Discrete time semi-Markov switching interest rate models

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Joint work with Pierre Devolder

January 29, 2009

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- To quote "The economist" of september 2008: "Nationalisation happening as fast as one can say Hugo Chavez"
- Size of the industry: hundreds of billions of dollars
- This illustrates the importance of accurate and realistic models !!!

What are we talking about?

- Idea: model future uncertainty about interest rates
- Zero coupon T-bond: a financial instrument that gives you one euro at time T
- Price at time t of a zero-coupon of time to maturity τ denoted by $P_t(\tau)$
- Link to interest rates: zero coupon yield is the interest rate $R_t(\tau)$ defined as $P_t(\tau) = \frac{1}{(1+R_t(\tau))^\tau}$

The notion of arbitrage

- A central notion in modern financial theory
- Arbitrage: a way of making money without risk and without initial investment
- A central hypothesis: markets are arbitrage free
- In practice this is more or less true

The model of Ho and Lee

- Model the prices of zero-coupon bonds in a "binomial" framework (discrete time approach)
- At time $t = 0$, they suppose as given a whole set of bond prices $P_0(\tau)$ for many τ's
- In a deterministic framework, it can be shown that $P_{t+1}(\tau - 1) = \frac{P_t(\tau)}{P_t(1)}$ for all τ
- In Ho and Lee, either $P_{t+1}(\tau - 1) = u(\tau)\frac{P_t(\tau)}{P_t(1)}$ or $P_{t+1}(\tau - 1) = d(\tau)\frac{P_t(\tau)}{P_t(1)}$ (for all τ)

The model of Ho and Lee

- They impose some no-arbitrage condition
- They impose path independence
- Condition 1 gives us the existence of a constant p such that $pu(\tau) + (1 - p)d(\tau) = 1$ (for all τ) [martingale measure]
- Condition 2 leads to $u(\tau) = \frac{1}{p+(1-p)\delta^\tau}$ and $d(\tau) = \delta^\tau u(\tau)$ for some constant δ

Some issues with the Ho and Lee model

- Although it is nice, it seems difficult to believe that the whole term structure be governed by only two constants (p and δ)
- Furthermore, these constants remain the same in good times or times of crisis......
- Our idea is to introduce an underlying process representing the state of the economy that would affect the value of these parameters

A brief reminder/introduction

- Idea: the semi-Markov process is used to represent the underlying state of the economy
- Define $E = 1, ..., m$, the possible states of the economy
- Let X_n be a r.v. taking value in E and T_n a r.v. taking values in $\mathbb{N}^+$ with $0 = T_0 \leq T_1 \leq T_2 ...$
- Then think of T_n as the n^{th} switching time of a system and X_n as the state at the n^{th} transition

A brief reminder/introduction

- (X,T) is called a homogeneous Markov renewal process if $\mathbb{P}[X_{n+1} = j, T_{n+1} - T_n \leq k | X_0, ..., X_n; T_0, ... T_n] = \mathbb{P}[X_{n+1} = j, T_{n+1} - T_n \leq k | X_n = i] = Q(i,j,k)$ for all (n,i,j,k) in $\mathbb{N}^+ \times E \times E \times \mathbb{N}^+$
- Define $\nu_k = \sup(n \geq 0 : T_n \leq k)$ with $n \in \mathbb{N}$ and $k \in \mathbb{N}$
- The process $Y_t = X_{\nu_t}$ with $t \in N^+$ is called a discrete time semi-Markov process

Some properties

- Generally Y_t is not Markovian......introduce process $K_t = t - T_{\nu_t}$, then (Y, K) is Markovian
- Markov processes are a subclass of semi-Markov processes
- Semi-Markov processes allow for more general duration distributions

The framework

- Discrete time financial market built on a probability space
- The market is assumed to carry two processes: a semi-Markov process Y_t and a vector process ζ_t
- ζ_t is a vector who can take two values $u_{Y_{t-1}}$ and $d_{Y_{t-1}}$ (the entries of the vector correspond to the different times to maturity)
- We define $\mathcal{F}_t = \sigma(Y_s, K_s, \zeta_s, 0 \leq s \leq t)$. We suppose that Y_t and ζ_t are conditionally independent given $\mathcal{F}_{t-1}$

Our model

- Given $Y_t = i, K_t = k$, the future is defined by the following set of events $A_{t+1}^{j,u} = \{\omega \in \Omega : \; Y_{t+1} = j, \zeta_{t+1} = u_i\}$ and $A_{t+1}^{j,d} = \{\omega \in \Omega : \; Y_{t+1} = j, \zeta_{t+1} = d_i\}$ for every possible state j.
- Given $Y_t = i, K_t = k$, the evolution of the value of a zero-coupon bond of time to maturity τ is given by

$$P_{t+1}(\tau) = u_i(\tau)\frac{P_t(\tau+1)}{P_t(1)}(\sum_{j=1}^{m} \mathbb{1}_{A_{t+1}^{j,u}}) + d_i(\tau)\frac{P_t(\tau+1)}{P_t(1)}(\sum_{j=1}^{m} \mathbb{1}_{A_{t+1}^{j,d}})$$

First implications

- Arbitrage: a way of making money without risk ("Free Lunch")
- Modern finance based on the no arbitrage assumption
- In order to avoid arbitrage, it can easily be shown that $u_i(\tau) > 1 > d_i(\tau)$ for every i and every τ
- Indeed, suppose $u_i(\tau) > d_i(\tau) > 1$. At t, buy a zero coupon of time to maturity τ. That costs $P_t(\tau)$. So borrow this quantity.
- At $t+1$, sell the bond and repay the loan. The net gain is $P_{t+1}(\tau-1) - \frac{P_t(\tau)}{P_t(1)}$ which is certainly positive given our model and $u_i(\tau) > d_i(\tau) > 1$.

A second look at no-arbitrage

- Given $Y_t = i, K_t = k$, build a portfolio comprising one bond of time to maturity τ and H bonds of time to maturity τ'
- At time t, the portfolio is worth $W_t = P_t(\tau) + HP_t(\tau')$
- Choose H such that the value of the portfolio is the same whether $\zeta_{t+1} = u_i$ or $\zeta_{t+1} = d_i$ ($W_{t+1}^{u_i} = W_{t+1}^{d_i}$)
- This portfolio is then a risk free asset and its present value should be equal to its future value properly discounted (no-arbitrage)

A second look at no-arbitrage

- This approach imposes the existence, for every state i, of a constant p_i such that for every τ:

$$p_i u_i(\tau) + (1 - p_i) d_i(\tau) = 1$$

- Very similar to Ho and Lee
- Link to martingale measures (sort of average)

The hunt for martingale measures

- Define $\pi_{Y_t j}(K_t) := \mathbb{P}[\zeta_{t+1} = u_{Y_t}, Y_{t+1} = j | \mathcal{F}_t]$
- Recall $A^{j,u}_{t+1} = \{\omega \in \Omega : \; Y_{t+1} = j, \zeta_{t+1} = u_{Y_t}\}$ and $A^{j,d}_{t+1} = \{\omega \in \Omega : \; Y_{t+1} = j, \zeta_{t+1} = d_{Y_t}\}$
- For each i, let us define a series of parameters $(p_{ij}(t), q_{ij}(t))_{j \in [1;m]; t \in \mathbb{T}}$ such that for every t, $\sum_{j=1}^{m} (p_{ij}(t) + q_{ij}(t)) = 1$
- Define $D_t = \prod_{s=0}^{t-1} \left(\sum_{j=1}^{m} \left[\frac{p_{Y_s j}(s)}{\pi_{Y_s j}(K_s)} \mathbb{1}_{A^{j,u}_{s+1}} + \frac{q_{Y_s j}(s)}{\kappa_{Y_s j}(K_s)} \mathbb{1}_{A^{j,d}_{s+1}} \right] \right)$

The hunt for martingale measures

- Define $\mathbb{P}^*$ as the the equivalent measure with density D_{T^*} with respect to $\mathbb{P}$
- Under the condition that $p_i = \sum_j p_{ij}(t)$ for every i and every t, $\mathbb{P}^*$ is an equivalent martingale measure meaning that the discounted value of every bond behaves like a $(\mathbb{P}^*,\mathcal{F}_t)$-martingale
- But there are of course an infinite of p_{ij} that will satisfy this condition: infinite number of martingale measures

Market completeness

- Loosely speaking, a market is said to be complete if every asset can be "replicated" by the other assets present on the market
- It has been shown that this is linked to the uniqueness of the martingale measure
- An infinite number of measures implies (again loosely speaking) an incomplete market

Market completeness

- In Ho and Lee, unique martingale measure, market completeness. Here more sources of risk (the underlying process), market incompleteness seems logical
- In practice, market incompleteness is good and bad. Bad since there is no unique way of pricing assets. Good since otherwise all derivatives are useless.

What the past tells us

- In Ho and Lee, binomial tree: recombining (up then down = down then up)
- In our case, this is more complicated: possibility of regime switches
- Is this idea still interesting?

First case: no switch

- At time t, state $Y_t = i$. We suppose that the state doesn't change for at least one period (i.e. $Y_{t+1} = i$).
- Then we impose that "up then down= down then up"
- This leads to the following relation $\frac{d_i(\tau)u_i(\tau+1)}{u_i(1)} = \frac{u_i(\tau)d_i(\tau+1)}{d_i(1)}$
- Eliminating d_i via the relation $p_i u_i(\tau) + (1 - p_i)d_i(\tau) = 1$ yields $u_i(\tau) = \frac{1}{p_i+(1-p_i)\delta_i^\tau}$ and $d_i(\tau) = \delta_i^\tau u_i(\tau)$ for some constant δ_i.

Second case: regime switches

- Applying the same intuitive condition in the presence of regime changes yields $\frac{u_i(\tau)u_i(1)}{u_i(\tau+1)} = \frac{d_j(\tau)d_j(1)}{d_j(\tau+1)}$ for any pair of states i, j
- One can show that this relation implies that $u_i = u_j$ and $d_i = d_j$
- This means that if we apply this condition (recombining trees) in the presence of regime changes, all states have to have the same impact on the term structure: this condition makes regime switching useless
- So we choose not to apply this condition in the presence of switches

Conclusions

- We have presented an alternative model to the Ho and Lee model
- This model can be completely characterized by parameters p_{ij} and δ_i for all i
- The model can be made to be arbitrage free but is incomplete

Future work

- Learn more about models with non-recombining trees and see how this applies or is linked
- Numerical simulation and testing
- Hopefully this will lead to an interesting paper

The end