IMPACT OF UNDERWRITING CYCLES ON THE SOLVENCY OF AN INSURANCE COMPANY

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Context:

- Ruin probabilities = the core of actuarial mathematics
- The most of the results require
  - Independence assumption for aggregate claim amounts
  - BUT unrealistic with the presence of underwriting cycles

Organization:

- Underwriting cycles
- Dynamics of the insurer’s surplus
- Ruin probability for light-tailed claims
- Comparison with the classical case for light-tailed claims
- Numerical illustration
### Description

- **Four stages**
  - Several years of low profitability
  - Change to rapidly increase the profitability
  - Several years of high profitability
  - Decline to return to a period of low profitability

### Most important explanation

**Related with underwriting strategies**

- Competition encourages underwriters to ease up on underwriting standards
  - Adverse loss experience ⇒ cycle turns down
- Tightened underwriting standards and rate increases stop the decline
- Cycle goes up
  - Insurers decide to cut prices or raise rates.
Model

Let $\Pi_t$ the ratio of premiums to losses.

Assumptions:

- **H1** Insurance premiums reflect rational expectations
- **H2** Information is available with lags of about two years

Consequence:

$$\Pi_t = a_0 + a_1 \Pi_{t-1} + a_2 \Pi_{t-2} + \epsilon_t$$  \hspace{1cm} (1)

Observation:

With $a_1 > 0$, $a_2 < 0$ and $a_1^2 + 4a_2 < 0$ : cyclical dynamics

Period equals to $2\pi / \arccos(a_1/2\sqrt{-a_2})$
Additional assumptions for the following

- Stationarity of the \( \{ \Pi_t \} \) process

  \[
  \begin{align*}
  a_1 &> 0 \\
  -1 &< a_2 < 0 \\
  a_1^2 + 4a_2 &< 0
  \end{align*}
  \]

- Error terms are finished

  \[ \Pr[|\epsilon_t| < d] = 1 \]

- Viability of the market

  \[ \Rightarrow E[\Pi_t] = \frac{a_0}{1 - a_1 - a_2} > 1. \quad (2) \]
Insurance's surplus

Assumptions:

H1 A small or medium size company

H2 The company aims to maintain its existing portfolio

Consequences:

C1 \( U_n = u + \sum_{k=1}^{n} C_k - \sum_{k=1}^{n} Y_k \),
where \( C_k = E[Y_k] \Pi_k \).
\( \Rightarrow C_i = a_1 C_{i-1} + a_2 C_{i-2} + \tilde{\epsilon}_i \),
where \( \tilde{\epsilon}_i = E[Y](a_0 + \epsilon_i) \).

C2 The processes \( \{Y_k, k = 1, 2, \ldots\} \) and \( \{\tilde{\epsilon}_k, k = 1, 2, \ldots\} \) are mutually independent.
Ruin probability for light-tailed claims

Expression for the ultimate ruin probability

Context

- Same approach as GERBER (1982) when annual gains obey an autoregressive process.
- BUT be careful, here:
  - the premiums $C_i$ follow an AR(2) process
  - the annual claim amounts $Y_i$ are i.i.d.

Notation

$$\psi(u, c_0, c_{-1}) = \Pr[U_n \leq 0 \text{ for some } n| U_0 = u, C_0 = c_0, C_{-1} = c_{-1}]$$
Lemma

Assume that the moment generating function $E(e^{rY})$ exists for all $r$ in the neighborhood of the origin and is steep and the moment generating function $E(e^{r\tilde{c}})$ exists for some $r < 0$ and is steep. If the insurance market is viable, that is, if the inequality (2) is satisfied, then there exists a unique positive constant $\rho$ such that

$$E\left[e^{\left(\frac{-\rho\tilde{c}}{1-a_1-a_2}\right)}\right] E[e^{\rho Y}] = 1.$$  

(3)

Furthermore, we have

$$\rho(\eta) = \frac{2E[Y]}{V[Y] + \frac{E^2[Y]V[\epsilon]}{(1-a_1-a_2)^2}}\eta + O(\eta^2).$$  

(4)

with $\eta = E[\Pi] - 1$. 


Theorem

Under the assumptions of Lemma 1 with some $\rho > 0$ satisfying (3), we have

$$\psi(u, c_0, c_{-1}) = \frac{e^{-\rho \hat{u}}}{E[e^{-\rho \hat{U}_T} \mid T < \infty]},$$

(5)

where

$$\begin{cases}
\hat{U}_n = U_n + \frac{a_1 + a_2}{1-a_1-a_2} C_n + \frac{a_2}{1-a_1-a_2} C_{n-1} \\
\hat{u} = \hat{U}_0 \\
T = \inf\{n \mid U_n \leq 0\}
\end{cases}$$
Lundberg-type inequality

Recall that $a_1 + a_2 < 1$, $a_2 < 0$ and $\Pr[UT < 0| T < \infty] = 1$. With

\[
\begin{align*}
C_{T-1} &> 0 & \text{a.s.} \\
C_T &> 0 & \text{a.s.} \\
a_1 + a_2 &< 0 \quad \text{(additional condition)}
\end{align*}
\]

\[\Rightarrow \psi(u, c_0, c_{-1}) \leq e^{-\rho \hat{u}}. \quad (6)\]
Affirmation

- The exponential decay rate of the numerator of (5) provides the significant asymptotic behavior in $u$.

Explanation

1. Let $Z_n = \sum_{i=1}^{n}(Y_i - C_i)$. If $E[\exp(rZ_n)] < \infty$ for $0 < r < r_0$, we have

   \[
   \kappa(r) := \lim_{n \to \infty} \frac{1}{n} \kappa_n(r) = \lim_{n \to \infty} \frac{1}{n} \ln E[\exp(rZ_n)]
   \]
   
   exists for $0 < r < r_0$,

   \[
   \text{(A1)} \quad \kappa(r) := \lim_{n \to \infty} \frac{1}{n} \kappa_n(r) = \lim_{n \to \infty} \frac{1}{n} \ln E[\exp(rZ_n)]
   \]

2. By the Gärtner-Ellis theorem,

   \[
   \lim_{u \to \infty} \frac{1}{u} \ln \psi(u, c_0, c_{-1}) = -\rho.
   \]
Let $Y \sim \text{Exp}(\alpha)$.

**Ultimate ruin probability**

$|U_T|$ is again exponential, and independent of the other quantities.

$$\Rightarrow \psi(u, c_0, c_{-1}) = \frac{(\alpha - \rho)e^{-\rho \hat{u}}}{\alpha E \left[ e^{-\rho \left( \frac{a_1+a_2}{1-a_1-a_2} C_T + \frac{a_2}{1-a_1-a_2} C_{T-1} \right)} \mid T < \infty \right]}.$$ 

**Upper bound**

The upper bound can be improved to $\left( \frac{\alpha - \rho}{\alpha} \right) e^{-\rho \hat{u}}$.

**Asymptotic result**

The dependence on $u$ of $C_T, C_{T-1}$ is quite weak $\Rightarrow \psi(u) \sim \text{cst.} e^{-\rho u}$.
Comparison with the classical case for light-tailed claims

**Classical case**

\[ \tilde{\psi}(u) = \Pr[\tilde{U}_n \leq 0 \text{ for some } n \mid \tilde{U}_0 = u], \]
where \( \tilde{U}_n = u + nE[Y]E[\Pi] - \sum_{k=1}^{n} Y_k, \ n = 1, 2, \ldots \).

**Comparison**

We have

\[
\lim_{u \to \infty} \frac{1}{u} \ln \psi(u, c_0, c_{-1}) = -\rho \text{ for all } c_0, c_{-1},
\]
\[
\lim_{u \to \infty} \frac{1}{u} \ln \tilde{\psi}(u) = -\tilde{\rho},
\]

where \( \rho \) and \( \tilde{\rho} \) are respectively solutions of

\[
h(r) = E \left[ e^{\frac{-r\tilde{e}}{1-a_1-a_2}} \right] E[e^{rY}] = 1,
\]
\[
\tilde{h}(r) = e^{-rE[Y]E[\Pi]} E[e^{rY}] = 1.
\]
Comparison

- **Proposition:**
  The inequality $\tilde{\rho} > \rho$ holds true.

- **Proof:**
  We can show that $\tilde{h}(r) < h(r)$ for $r > 0$. Since $\tilde{h}(0) = h(0) = 1$; $\tilde{h}'(0) < 0$ and $h'(0) < 0$; $\tilde{h}''(r) > 0$ and $h''(r) > 0$, we necessarily have $\tilde{\rho} > \rho$.

$\Rightarrow$ there exists $u_0$ such that for all $u \geq u_0$, $\tilde{\psi}(u) \leq \psi(u, c_0, c_{-1})$ whatever $c_0$ and $c_{-1}$.

$\Rightarrow$ Market cycles increase asymptotically the risk.
Market

- Data: from Canadian motor insurance market.
- Parameters:

\[
\begin{align*}
\hat{a}_0 &= 0.6197 \\
\hat{a}_1 &= 0.9945 \\
\hat{a}_2 &= -0.4699 \\
\hat{\sigma}^2 &= 0.01602
\end{align*}
\]
Assume that $Y \sim Gamma(q_1, q_2)$, with $E[Y] = q_1 q_2$ and $V[Y] = q_1 q_2^2$.

**Assumptions to fix $q_1$ and $q_2$**

Let

- $Y^{\text{market}} = \text{the annual claim amount for the entire Canadian market}$
- $n = \text{the number of insured vehicles in the Canadian market}$

If

- $K_i = \text{annual claim amount of vehicle } i \Rightarrow Y^{\text{market}} = \sum_{i=1}^{n} K_i$.
- $K_i$’s are i.i.d
- $nE[K] = 1000$
- the market share of the company = $x\%$
Assumptions to fix $q_1$ and $q_2$

then

\[
\begin{align*}
    E[Y] &= x\% n E[K] = x\% 1000 \\
    \frac{\sqrt{V[Y]}}{E[Y]} &= \frac{\sqrt{V[\sum_{i=1}^{x\% n} K_i]}}{E[\sum_{i=1}^{x\% n} K_i]} = \frac{\sqrt{\sum_{i=1}^{x\% n} V[K_i]}}{\sum_{i=1}^{x\% n} E[K_i]} = \frac{\sqrt{x\% n V[K]}}{x\% n E[K]} = \frac{1}{\sqrt{x\%}} \frac{\sqrt{V[Y_{\text{market}}]}}{E[Y_{\text{market}}]}.
\end{align*}
\]

$\Rightarrow q_1$ and $q_2$ are deduced from $CV_{\text{market}} = \frac{\sqrt{V[Y_{\text{market}}]}}{E[Y_{\text{market}}]} = 0.17$. 
Market share = 5% ⇒ $q_1 = 1.71$ and $q_2 = 29.24$.

### Numerical results:

<table>
<thead>
<tr>
<th>$(c_{-1}, c_0)$</th>
<th>$u=0$</th>
<th>$u=50$</th>
<th>$u=100$</th>
<th>$u=150$</th>
<th>$u=200$</th>
<th>$u=250$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(62.5, 64.0)$</td>
<td>0.7746</td>
<td>0.3802</td>
<td>0.1772</td>
<td>0.0841</td>
<td>0.0410</td>
<td>0.0194</td>
</tr>
<tr>
<td>$(64.0, 57.5)$</td>
<td>0.7513</td>
<td>0.3827</td>
<td>0.1777</td>
<td>0.0777</td>
<td>0.0385</td>
<td>0.0192</td>
</tr>
<tr>
<td>$(57.5, 60.5)$</td>
<td>0.7661</td>
<td>0.3772</td>
<td>0.1794</td>
<td>0.0831</td>
<td>0.0440</td>
<td>0.0209</td>
</tr>
<tr>
<td>$(60.5, 58.0)$</td>
<td>0.7672</td>
<td>0.3676</td>
<td>0.1836</td>
<td>0.0678</td>
<td>0.0375</td>
<td>0.0185</td>
</tr>
<tr>
<td>$(58.0, 58.0)$</td>
<td>0.7574</td>
<td>0.3811</td>
<td>0.1778</td>
<td>0.0898</td>
<td>0.0418</td>
<td>0.0207</td>
</tr>
<tr>
<td>$(58.0, 61.5)$</td>
<td>0.7760</td>
<td>0.3848</td>
<td>0.1839</td>
<td>0.0668</td>
<td>0.0351</td>
<td>0.0166</td>
</tr>
<tr>
<td>$(61.5, 74.5)$</td>
<td>0.7523</td>
<td>0.3769</td>
<td>0.1772</td>
<td>0.0879</td>
<td>0.0426</td>
<td>0.0204</td>
</tr>
<tr>
<td>$(74.5, 77.0)$</td>
<td>0.7566</td>
<td>0.3786</td>
<td>0.1804</td>
<td>0.0653</td>
<td>0.0305</td>
<td>0.0162</td>
</tr>
<tr>
<td>$(58.0, 77.0)$</td>
<td>0.5059</td>
<td>0.2528</td>
<td>0.1256</td>
<td>0.0587</td>
<td>0.0297</td>
<td>0.0141</td>
</tr>
</tbody>
</table>
1. $\psi(u, c_0, c_{-1})$ and $\tilde{\psi}(u)$ decrease as $u$ increases.
2. $\psi(u, c_0, c_{-1})$ depends on the cycle when the insurer starts its business.

BUT, also for small $u$, $\psi(u, c_0, c_{-1}) \geq \tilde{\psi}(u)$ whatever $c_0$ and $c_{-1}$. 
We have studied the ruin probability when the premium income is subject to underwriting cycles. We derived:

* An explicit expression,
* An upper bound.

We have characterized the asymptotic behavior in $u$ of the ultimate ruin probability.

A comparison was performed with the classical case $\Rightarrow$ underwriting cycles increase the risk.