NONPARAMETRIC ESTIMATION OF EXTREME VALUE COPULAS IN ARBITRARY DIMENSIONS

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Introduction

Let $\mathbf{X}_i = (X_{i1}, \ldots, X_{ip})$ i.i.d. p-variate random vectors with joint df F. Define:

$$\begin{aligned} \mathbf{M}_n &= (M_{n1}, \ldots, M_{np}) \\ &= (\vee_{i=1}^n X_{i1}, \ldots, \vee_{i=1}^n X_{ip}) \end{aligned}$$

If for some normalizing constants $(a_{n,1}, b_{n,1}), \ldots, (a_{n,p}, b_{n,p})$:

$$\begin{aligned} & \lim_{n\to\infty} \mathbf{P}\left(\frac{M_{n,1} - a_{n,1}}{b_{n,1}} \leq x_1, \ldots, \frac{M_{n,p} - a_{n,p}}{b_{n,p}} \leq x_p \right) \\ = \; & \lim_{n\to\infty} F^n(a_{n,1}x_1 + b_{n,1}, \ldots, a_{n,p}x_p + b_{n,p}) = G(x_1, \ldots, x_p) \end{aligned}$$

then F is in the domain of attraction of G ($F \in \mathrm{D}(G)$), if G is a p-variate distribution function with nondegenerate marginals. G is called a p-variate extreme-value distribution.

Introduction

Definition

G is a multivariate extreme value distribution if and only if:

1. its margins $G_1, \ldots, G_p$ are univariate extreme value distributions (similar definition with $p = 1$)
2. $$G(\mathbf{x}) = \exp[-l\{-\log G_1(x_1), \ldots, -\log G_p(x_p)\}] \qquad (1)$$

   where l is the so-called stable tail dependence function.

   $$l(\mathbf{y}) = \lim_{s \downarrow 0} \frac{1}{s} \Pr[\textstyle\bigcup_{j=1}^{p} \{F_j(X_j) > 1 - s y_j\}] \qquad (2)$$

Introduction

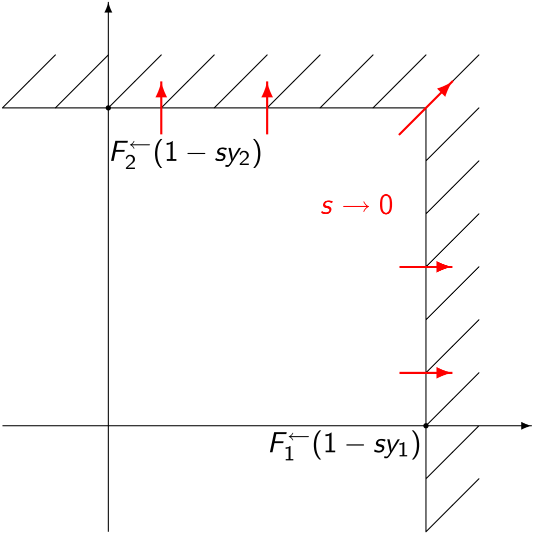


Figure: Graphical interpretation of stable dependence function l. Name is deduced from the L-shape of the probability domain. ($F^{\leftarrow}$: generalized inverse.)

Introduction

Definition

The restriction of l to the unit simplex

$$\Delta_p = \{(w_1, \dots, w_p) : \sum_{i=1}^{p} w_i = 1, w_j \geq 0, \forall j = 1, \dots, p\}$$

is called Pickands dependence function A.

A satisfies the following properties:

- $A(e_j) = 1 \quad \forall j = 1, \dots, p$
- A is convex
- $\max(w_1, \dots, w_p) \leqslant A(\mathbf{w}) \leqslant 1$

Example: Gumbel or logistic dependence function

$$A(\mathbf{w}) = (w_1^r + \dots + w_p^r)^{1/r}, \quad r \geq 1$$

Introduction

Recall the definition of multivariate extreme-value distribution:

$$G(\mathbf{x}) = \exp[-l\{-\log G_1(x_1), \ldots, -\log G_p(x_p)\}]$$

The extreme-value copula C is defined by:

$$G(x) = C(G_1(x_1), \ldots, G_p(x_p)) \tag{3}$$

Definition

It follows immediately that the expression for C is given by:

$$\begin{aligned} C(\mathbf{u}) &= \exp\{-l(-\log u_1, \ldots, -\log u_p)\} \\ &= \exp\left\{\left(\sum_{j=1}^{p} \log u_j\right) A\left(\frac{\log u_1}{\sum_{j=1}^{p} \log u_j}, \ldots, \frac{\log u_{p-1}}{\sum_{j=1}^{p} \log u_j}\right)\right\} \end{aligned}$$

for $0 < u_j \leqslant 1$. Such copulas are called extreme-value copulas.

Introduction

An alternative definition for extreme-value copulas is given below:

Definition

A copula C is an extreme-value copula iff it is max-stable

$$\forall t > 0 : \{C(u_1^{1/t}, \ldots, u_p^{1/t})\}^t = C(u_1, \ldots, u_p) \tag{4}$$

Example: Gumbel (logistic) dependence function:
$A(\mathbf{w}) = (w_1^r + \cdots + w_p^r)^{1/r}, \quad r \geq 1$ For all $t > 0$:

$$\begin{aligned} C^t(u_1^{1/t}, \ldots, u_p^{1/t}) &= \exp\left\{ -[(-\log u_1^{1/t})^r + \cdots + (-\log u_p^{1/t})^r]\right\}^{t/r} \\ &= \exp\left\{ -\frac{1}{t}[(-\log u_1)^r + \cdots + (-\log u_p)^r]\right\}^{t/r} \\ &= C(u_1, \ldots, u_p) \end{aligned}$$

Introduction

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a p-variate random sample from an unknown distribution F with continuous margins $F_1, \ldots, F_p$ and extreme value copula C.

Problem

Estimate C or, equivalently, its Pickands dependence function A in the case **the margins $F_1, \ldots, F_p$ are known.**

Remark:
For $p = 2$, the case with unknown margins is treated in:
Genest C. and Segers J. (2009) Rank-based inference for bivariate extreme-value copulas *The Annals of Statistics*
It consists in replacing $U_{ij} = F(X_{ij})$ by the rank estimator $\hat{U}_{ij} = \frac{R_i^{X_j}}{n+1}$.

Literature

Different estimators for dimension $p > 2$ can be extended to higher dimensions:

- Deheuvels (1991)
- Hall & Tajvidi (2000)
- Pickands (1981)
- ZWP (2008)

ZWP-estimator: extension of the CFG estimator to higher dimensions.

ZWP-estimator

Define:

$$Z_{ij} = \frac{\bigvee_{l \neq j} \frac{\log F_l(X_{il})}{w_l}}{\frac{\log F_j(X_{ij})}{1-w_j} + \bigvee_{l \neq j} \frac{\log F_l(X_{il})}{w_l}} \tag{5}$$

One can show that:

$$\begin{aligned} &\mathrm{P}(Z_{ij} \leq z) = z + z(1-z) \times \\ &\frac{\partial}{\partial z} \log A\left(\frac{zw_1}{1-w_j}, \ldots, \frac{zw_{j-1}}{1-w_j}, 1-z, \frac{zw_{j+1}}{1-w_j}, \ldots, \frac{zw_{p-1}}{1-w_j}\right) \end{aligned} \tag{6}$$

Taking into account the boundary conditions, the preceding P.D.E. admits the following solution:

$$\log A(w_1, \ldots, w_{p-1}) = \int_0^{1-w_j} \frac{\mathrm{P}(Z_{ij} \leq z) - z}{z(1-z)} dz \tag{7}$$

Estimator:

1. Replace $\mathrm{P}(Z_{ij} \leq z)$ by its empirical distribution.
2. Add weight functions $\lambda_j(\cdot)$ to satisfy boundary conditions, i.e $\hat{A}(\mathbf{e}_j) = 1$.

ZWP-estimator

Definition

The ZWP-estimator is defined as follows:

$$\log \hat{A}_n^{ZWP}(\mathbf{w}) = \sum_{j=1}^{p} \lambda_j(\mathbf{w}) \frac{1}{n} \sum_{i=1}^{n} \int_0^{1-w_j} \frac{\mathbf{1}(Z_{ij} \leq z) - z}{z(1-z)} dz$$

with weight functions:

- $\lambda_j(\mathbf{w}) \geq 0 \quad \forall j = 1, \dots, p$
- $\sum_{j=1}^{p} \lambda_j(\mathbf{w}) = 1$

Main disadvantage: No real statistic criterion for choosing the optimal weight functions.

Naive estimator

Observe that:

$$\xi_i(\mathbf{w}) = -\bigvee_{j=1}^{p} \frac{\log(U_{ij})}{w_j} \sim \mathrm{Exp}\Big(A(\mathbf{w})\Big) \left(\mathrm{E}\xi_i(\mathbf{w}) = \frac{1}{A(\mathbf{w})}\right) \quad (8)$$

Exploit the relationship between Gumbel and Exponential distributions:

$$-\log \xi_i(\mathbf{w}) = \underbrace{-\log\left(\frac{\xi_i(\mathbf{w})}{1/A(\mathbf{w})}\right)}_{\text{Standard Gumbel}} + \log A(\mathbf{w})$$

From where we deduce that:

$$\log A(\mathbf{w}) = \mathrm{E}(-\log \xi_i(\mathbf{w}) - \gamma)$$

Naive estimator

Definition

Naive estimator inspired from Segers(2007):

$$\log\left(\hat{A}_n(\mathbf{w})\right) = -\frac{1}{n}\sum_{i=1}^{n}\log\left(\xi_i(\mathbf{w})\right) - \gamma \tag{9}$$

with $\gamma = 0.577\ldots$ (= mean of standard Gumbel)

Naive in the sense that this estimator is unbiased but does not necessarily satisfy boundary conditions, i.e.

$$\hat{A}_n(\mathbf{e}_j) \neq 1 \quad \forall j = 1, \ldots, p$$

Lemma (Gudendorf & Segers, 2009)

Let $\mathbf{w} \in \Delta_p$ and the constraints on the weight functions be still valid, then:

$$\log \hat{A}_n^{ZWP}(\mathbf{w}) = \log \hat{A}_n(\mathbf{w}) - \sum_{j=1}^{p} \lambda_j(\mathbf{w}) \log \hat{A}_n(\mathbf{e}_j) \qquad (10)$$

Remember:

$$\log\left(\hat{A}_n(\mathbf{w})\right) = -\frac{1}{n}\sum_{i=1}^{n}\log\left(\xi_i(\mathbf{w})\right) - \gamma \tag{11}$$

The preceding estimator:

$$\log\left(\hat{A}_n^{ZWP}(\mathbf{w})\right) = \log\left(\hat{A}_n(\mathbf{w})\right) - \sum_{j=1}^{p}\lambda_j(\mathbf{w})\log\hat{A}_n(\mathbf{e}_j)$$

$\Rightarrow$ Formula resembles the intercept estimator in linear regression model. Or expressed alternatively:

$$\log\left(\hat{A}_n(\mathbf{w})\right) = \log\left(\hat{A}_n^{ZWP}(\mathbf{w})\right) + \sum_{j=1}^{p}\lambda_j(\mathbf{w})\log\hat{A}_n(\mathbf{e}_j)$$

Independent of the values for the weight functions $\lambda_j(\cdot)$, we have that $\log \hat{A}_n^{ZWP}$ is an unbiased estimator:

$$\mathrm{E} \log \hat{A}_n^{ZWP}(\mathbf{w}) = \log A(\mathbf{w})$$

because

$$\begin{aligned} \mathrm{E} \log \hat{A}_n(\mathbf{e}_j) &= \mathrm{E}(-\log \xi_i(\mathbf{e}_j) - \gamma) \\ &= \log A(\mathbf{e}_j) + \gamma - \gamma \\ &= \log 1 + 0 = 0 \end{aligned}$$

Consider the following regression model:

$$-\log\{\xi_i(\mathbf{w})\} - \gamma = \lambda_0 + \lambda_1(-\log \xi_i(\mathbf{e}_1) - \gamma) \\ + \cdots + \lambda_p(-\log \xi_i(\mathbf{e}_p) - \gamma) + \epsilon_i \quad (12)$$

where we deduce that the intercept can be considered as an estimate for $\log A(\mathbf{w})$.

Definition (Gudendorf & Segers, 2009)

We define the OLS-estimator for the stable tail dependence function $A(\mathbf{w})$ as follows:

$$\hat{\lambda}_0 = \log \hat{A}_n^{OLS}(\mathbf{w}) = \log \hat{A}_n(\mathbf{w}) - \sum_{j=1}^{p} \hat{\lambda}_j(\mathbf{w}) \log \hat{A}_n(\mathbf{e}_j) \quad (13)$$

Properties:

1. Constraints on weight functions no longer necessary and also not satisfied by OLS.
2. Big advantage: Gauss-Markov theorem guarantees efficiency in the class of linear unbiased estimators.
3. The OLS estimator $\log \hat{A}_n^{OLS}(\mathbf{w})$ satisfies the boundary conditions.

Simulations

Logistic (Gumbel) dependence function:

$$A(\mathbf{w}) = (w_1^5 + w_2^5 + w_3^5)^{1/5}, \qquad (14)$$

Our simulations:

- 1000 data sets (Monte-Carlo)
- Sample size $n = 25, 50, 75, 100$
- Estimated for values $w_1 = w_2$

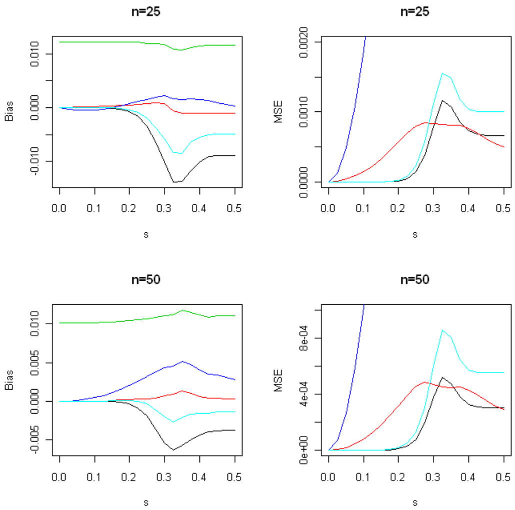
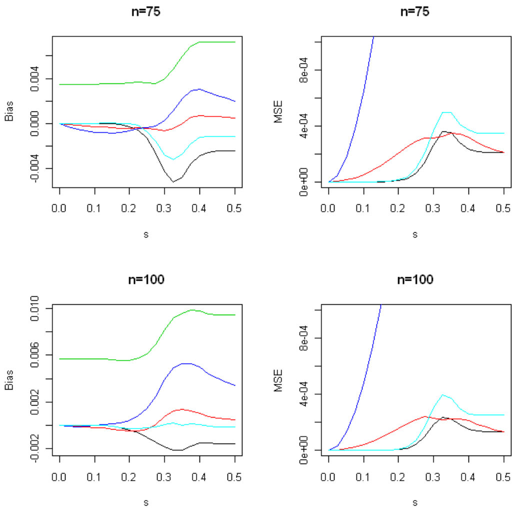


Figure: Black: OLS, Red: ZWP, Green: Pickands , Blue: Deheuvels, Blue(light):Hall & Tajvidi



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