IMPACT OF UNDERWRITING CYCLES ON THE SOLVENCY OF AN INSURANCE COMPANY

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Context :

- Ruin probabilities = the core of actuarial mathematics
- The most of the results require
  - Independence assumption for aggregate claim amounts
  - BUT unrealistic with the presence of underwriting cycles

Organization :

- Underwriting cycles
- Dynamics of the insurer's surplus
- Ruin probability for light-tailed claims
- Comparison with the classical case for light-tailed claims
- Numerical illustration

Description

- Four stages
  - Several years of low profitability
  - Change to rapidly increase the profitability
  - Several years of high profitability
  - Decline to return to a period of low profitability

Most important explanation

Related with underwriting strategies

- Competition encourages underwriters to ease up on underwriting standards
- $\Rightarrow$ Adverse loss experience $\Rightarrow$ cycle turns down
- Tightened underwriting standards and rate increases stop the decline
- Cycle goes up

$\Rightarrow$ Insurers decide to cut prices or raise rates.

Model

Let Π_t the ratio of premiums to losses.

Assumptions :

- H1 Insurance premiums reflect rationnal expectations
- H2 Information is available with lags of about two years

Consequence :

$$\Pi_t = a_0 + a_1\Pi_{t-1} + a_2\Pi_{t-2} + \epsilon_t \tag{1}$$

Observation :

With $a_1 > 0$, $a_2 < 0$ and $a_1^2 + 4a_2 < 0$: cyclical dynamics

Period equals to $2\pi / \arccos(a_1/2\sqrt{-a_2})$

Additional assumptions for the following

- Stationarity of the $\{\Pi_t\}$ process

$$\Rightarrow \begin{cases} a_1 > 0 \\ -1 < a_2 < 0 \\ a_1^2 + 4a_2 < 0 \end{cases}$$

- Error terms are finished

$$\Pr[|\epsilon_t| < d] = 1$$

- Viability of the market

$$\Rightarrow E[\Pi_t] = \frac{a_0}{1 - a_1 - a_2} > 1. \tag{2}$$

Insurer's surplus

Assumptions :

- H1 A small or medium size company
- H2 The company aims to maintain its existing portfolio

Consequences :

- C1 $U_n = u + \sum_{k=1}^{n} C_k - \sum_{k=1}^{n} Y_k$,
  where $C_k = E[Y_k]\,\Pi_k$.
  $\Rightarrow C_i = a_1 C_{i-1} + a_2 C_{i-2} + \tilde{\epsilon}_i$,
  where $\tilde{\epsilon}_i = E[Y](a_0 + \epsilon_i)$.
- C2 The processes $\{Y_k,\ k = 1, 2, \ldots\}$ and $\{\tilde{\epsilon}_k,\ k = 1, 2, \ldots\}$ are mutually independent.

Context

- Same approach as GERBER (1982) when annual gains obey an autoregressive process.
- BUT be careful, here :
  - the premiums C_i follow an AR(2) process
  - the annual claim amounts Y_i are i.i.d.

Notation

$$\psi(u, c_0, c_{-1}) = \Pr[U_n \leq 0 \text{ for some } n | U_0 = u, C_0 = c_0, C_{-1} = c_{-1}].$$

Lemma

Assume that the moment generating function $E(e^{rY})$ exists for all r in the neighborhood of the origin and is steep and the moment generating function $E(e^{r\tilde{\epsilon}})$ exists for some $r < 0$ and is steep. If the insurance market is viable, that is, if the inequality (2) is satisfied, then there exists a unique positive constant ρ such that

$$E\left[e^{\left(\frac{-\rho\tilde{\epsilon}}{1-a_1-a_2}\right)}\right] E[e^{\rho Y}] = 1. \tag{3}$$

Furthermore, we have

$$\rho(\eta) = \frac{2E[Y]}{V[Y] + \frac{E^2[Y]V[\epsilon]}{(1-a_1-a_2)^2}}\eta + O(\eta^2). \tag{4}$$

with $\eta = E[\Pi] - 1$.

Theorem

Under the assumptions of Lemma 1 with some $\rho > 0$ *satisfying* (3), *we have*

$$\psi(u, c_0, c_{-1}) = \frac{e^{-\rho \hat{u}}}{E[e^{-\rho \hat{U}_T} | T < \infty]}, \qquad (5)$$

where

$$\begin{cases} \hat{U}_n = U_n + \frac{a_1 + a_2}{1 - a_1 - a_2} C_n + \frac{a_2}{1 - a_1 - a_2} C_{n-1} \\ \hat{u} = \hat{U}_0 \\ T = \inf\{n | U_n \leq 0\} \end{cases}$$

Lundberg-type inequality

Recall that $a_1 + a_2 < 1$, $a_2 < 0$ and $\Pr[U_T < 0 | T < \infty] = 1$.
With

$$\begin{cases} C_{T-1} > 0 & \text{a.s.} \\ C_T > 0 & \text{a.s.} \\ a_1 + a_2 < 0 \text{ (additional condition)} & \end{cases}$$

$$\Rightarrow \psi(u, c_0, c_{-1}) \leq e^{-\rho \hat{u}}. \tag{6}$$

Affirmation

- The exponential decay rate of the numerator of (5) provides the significant asymptotic behavior in u.

Explanation

1. Let $Z_n = \sum_{i=1}^{n}(Y_i - C_i)$. If $E[\exp(rZ_n)] < \infty$ for $0 < r < r_0$, we have
   - (A1) $\kappa(r) := \lim_{n\to\infty} \frac{1}{n}\kappa_n(r) = \lim_{n\to\infty} \frac{1}{n} \ln E[\exp(rZ_n)]$ exists for $0 < r < r_0$,
   - (A2) there exists a unique $0 < \rho < r_0$ such that $\kappa(\rho) = 0$.
2. By the Gärtner-Ellis theorem,

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi(u, c_0, c_{-1}) = -\rho. \quad (7)$$

Let $Y \sim Exp(\alpha)$.

Ultimate ruin probability

$|U_T|$ is again exponential, and independent of the other quantities.

$$\Rightarrow \psi(u, c_0, c_{-1}) = \frac{(\alpha - \rho)e^{-\rho\hat{u}}}{\alpha\, E\left[e^{-\rho\left(\frac{a_1+a_2}{1-a_1-a_2}C_T + \frac{a_2}{1-a_1-a_2}C_{T-1}\right)} \middle| T < \infty\right]}.$$

Upper bound

The upper bound can be improved to $\left(\frac{\alpha-\rho}{\alpha}\right)\, e^{-\rho\hat{u}}$.

Asymptotic result

The dependence on u of C_T, C_{T-1} is quite weak $\Rightarrow \psi(u) \sim cst.e^{-\rho u}$.

Classical case

$\widetilde{\psi}(u) = \Pr[\widetilde{U}_n \leq 0 \text{ for some } n | \widetilde{U}_0 = u]$,
where $\widetilde{U}_n = u + nE[Y]\,E[\Pi] - \sum_{k=1}^{n} Y_k, \; n = 1, 2, \ldots$.

Comparison

We have

$$\lim_{u\to\infty} \frac{1}{u} \ln \psi(u, c_0, c_{-1}) = -\rho \text{ for all } c_0, c_{-1},$$
$$\lim_{u\to\infty} \frac{1}{u} \ln \widetilde{\psi}(u) = -\widetilde{\rho},$$

where ρ and $\widetilde{\rho}$ are respectively solutions of

$$h(r) = E\left[e^{\frac{-r\tilde{\epsilon}}{1-a_1-a_2}}\right] E[e^{rY}] = 1,$$
$$\widetilde{h}(r) = e^{-rE[Y]E[\Pi]}\,E[e^{rY}] = 1.$$

Comparison

- Proposition :
  The inequality $\widetilde{\rho} > \rho$ holds true.
- Proof :
  We can show that $\widetilde{h}(r) < h(r)$ for $r > 0$. Since $\widetilde{h}(0) = h(0) = 1$; $\widetilde{h}'(0) < 0$ and $h'(0) < 0$; $\widetilde{h}''(r) > 0$ and $h''(r) > 0$, we necessarily have $\widetilde{\rho} > \rho$.

$\Rightarrow$ there exists u_0 such that for all $u \geq u_0$, $\widetilde{\psi}(u) \leq \psi(u, c_0, c_{-1})$ whatever c_0 and c_{-1}.
$\Rightarrow$ Market cycles increase asymptotically the risk.

Market

- Data : from Canadian motor insurance market.
- Parameters :

$$\begin{cases} \widehat{a}_0 = 0.6197 \\ \widehat{a}_1 = 0.9945 \\ \widehat{a}_2 = -0.4699 \\ \widehat{\sigma}^2 = 0.01602 \end{cases}$$

Assume that $Y \sim Gamma(q_1, q_2)$, with $E[Y] = q_1 q_2$ and $V[Y] = q_1 q_2^2$.

Assumptions to fix q_1 and q_2

Let

- Y^{market} = the annual claim amount for the entire Canadian market
- n = the number of insured vehicles in the Canadian market

If

- K_i = annual claim amount of vehicle $i \Rightarrow Y^{\text{market}} = \sum_{i=1}^{n} K_i$.
- K_i's are i.i.d
- $nE[K] = 1000$
- the market share of the company = $x\%$

Assumptions to fix q_1 and q_2

then

$$\begin{cases} E[Y] = x\% n E[K] = x\% 1000 \\ \frac{\sqrt{V[Y]}}{E[Y]} = \frac{\sqrt{V[\sum_{i=1}^{x\% n} K_i]}}{E[\sum_{i=1}^{x\% n} K_i]} = \frac{\sqrt{\sum_{i=1}^{x\% n} V[K_i]}}{\sum_{i=1}^{x\% n} E[K_i]} = \frac{\sqrt{x\% \, n \, V[K]}}{x\% \, n \, E[K]} = \frac{1}{\sqrt{x\%}} \frac{\sqrt{V[Y^{\text{market}}]}}{E[Y^{\text{market}}]}. \end{cases}$$

$\Rightarrow$ q_1 and q_2 are deduced from $CV^{\text{market}} = \frac{\sqrt{V[Y^{\text{market}}]}}{E[Y^{\text{market}}]} = 0.17.$

Market share = 5% $\Rightarrow q_1 = 1.71$ and $q_2 = 29.24$.

Numerical results :

Ultimate ruin probability							
	(c_{-1}, c_0)	u=0	u=50	u=100	u=150	u=200	u=250
ψ	(62.5, 64.0)	0.7746	0.3802	0.1772	0.0841	0.0410	0.0194
	(64.0, 57.5)	0.7513	0.3827	0.1777	0.0777	0.0385	0.0192
	(57.5, 60.5)	0.7661	0.3772	0.1794	0.0831	0.0440	0.0209
	(60.5, 58.0)	0.7672	0.3676	0.1836	0.0678	0.0375	0.0185
	(58.0, 58.0)	0.7574	0.3811	0.1778	0.0898	0.0418	0.0207
	(58.0, 61.5)	0.7760	0.3848	0.1839	0.0668	0.0351	0.0166
	(61.5, 74.5)	0.7523	0.3769	0.1772	0.0879	0.0426	0.0204
	(74.5, 77.0)	0.7566	0.3786	0.1804	0.0653	0.0305	0.0162
$\widetilde{\psi}$	/	0.5059	0.2528	0.1256	0.0587	0.0297	0.0141

1. $\psi(u, c_0, c_{-1})$ and $\tilde{\psi}(u)$ decrease as u increases.
2. $\psi(u, c_0, c_{-1})$ depends on the cycle when the insurer starts its business.

BUT, also for small u, $\psi(u, c_0, c_{-1}) \geq \widetilde{\psi}(u)$ whatever c_0 and c_{-1}.

- We have studied the ruin probability when the premium income is subject to underwriting cycles. We derived :
  - * An explicit expression,
  - * An upper bound.
- We have characterized the asymptotic behavior in u of the ultimate ruin probability.
- A comparison was performed with the classical case $\Rightarrow$ underwriting cycles increase the risk.