Options and semi-Markov regime switching

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Black-Scholes theory
Shortcomings of Black-Scholes
Semi-Markov processes
Regime switching models
Conclusions and future work

Introduction

Outline of the talk

1. Classical arbitrage theory
2. Shortcomings of Black-Scholes
3. Semi-Markov processes
4. Regime switching models
5. Conclusions and future work

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Black-Scholes theory

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- Stocks: primary assets worth S_t at time t
- Options: Example: european call option (a contract settled at time t that gives the holder the right but not the obligation to buy an asset at time $T > t$ for a prespecified price K)
- At time T, the payoff of such an option is $(S_T - K)^+$
- Aim: to price and hedge such contracts

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- Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}_t$
- In the Black-Scholes model, we consider 2 assets: the bank account S^0 and one risky asset S^1: the stock.
- We assume that the stock price S^1_t follows the process:

$$dS^1_t = S^1_t(\mu dt + \sigma dB_t)$$

- Randomness comes from the price of the asset

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- Absence of arbitrage: one can’t make money in the market with no risk and no initial investment
- Law of one price: two financial assets with same payoff must have same initial price
- Central idea of arbitrage pricing

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Martingale measure (local martingale measure):

- a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is an equivalent martingale measure if:

  1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$
  2. the discounted price process of all the assets are $\mathbb{Q}$-martingales.

  We write $\mathcal{P}$ the set of all equivalent martingale measures

- $\mathcal{P} \neq \emptyset \iff$ no arbitrage opportunities

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- To price an option: find a self-financing portfolio based on the riskless asset and the underlying that has the same payoff as the option.
- Law of one price: option and portfolio have the same price
- Under $\mathbb{Q}$, the discounted portfolio it is also a martingale and so the price of the option becomes

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}H(S_T)|\mathcal{F}_t]$$

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- How to obtain a martingale measure $\mathbb{Q}$: Girsanov's theorem

- How are we sure that there exists a self-financing replicating strategy? In complete markets by martingale representation theorems. Linked to the uniqueness of the martingale measure.

- How can we compute the conditional expectation: in this case it is possible to find a closed-form solution -the Black-Scholes formula- thanks to the Markov property.

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A nobel prize winning formula:

$$\begin{aligned} V_t &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S_T^1 - K)^+ | \mathcal{F}_t] \\ &= S_t^1 N(d_1) - Ke^{-r(T-t)} N(d_2) \end{aligned}$$

with:

$$\begin{aligned} d_1 &= \frac{\log(S_t^1/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

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Shortcomings of Black-Scholes

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Implicit properties of Black-Scholes:

- Normality of log-returns:

  1. Log-returns: $Q_{t+1;\,t} = \ln\left(\frac{S^1_{t+1}}{S^1_t}\right)$
  2. But, $S^1_t = S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t)$
  3. So in Black-scholes: $Q_{t+1;\,t} \sim \mathcal{N}(\mu - \frac{1}{2}\sigma^2; \sigma^2)$

- Variance of log-returns ($= \sigma^2$) is the square of volatility. In Black-Scholes, σ is a constant.

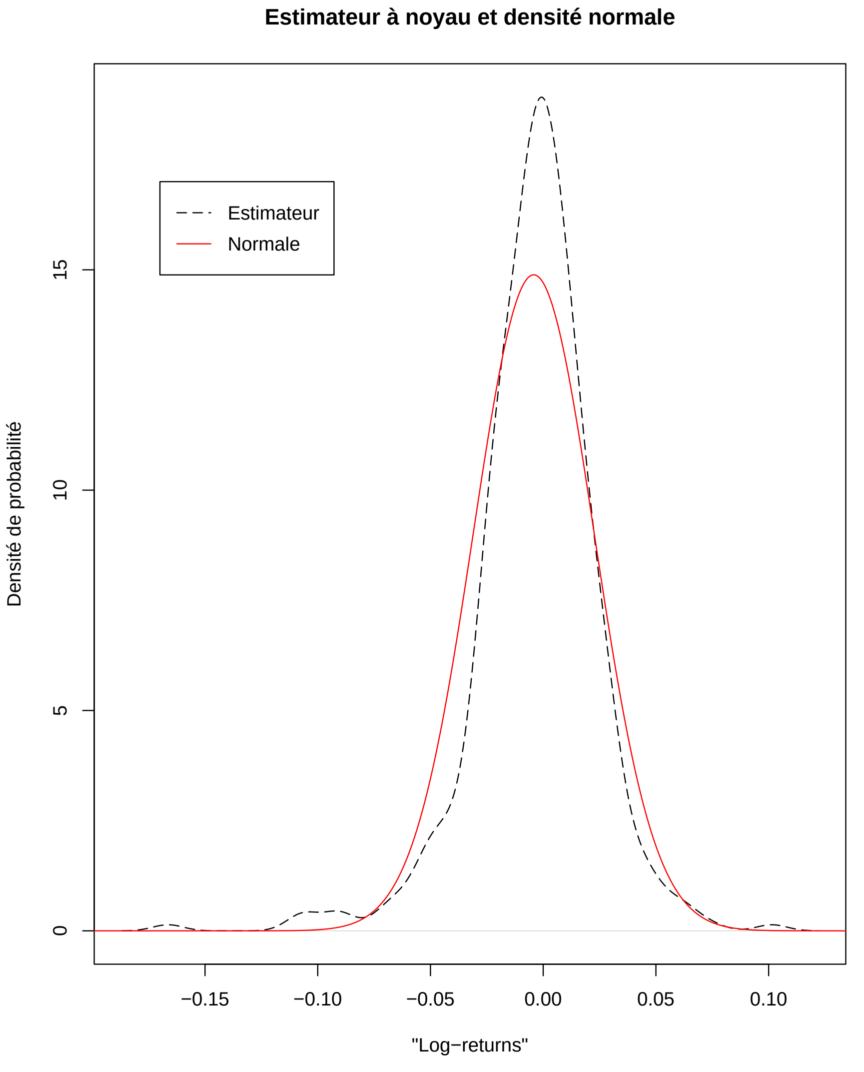
Estimateur à noyau et densité normale
Estimateur
Normale
Densité de probabilité
0
5
10
15
−0.15
−0.10
−0.05
0.00
0.05
0.10
"Log−returns"

Figure 1:

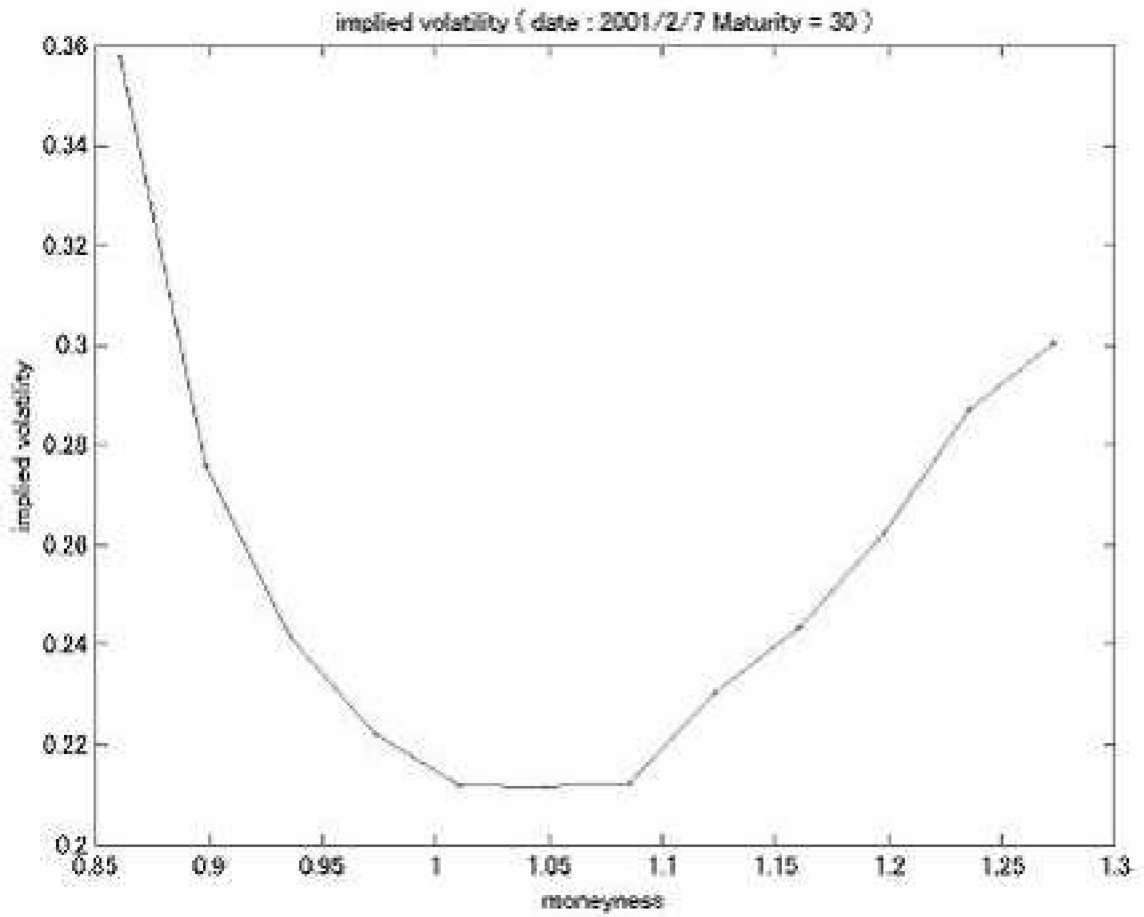
implied volatility (date : 2001/2/7 Maturity = 30)
0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
implied volatility
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25
1.3
moneyness

Figure 2:

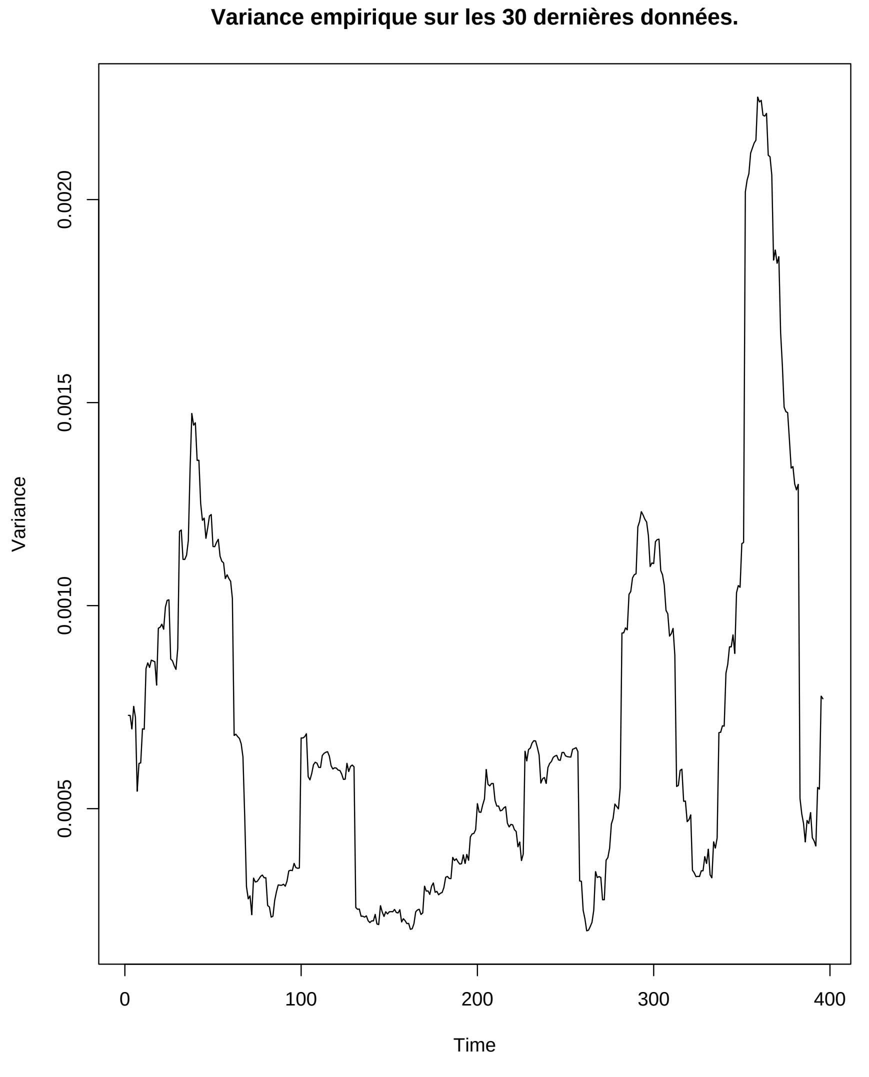
Variance empirique sur les 30 dernières données.
Variance
0.0005
0.0010
0.0015
0.0020
0
100
200
300
400
Time

Figure 3:

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Semi-Markov processes

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- Define the state-space by $E = \{x_1, ..., x_m\}$. Possible states of a system, for each $n \in \mathbb{N}$, a random variable X_n taking values in E.
- Random variable T_n taking values in $\mathbb{R}^+$ such that $0 = T_0 \leq T_1 \leq T_2 \leq ...$
- Interpret the sequence $(X_n)_{n \in \mathbb{N}}$ as being the successive states of a system and $(T_n)_{n \in \mathbb{N}}$ as the switching times of the system.

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- The stochastic process $(X, T) = \{X_n, T_n; n \in \mathbb{N}\}$ is said to be a homogeneous Markov renewal process with state space E provided that

$$\mathbb{P}[X_{n+1} = x_j, T_{n+1} - T_n \leq t | X_0, \ldots, X_n; T_0, \ldots T_n]$$

$$= \mathbb{P}[X_{n+1} = x_j, T_{n+1} - T_n \leq t | X_n = x_i] = Q(x_i, x_j, t)$$

- Q is called the semi-Markov kernel

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- Let us define ν by

$$\nu_t = \sup(n \geq 0 : T_n \leq t)$$

- Define Y as follows

$$Y_t = X_{\nu_t}$$

The process Y is called a semi-Markov process with kernel Q

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- We suppose the asset price dynamics are governed by (with Y_t a semi-Markov process)

$$dS_t^1 = S_t^1(\mu(Y_t)dt + \sigma(Y_t)dB_t)$$

- So the parameters of the model "switch", they are governed by the evolution of Y_t
- The aim is still to price and hedge options on S_t^1

- Some questions:
- Do the usual rules of stochastic calculus still apply?
- Is the market arbitrage free? Existence of martingale measures?
- Is the market complete? Under which conditions?
- How to price without the Markov property?

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- Define $Z = \{z_k; z_k = x_i - x_j : x_i,\ x_j \in E,\ \ i \neq j\}$
- The set Z contains $m(m-1)$ elements
- Define:

$$N_t(A) = \sum_{n\geq 1} 1_{\{T_n \leq t\}} 1_{\{Z_n \in A\}}$$

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- Then,

$$Y_t = \sum_{i=1}^{m(m-1)} \int_0^t z_i dN_t(z_i) = \sum_{i=1}^{m(m-1)} z_i N_t(z_i)$$

- This is very convenient since it implies that Y_t is a semi-martingale (as a process of finite-variation)
- So we can use the usual rules of stochastic calculus

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- By the girsanov theorem, it can be shown that there exists an infinite number of martingale measures.
- Furthermore, using the martingale representation theorem, we can show that market is incomplete.
- Can we complete the market? and under which conditions?

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- Let us add $m(m-1)$ assets with the following dynamics:

$$dC_t^i = C_{t^-}^i(\alpha_t^i dt + \sum_{j=1}^{m(m-1)} \gamma_t^i(z_j) dN_t(z_j))$$

- We have demonstrated that this implies uniqueness of the martingale measure if the following condition is verified: $det(G) \neq 0$ with $G = [g_{ij}]$ and $g_{ij} = \gamma^i(z_j)$ (for $1 \leq i, j \leq m(m-1)$)

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- We now work under the unique martingale measure
- By martingale representation theorems, we can show that the market is complete if $det(G^t) \neq 0$.
- But $det(G^t) = det(G)$ and so this condition is the same as the one assuring uniqueness of the martingale measure

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- Although this result may seem simple it is in fact quite profound as it links the uniqueness of the martingale measure to the completeness of the market. This is closely related to the so-called second fundamental theorem of asset pricing although this theorem is more a rule-of-thumb than a theorem since it has only been proved in some special cases.
- Might not work so well if state-space if infinite (work in progress)

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- What about the pricing and the hedging of options?
- Let $\mathbb{Q}$ be any martingale measure, the price of the option is still given by

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}H(S_T)|\mathcal{F}_t]$$

- However, because the process (S_t^1, Y_t) is not Markovian anymore, this becomes difficult to calculate.

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- This is still work in progress but one idea is the following.
- Define the process $K_t = t - T_{N_t(Z)}$
- The process (S_t, Y_t, K_t) is Markovian. So we need to identify the martingale measures for (S_t, Y_t, K_t) and then apply a feynman-Kac lemma to our conditional expectation (work in progress)

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- There is still a lot to be done in terms of hedging and pricing of options in semi-Markov regime switching models.
- One can also try and build term-structure models in a semi-Markov regime switching framework
- Finally, the aim is to estimate these models and compare them with existing models but this can only be done once a convincing theory is available