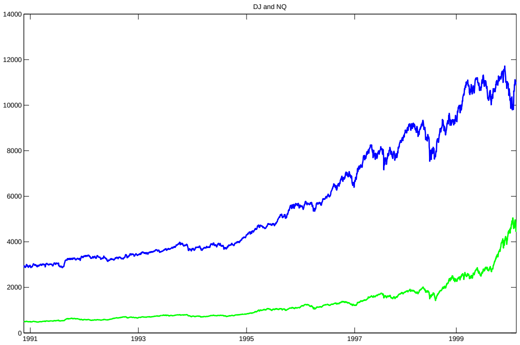


Figure: Dow Jones (blue line) and NQ (green line), daily data: April 1991 - March 2000

Data set

Log returns and estimate of $h_t^{1/2}$ from GARCH(1,1)

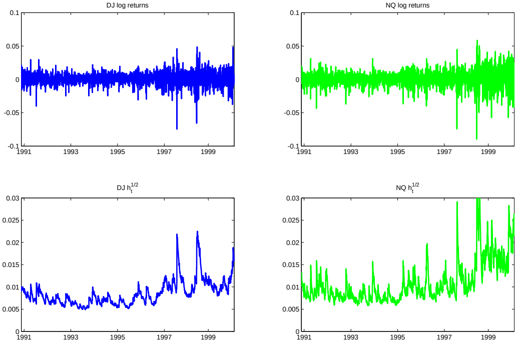


Figure: Dow Jones (blue line) and NQ (green line), daily data: April 1991 - March 2000

Local Change Point procedure

Estimated θ parameter and intervals of time homogeneity via V. Spokoiny

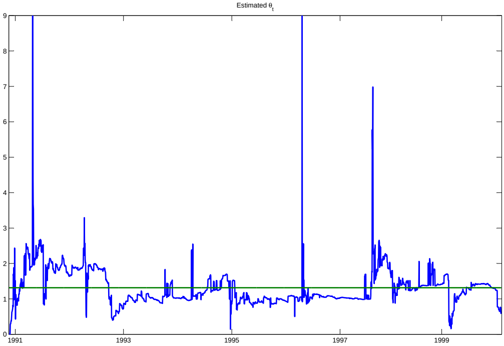


Figure: Estimated θ_t (upper panel) and intervals of time homogeneity (lower panel): April 1991 - March 2000

Parametric time variation approach

Estimated θ parameter via method of A. Patton: $\theta_t \sim ARMA(1, 10)$

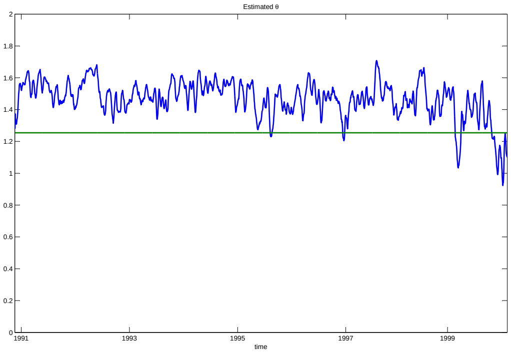


Figure: Time-varying θ_t (blue line) and time-invariant θ (red line): April 1991 - March 2000

Conclusions

- ***LCP procedure*** does not require any assumptions on Marginal distributions and is more flexible
- ***LCP procedure*** is computationally more laborious - hard to apply to long time-series.

- ***Parametric approach*** by Patton is clear and easy to implement
- Need to fine-tune the proper functional form for θ_t and transformation function $\Lambda(x)$

- **Further research**:
- Run simulations to compare results of LCP procedure vs. Parametric approach
- Compare both models to one of frequently used correlation models, i.e. DCC, CCC, BEKK and etc.

For Further Reading I

Andrew J.Patton
Modeling asymmetric exchange rate dependence
INTERNATIONAL ECONOMIC REVIEW, Vol. 47, No. 2, May 2006

D.Mercurio and V.Spokoiny
Estimation of time dependent volatility via LCP analysis with applications to VaR
Annals of Statistics, 32:558-602,2004

E.Giacomini, W.Härdle, E.Ignatieva, and V.Spokoiny.
Inhomogeneous Dependency Modelling with Time Varying Copulae.
SFB 649 Discussion Paper 2006-075.