

# **Stochastic Frontier Analysis of Efficiency of Moroccan Municipalities**

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# Efficiency analysis

- Basic idea

- Comparison between the Decision Making Units DMU (firms, for example) in order to know how the inputs are used to produce outputs.

- Nonparametric Data Envelopment Analysis (DEA)  
or the Free Disposal Hull (FDH),
    - Stochastic Parametric Frontier Analysis (FSA)

- ...

# Stochastic Frontier Analysis (SFA)

- Error term is divided in two **independent** terms,

$$\varepsilon_i = v_i - u_i$$

$i$  : Decision Making unit number  $i$  (DMUi)

- $v_i$  reflects the pure randomness or the usual **statistic noise**,  $v_i \sim N(0, \sigma_v^2)$

- $u_i$  reflects the **technical inefficiency**,

and  $u$  is a non-negative error term ( $u_i \geq 0$ )

$u_i = 0$  for a technically efficient decision unit

# Stochastic Frontier Analysis (SFA)

- Some possible distribution of  $u_i$ 
  - Half-Normal  
 $u_i \sim iidN^+(0, \sigma_u^2)$
  - Truncated Normal  
 $u_i \sim iidN(\mu, \sigma_u^2)$  truncated at zero
  - Exponentiel  
 $u_i \sim iidExp\left(\frac{1}{\sigma_u}\right)$
  - Gamma ...

# Stochastic Frontier Analysis (SFA)

- Stochastic model (Production Function) with cross-sectional data

$$y_i = f(x_i, \beta) + v_i - u_i$$

Where

$f(x_i, \beta)$  : The production technology, it is assumed either a [Cobb-Douglas](#) or a [translog](#) function (less restrictive).

$y_i$  : The observed output for observation  $i$  (one logged output);

$x_i$  : A vector of inputs for observation  $i$  (logged);

$\beta$  : A vector of parameters to be estimated;

# Estimation of the stochastic model

- Estimation can be made using
  - Maximum Likelihood (**ML**) method
  - Corrected Ordinary Least Squares (**COLS**) method. Greene (1980) proposes to correct the bias by shifting  $\beta_0$  ,

$$\hat{\beta}_0^* = \hat{\beta}_0 + \varepsilon^*$$

where  $\varepsilon^* = \max(\varepsilon)$

# Estimation of the stochastic model

- How to separate the error term into two components  $v$  and  $u$ 
  - $\varepsilon_i$  can be estimated as  $\hat{\varepsilon}_i = y_i - f(x_i, \hat{\beta})$
  - Jondrow, Knox Lovell, Materov and Schmidt (1981) have proposed a decomposition by considering the expected value of  $u$  , conditional on  $\varepsilon = v - u$

$$E\left(\frac{u}{\varepsilon}\right)$$

# Estimation of the stochastic model

- The half-normal distribution (For example)

$$h(u) = \frac{2}{\sqrt{2\pi}\sigma_u} \exp\left\{-\frac{1}{2\sigma_u^2}u^2\right\}, \quad u \geq 0$$

$$g(v, u) = f(v).h(u) = \frac{1}{\pi\sigma_v\sigma_u} \left[ \exp\left(-\frac{1}{2\sigma_v^2}v^2 - \frac{1}{2\sigma_u^2}u^2\right) \right]$$

Replace  $v = u + \varepsilon$  to obtain  $g(u, \varepsilon)$

$$g(\varepsilon) = \int_0^{+\infty} g(u, \varepsilon) du = \frac{2}{\sqrt{2\pi}\sigma} (1 - \Phi) \left[ \exp\left(-\frac{1}{2\sigma^2}\varepsilon^2\right) \right]$$

$$g\left(\frac{u}{\varepsilon}\right) = \frac{g(u, \varepsilon)}{g(\varepsilon)} \quad \Rightarrow \quad E\left(\frac{u}{\varepsilon}\right) = \mu_* + \sigma_* \frac{\phi\left(-\frac{\mu_*}{\sigma_*}\right)}{1 - \Phi\left(-\frac{\mu_*}{\sigma_*}\right)}$$



# Estimation of the stochastic model

- The half-normal distribution (cont.)

- $\hat{u} = E\left(\frac{u}{\hat{\varepsilon}}\right)$

avec  $\mu_* = -\frac{\sigma_u^2 \varepsilon}{\sigma^2}$  ,  $\sigma_*^2 = \frac{\sigma_v^2 \sigma_u^2}{\sigma^2}$  ,  $\sigma^2 = \sigma_v^2 + \sigma_u^2$

and  $\lambda = \frac{\sigma_u}{\sigma_v} \geq 0$  is a measure of asymmetry

(Skewness) of the disturbance term  $\varepsilon$  .

# Estimation of the stochastic model

- The maximum likelihood estimator (MLE) of  $\varphi = (\sigma, \lambda, \beta)$  is

$$\hat{\varphi}_{ML} = \arg \max_{\varphi} l(\varphi)$$

and  $\widehat{TE} = \exp(-\hat{u})$

# Estimation of the stochastic model

- Estimation methods
  - Analytical estimation (Not usually possible)  
Alternative:
  - Numerical estimation;
  - Monte Carlo Simulations.

# Application

- DMUs : 91 (1298) municipalities (DMUs)
- One input : Recipe of functioning (Urban tax, tax on the collection of the waste, subsidies...)
- One output : Financial autonomy

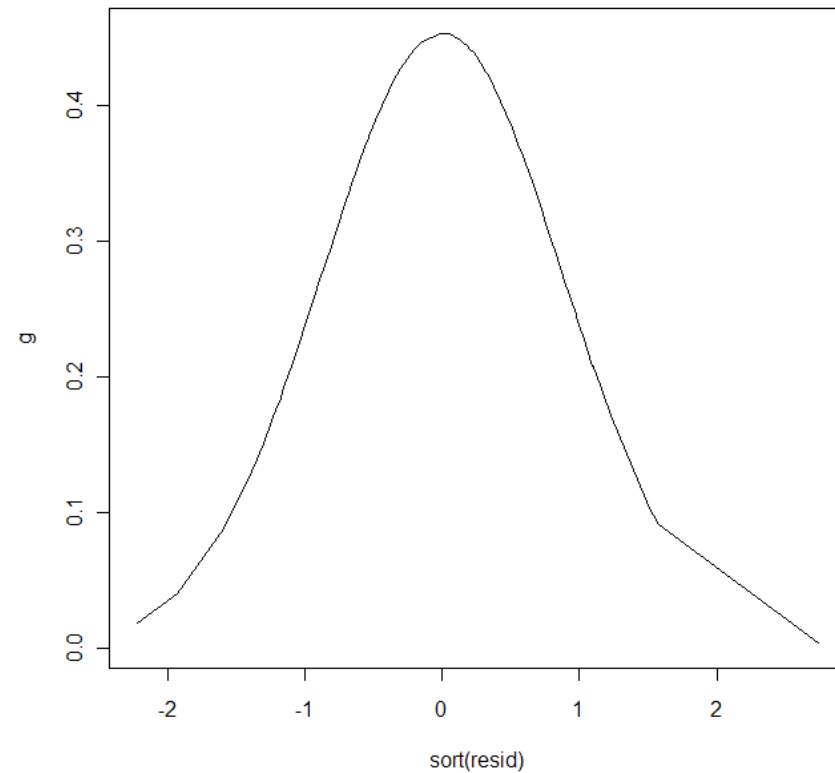
# Estimation of the stochastic model

- The half-normal distribution of  $u_i$  (a and b)  
 $v_i \sim iidN(0, \sigma_v^2)$  and  $u_i \sim iidN^+(0, \sigma_u^2)$
- The truncated-normal distribution of  $u_i$  (c and d)  
 $v_i \sim iidN(0, \sigma_v^2)$  and  $u_i \sim iidN(\mu, \sigma_u^2)$  truncated at zero
- $u_i \sim iidN(m_i, \sigma_u^2)$  truncated at zero where  $m_i = z_i \delta$   
(e and f)

# Results

- N=91 : does not provides a valid  $TE_i$  due to the positive Skewness of the distribution of  $\mathcal{E}$

Fig.1: Density of mle residuals



# Results

Table1: Efficiencies of the six models

Obs	Eff. Half-N. Cobb D. (a)	Eff. Half-N. Translog (b)	Eff. Trunc-N. Cobb D. (c)	Eff. Trunc-N. Translog (d)	Eff. $m_i = z_i \delta$ Cobb D. (e)	Eff. $m_i = z_i \delta$ Translog (f)
1	0,097	0,092	0,122	0,121	0,133	0,132
2	0,094	0,089	0,117	0,116	0,128	0,126
3	0,095	0,090	0,119	0,117	0,129	0,128
4	0,092	0,088	0,114	0,113	0,124	0,123
5	0,087	0,084	0,108	0,107	0,117	0,116
6	0,550	0,544	0,641	0,642	0,671	0,669
⋮	⋮	⋮	⋮	⋮	⋮	⋮
244	0,919	0,917	0,916	0,915	1,000	1,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
1296	0,435	0,432	0,538	0,541	0,577	0,575
1297	0,691	0,693	0,743	0,747	0,761	0,761
1298	0,257	0,241	0,339	0,338	0,380	0,376
-logL	-1771.50	-1771.165	-1764.131	-1764.117	-1763.299	-1763.286
mean	0.478	0.475	0.558	0.560	0.587	0.586

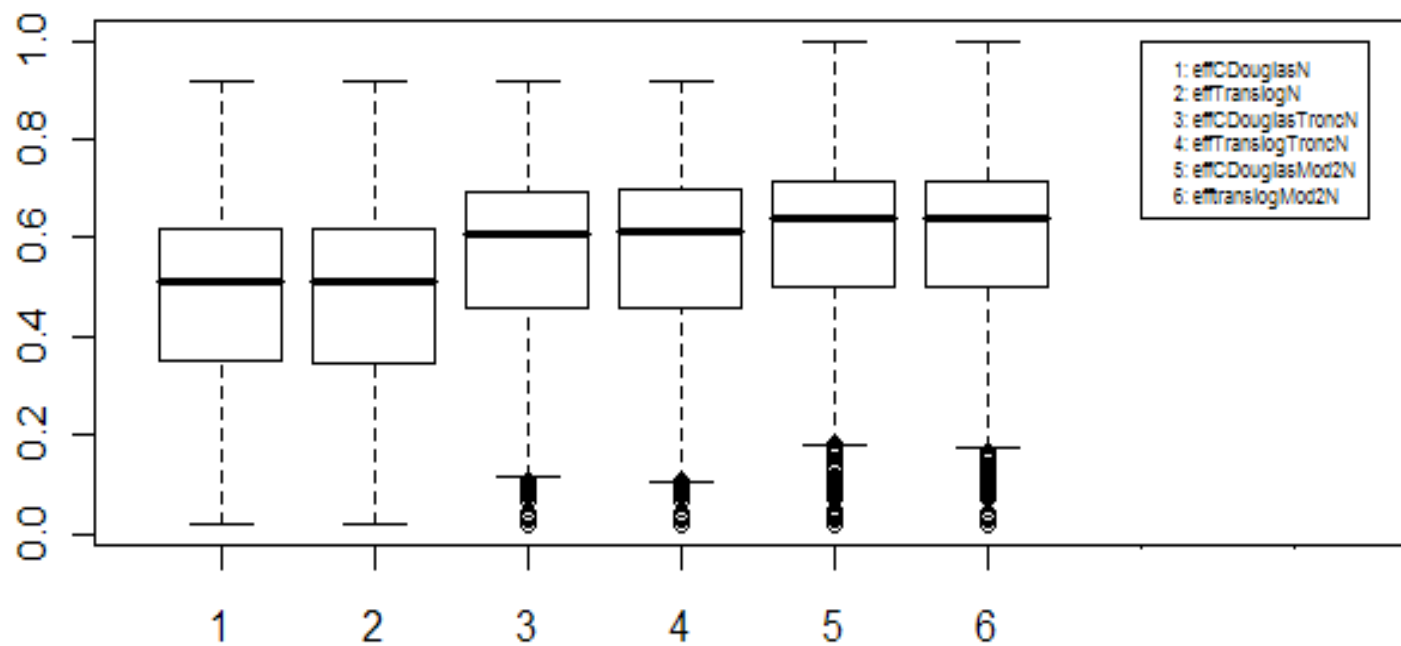
# Results

- Table 1 and Fig. 2 indicate
  - Efficiencies are different according to distribution of  $u_i$ ;
  - Efficiencies are not more different according to Cobb-Douglas or translog functions;
  - The two models with the half-normal distribution (a and b) provide smaller technical efficiencies than the four others;
  - $\lambda = 3.537 \neq 0$ , no problem of skewness.



# Results

Fig2 Boxlot for all models



# Include the Instrumental Variable in the estimation

- When  $\text{cov}(x, \varepsilon) \neq 0$  and  $E(v/Z) = 0$

Estimation in two Stages

- $x = Z' \Pi + v$  is estimated by OLS
- $y = \hat{x}' \beta + \varepsilon$  is estimated by OLS (ML)

$$\hat{\beta}_{IV} = (x' P_Z x)^{-1} x' P_Z y \quad \text{with} \quad P_Z = Z(Z' Z)^{-1} Z'$$

# Include the Copula in the estimation

- If  $U \perp V$  it is recommended to find the joint density  $g(u, v)$  with copula

Then

$$G(u, v) = C_{\theta}(F_1(u), F_2(v))$$

or 
$$g(u, v) = f_1(u) \cdot f_2(v) \cdot c_{\theta}(F_1(u), F_2(v))$$

# Include the Copula in the estimation

- Farlie-Gumbel-Morganstern (FGM) copula

$$\begin{aligned}g_{\theta}(u, v) &= f_1(u).f_2(v).c_{\theta}(F_1(u), F_2(v)) \\&= f_1(u).f_2(v).[1 + \theta - 2\theta F_1(u) - 2\theta F_2(v) + 4\theta F_1(u)F_2(v)] \\&= \left(\frac{2}{\sigma_u} \phi\left(\frac{u}{\sigma_u}\right)\right) \cdot \phi\left(\frac{v}{\sigma_v}\right) \\&\quad \cdot \left[1 + \theta - 2\theta \left(\frac{2}{\sigma_u} \Phi\left(\frac{u}{\sigma_u}\right)\right) - 2\theta \Phi\left(\frac{v}{\sigma_v}\right) + 4\theta \left(\frac{2}{\sigma_u} \Phi\left(\frac{u}{\sigma_u}\right)\right) \Phi\left(\frac{v}{\sigma_v}\right)\right].\end{aligned}$$