Efficient estimation of a semiparametric dynamic copula model

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Outline

Problems and Solutions

Problems

- Modeling dependence is critical for financial time series
- Model volatility of returns of financial assets
- Different approaches are used to model dynamic correlations
- **BUT:** return distributions often reveal skewness and lower-tail dependencies

Copula

- Allow to model nonlinear dependence
- Look beyond correlation (i.e., linear dependence), which is required for non-elliptical distributions

What are Copulas?

Copulas allow us to model the dependence relationships among r.v. independently of their marginal distributions [1]

Definition:

Function $C : [0,1]^2 \longrightarrow [0,1]$ such that

$$F(x,y) = C\{F_1(x), F_2(y)\}$$

is a copula

Example: Clayton copula

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

where $\theta \in (0,\infty)$ - dependence parameter

[1] For more details see Embrechts, Lindskog and McNeil (2001)

Outline

Motivation

- Dependence may vary over time
  For $C = \{C_\theta, \theta \in \Theta\}$ we allow θ to be time-varying
- **Patton (2006)**: dependence parameter is a parametric function of lagged u_t, v_t
- **Here:** we assume θ to be a smooth function of time

Outline

Modeling of marginal distributions

- **Assume**

  $\{X_t\}$ - a stochastic processes, e.g. $X_t = (X_{1t}, X_{2t})'$

  $X_{i,t}|\mathcal{F}_{t-1} \sim N(\mu_{it}, \sigma^2_{it}), i = 1, 2$

  - where
    $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots, X_0)$
    μ_{it} e.g. ARMA
    σ^2_{it} e.g. GARCH

- **Estimate** the parameter vector

$$\phi = (\phi_1', \phi_2')'$$

- **Standardize**

$$z_{it} = \frac{X_{it} - \mu_{it}}{\sigma_{it}} \sim N(0, 1)$$

Estimating a copula model

- **Joint distribution of z_{1t}, z_{2t}**

$$F(z_{1t}, z_{2t}) = C(\Phi(z_{1t};\phi_1), \Phi(z_{2t};\phi_2);\theta)$$

- **The joint log-likelihood**

$$\begin{aligned}\mathbb{L}(\theta,\phi) &= \sum_{t=1}^{T} \ln c(\Phi(z_{1t};\phi_1), \Phi(z_{2t};\phi_2);\theta) \\ &\quad + \sum_{t=1}^{T} \ln \varphi(z_{1t};\phi_1) + \sum_{t=1}^{T} \ln \varphi(z_{2t};\phi_2) \\ &= \mathbb{L}_C(\theta,\phi) + \mathbb{L}_V(\phi)\end{aligned}$$

$(\phi,\theta) = [\phi_1', \phi_2', \theta]'$ is the parameter vector to be estimated

$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$

Estimating a copula model

Two-step Maximum likelihood (ML)

First step

$$\widetilde{\phi} = \arg\max_{\phi \in \Phi} \mathbb{L}_V(\phi)$$

Second step

$$\widetilde{\theta} = \arg\max_{\theta \in \Theta} \mathbb{L}_C(\theta, \widetilde{\phi})$$

$$\widetilde{\theta} = \arg\max_{\theta \in \Theta} \mathbb{L}_C(\theta, \widetilde{\phi})$$

$$\widetilde{\theta}(\tau) = \arg\max_{\theta} \sum_{t=1}^{T} \ln c(\Phi_1(\cdot), \Phi_2(\cdot); \theta) \cdot K_h(t/T - \tau)$$

Efficient estimator:

Outline

Asymptotic theory

Let

$$\ell_t(\theta) = \ln c(\Phi(z_{1t};\phi_1), \Phi(z_{2t};\phi_2);\theta)$$

Define

$$s(\tau) = E\left[(\ell_t'(\theta))^2 | t/T = \tau\right]$$
$$J(\tau) = E\left[\ell_t''(\theta) | t/T = \tau\right]$$
$$g(\tau) = E\left[\ell_t'''(\theta) | t/T = \tau\right]$$

Theorem 1: Under certain assumptions

$$\sqrt{Th}\left(\widetilde{\theta}(\tau) - \theta(\tau) - h^2 b(\tau)\right) \rightarrow_d N(0, V_\theta(\tau)),$$

where

$$b(\tau) = \left\{\frac{\theta''(\tau)}{2} + J(\tau)^{-1}\frac{g(\tau)}{2}\theta'(\tau)^2\right\}\mu_2(K)$$
$$V_\theta(\tau) = J^{-2}(\tau)s(\tau)||K||^2$$

$K_h(\cdot)$ is a Kernel

h is a bandwidth

Outline

Local likelihood estimation

Bandwidth selection

We need to choose bandwidth h to balance the bias and variance.

- **Estimator for MSE is**

$$\widehat{MSE}_p(\tau; h) = \widehat{B}_p^2(\tau; h) + \widehat{V}_p(\tau; h)$$

- **Bandwidth**

$$\widehat{h}_p = \underset{h}{\operatorname{argmin}} \left\{ \int \widehat{MSE}_p(x; h) w(x) dx \right\}$$

where $w(x)$ is a weight function

Outline

Simulations design

Step 1: **Simulate** $\{r_{1,t}, r_{2,t}\}_{t=1}^{T}$ such that:

$$\begin{aligned} \epsilon_{i,t} &\sim GARCH(1,1), i = 1,2 \\ z_{i,t} &\sim N(0,1) \\ F(z_{1,t}, z_{2,t}) &= C\left(\Phi(z_{1,t}), \Phi(z_{2,t}); \theta\right) \\ \theta &= \theta(t/T) \end{aligned}$$

- $\#$ replicas $K = 100$
- $\#$ observations $T = 100, 500$ and 1000

Step 2: **Estimate** $\widetilde{\phi}$, $\widehat{\theta}_t$ and $\widehat{\phi}$

Step 3: Test the performance of the model: **MSE** and dynamic quantile (**DQ**) test of Engle and Manganelli (2004)

Simulations design

Functions for the dependence parameter:

1. **Constant**: $\theta(u) = 0, 1, 2$ and 3
2. **Step**: $\theta(u) = 2 + 1(u > 0.5)$
3. **Slow sine** $\theta(u) = 2 + \sin(50u/3)$
4. **Fast sine** $\theta(u) = 2 + \sin(50u)$
5. **Slow sine, big amplitude** (Slow BA)
   $\theta(u) = 2 + 1.9 \sin(50u/3)$
6. **Fast sine, big amplitude** (Fast BA)
   $\theta(u) = 2 + 1.9 \sin(50u)$

Simulations design

One replica example

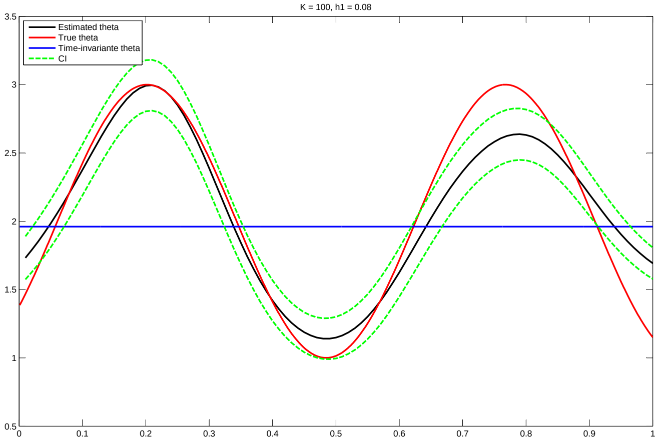


Figure: True θ (red line), time-varying $\widehat{\theta}_t$ (black line), time-invariant $\widehat{\theta}$ (blue line) and 95% confidence intervals

Simulations results

Constant models

	MSE*1000						MSE	
Model	$\widetilde{\omega}$	$\widetilde{\alpha}$	$\widetilde{\beta}$	$\widehat{\omega}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widetilde{\theta}$	$\widehat{\theta}$
$\theta = 0$	5.1$E-8$	0.62	5.46	5.0$E-8$	0.57	5.38	0.01	0.01
$\theta = 1$	5.9$E-8$	0.66	6.08	4.8$E-8$	0.56	5.39	0.05	0.04
$\theta = 2$	5.0$E-8$	0.50	5.09	3.6$E-8$	0.30	4.38	0.11	0.09
$\theta = 3$	5.1$E-8$	0.48	5.25	2.9$E-8$	0.26	3.81	0.21	0.16
$\theta = 2$ to 4	5.6$E-8$	0.71	5.78	3.6$E-8$	0.32	4.55	0.17	0.12

Table: Mean squared error *MSE* for copula dependency parameter $\widehat{\theta}$ and for parameter vectors $\widetilde{\phi}$ and $\widehat{\phi}$.

Simulations results

Sine models

	MSE*1000						MSE	
Model	$\widetilde{\omega}$	$\widetilde{\alpha}$	$\widetilde{\beta}$	$\widehat{\omega}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widetilde{\theta}$	$\widehat{\theta}$
Slow sine	$6.3E{-8}$	0.66	6.53	$4.1E{-8}$	0.36	5.49	0.21	0.15
Slow BA sine	$5.2E{-8}$	0.46	5.23	$3.6E{-8}$	0.29	4.53	0.47	0.43
Fast sine	$5.6E{-8}$	0.67	6.35	$4.0E{-8}$	0.34	5.09	0.60	0.54
Fast BA sine	$5.6E{-8}$	0.52	5.68	$4.7E{-8}$	0.39	5.02	2.01	1.96

Table: Mean squared error *MSE* for copula dependency parameter $\widehat{\theta}$ and for parameter vectors $\widetilde{\phi}$ and $\widehat{\phi}$.

Simulations results

Time increment models

	MSE*1000						MSE	
Model	$\widetilde{\omega}$	$\widetilde{\alpha}$	$\widetilde{\beta}$	$\widehat{\omega}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widetilde{\theta}$	$\widehat{\theta}$
T=100	$1.2E-7$	1.40	7.90	$9.0E-8$	0.97	6.93	0.68	0.62
T=500	$6.3E-8$	0.66	6.53	$4.1E-8$	0.36	5.49	0.21	0.15
T=1000	$3.7E-8$	0.43	4.66	$2.0E-8$	0.18	2.85	0.12	0.11

Table: Mean squared error *MSE* for copula dependency parameter $\widehat{\theta}$ and for parameter vectors $\widetilde{\phi}$ and $\widehat{\phi}$.

Outline

Data set: MSCI Germany and UK

- **Data:**
  - Morgan Stanley Capital International (MSCI) index for Germany and UK
  - weekly quotations, 11 October 1989 - 15 October 2008
- **Model:**
  - r_G, r_{UK} - log-returns
  - r_G and r_{UK} show presence of autocorrelation

$$\begin{aligned} r_{G,t} &= \underset{(0.03)}{-0.08} r_{G,t-1} + \epsilon_{G,t} \\ r_{UK,t} &= \underset{(0.03)}{-0.10} r_{UK,t-1} + \epsilon_{UK,t} \end{aligned}$$

  - for $\widehat{\epsilon}_{G,t}$ and $\widehat{\epsilon}_{UK,t}$ build GARCH(1,1) with Student errors

$$\begin{aligned} h_{G,t} &= \underset{(0.08E-4)}{0.13E-4} + \underset{(0.03)}{0.10} \epsilon^2_{G,t-1} + \underset{(0.03)}{0.89} h_{G,t-1}, \\ &\quad \nu_G = \underset{(1.77)}{8.05} \\ h_{UK,t} &= \underset{(0.24E-4)}{0.32E-4} + \underset{(0.05)}{0.13} \epsilon^2_{UK,t-1} + \underset{(0.09)}{0.82} h_{UK,t-1}, \\ &\quad \nu_{UK} = \underset{(3.30)}{11.16} \end{aligned}$$

Data set: MSCI Germany and UK

Volatilities estimated from univariate GARCH models with Student errors.

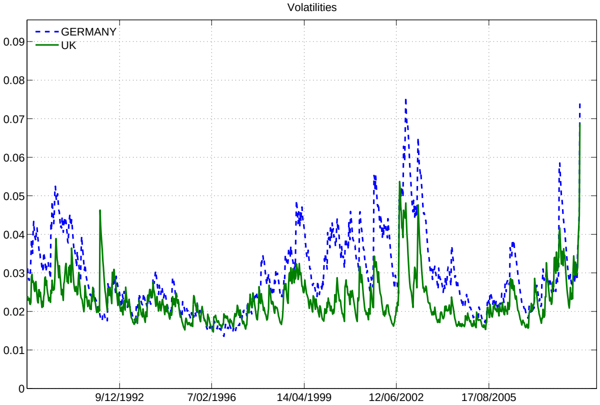
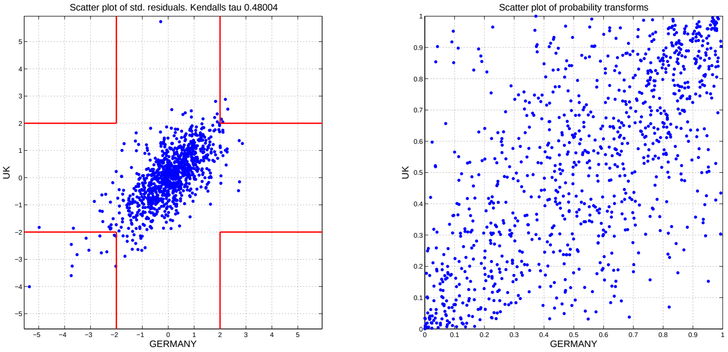


Figure: Germany (blue line) and UK (green line)

Data set: MSCI Germany and UK

Scatter plots



(a) z_G vs. z_{UK} (b) $F(z_G)$ vs. $F(z_{UK})$

Figure: Scatter plots for the standardized returns r_G versus r_{UK} and for $F(r_G;\nu_G)$ versus $F(r_{UK};\nu_{UK})$, where $F(\cdot;\nu)$ is a Student distribution with ν d.o.f.

Data set: MSCI Germany and UK

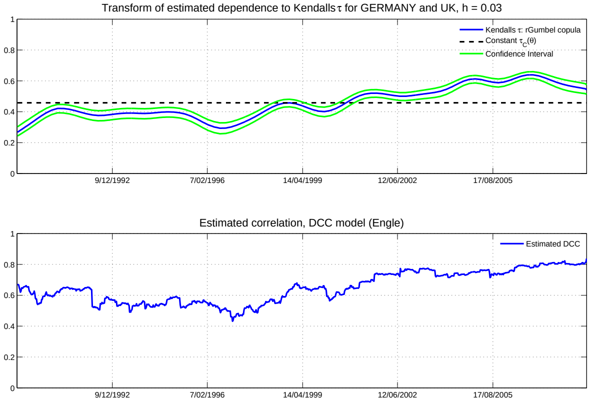


Figure: Estimated dependence $\widehat{\theta}(t)$ and 95% confidence intervals (upper panel) transformed to Kendall's tau and estimated corelation via Dynamic Conditional Correlation (DCC) model (lower panel)

Data set: MSCI Germany and UK

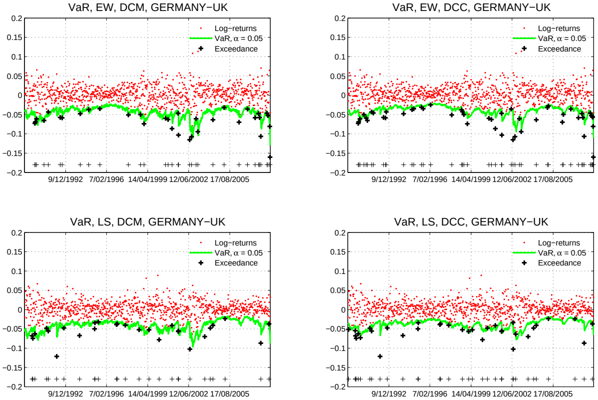


Figure: Value-at-Risk for Dynamic Copula model (left panels) and for Dynamic Conditional Correlation (DCC) model (right panels) for equal weighted (EW) (upper panels) and long-short (LS) (lower panels) portfolios

Finally

- **What has been done:**

  Simulation results for different copulas

  Theory for efficient estimator of ϕ

  Asymptotic theory for the estimator of θ

- **In progress:**

  Goodness-of-fit test for time-varying copulas

  Extend to higher dimensions

  Improved DCC model for higher dimensions

For Further Reading I

J. Fan, M. Farmen, I. Gijbels
Local maximum likelihood estimation and inference
J. R. Statist. Soc., 60, Part 3, pp. 591-608, 1998

A.J.Patton
Modeling asymmetric exchange rate dependence
INTERNATIONAL ECONOMIC REVIEW, Vol. 47, No. 2, May 2006

R. Dahlhaus
A likelihood approximation for locally stationary processes
The Annals of Statistics, 28, 1762-1794, 2000

F.C. Drost, C.A.J. Klaassen, and B.J.M. Werker
Adaptive estimation in time-series models
The Annals of Statistic, 25, 786-817, 1997