

Heavy tailed functional linear processes

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- A measurable function $f : [a, \infty) \rightarrow \mathbb{R}_+$ is *regularly varying* with index $\alpha \in \mathbb{R}$ if

$$\lim_{t \rightarrow \infty} \frac{f(xt)}{f(t)} = x^\alpha.$$

This means that, at infinity, the behavior of f is like the one of x^α .

- Basic link with *extreme value theory*:

Given $\gamma > 0$, a positive random variable X belongs to the domain of attraction of $G_\gamma \Leftrightarrow 1 - F_X$ is regularly varying with index $-1/\gamma$.

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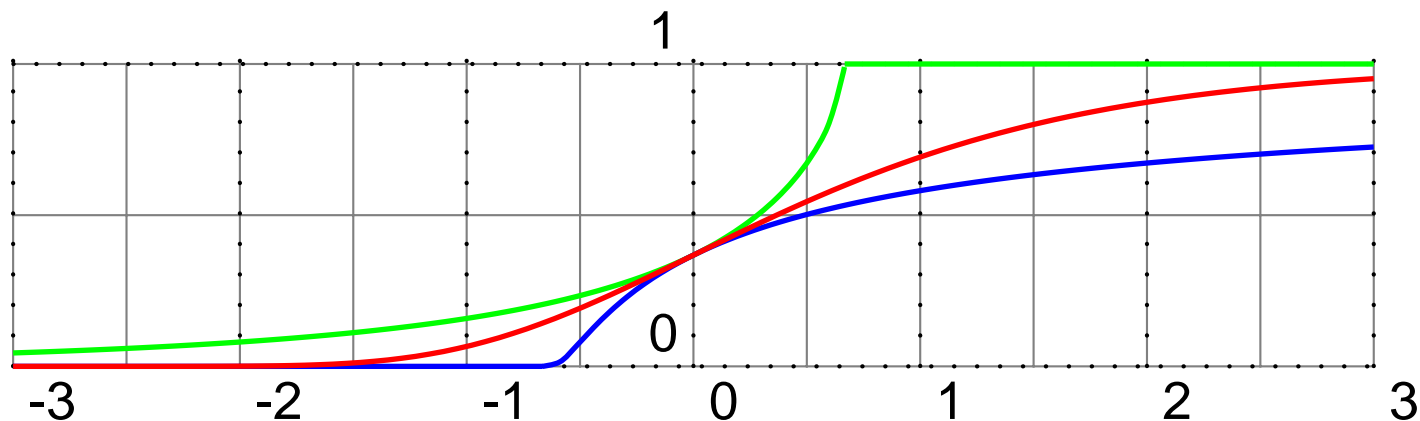
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■ Graphics of G_γ for different values of $\gamma \in \mathbb{R}$:

— $\gamma = -1.5$ (reverse-Weibull type)
— $\gamma = 0$ (Gumbel distribution)
— $\gamma = 1.5$ (Fréchet type)



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- Heavy tail = regularly varying behavior

- Two distinct cases:
 - *the observed process itself is heavy tailed*
(*finance, telecommunication, environment ...*)

 - *a transformation of observed process is heavy tailed*
(*to have unit-Fréchet margins, to study the extremal properties*)

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- A random *variable* represents a single observation.
- A random *vector* in \mathbb{R}^d represents d observations, for instance at d meteorological stations.
- Sometimes it may be better to consider the information *everywhere*.



For these reasons we are led to consider *random functions*.

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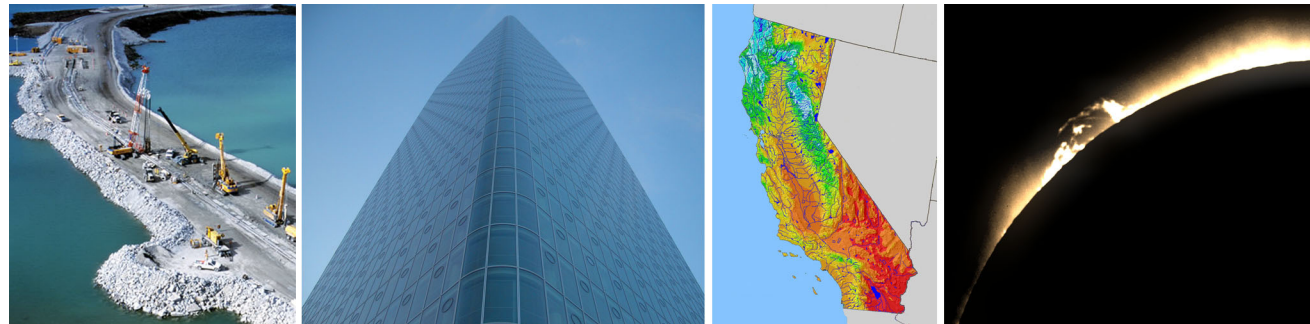
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Examples of random functions: *spatial phenomena*.

- sea level along a dike
- windspeed along the facade of a building
- precipitation in a geographical region
- protuberances on the surface of the sun



- Functions are *infinite dimensional objects*.

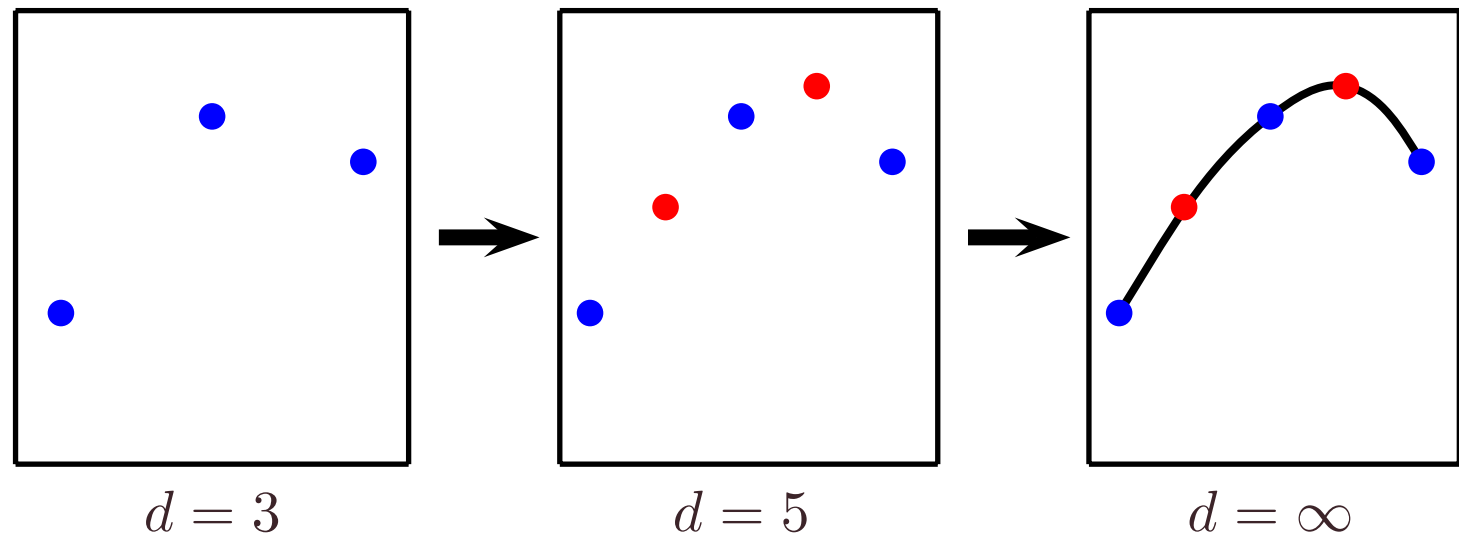


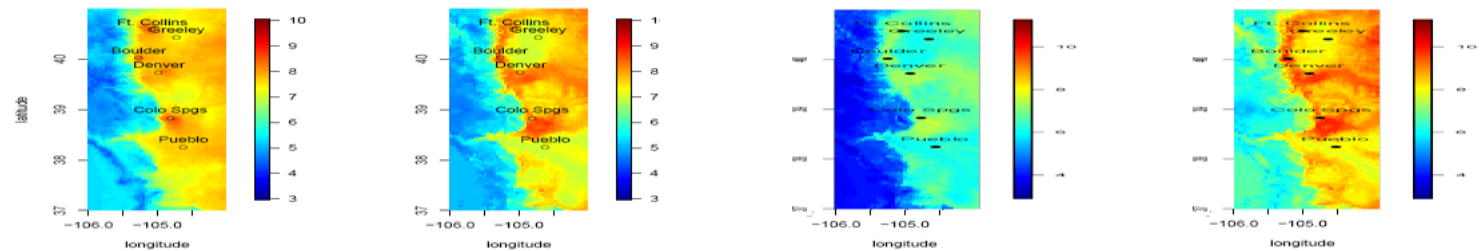
Figure 1: Functional observations \Leftrightarrow “ ∞ measurement points”.

- Sets of functions: $C([0, 1]^d, \mathbb{R})$, $D([0, 1]^d, \mathbb{R})$, $USC([0, 1]^d)$...
- In this talk we focus on $C([0, 1]^d, \mathbb{R})$.

- A sequence of observations of spatial phenomena is a *time series of function-valued data*.
- Goal: to deal with *space and time dependencies for extremes*.

Space Cross-sectional *tail dependence*

Time Temporal *clusters of extremes*



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- *Functional linear processes*: definition and motivation
- For stationary processes in $C([0, 1]^d, \mathbb{R})$:
 - *spectral measure*
 - *spectral process*
- Illustration of these concepts on linear processes

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- Let C_1 and C_2 be two spaces of continuous functions.

- A *linear process* $(X_t)_{t \in \mathbb{Z}}$ is of the form

$$X_t = \sum_{i \in \mathbb{Z}} T_i(Z_{t-i}), \quad t \in \mathbb{Z},$$

where

- Z_t are iid regularly varying *innovations in C_1* ,
- $T_i : C_1 \rightarrow C_2$ are linear and continuous.

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- If there exists $0 < p < \min(1, \alpha)$

$$\sum_{i \in \mathbb{Z}} \|T_i\|^p < \infty$$

then the sum

$$X_t = \sum_{i \in \mathbb{Z}} T_i(Z_{t-i})$$

is almost surely convergent and the process $(X_t)_{t \in \mathbb{Z}}$ is stationary.

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- The expression

$$X_t = \sum_{i \in \mathbb{Z}} T_i(Z_{t-i}), \quad t \in \mathbb{Z},$$

generalizes real-valued *MA*(∞) processes.

- If $C_1 = C_2$: *autoregressive processes*

$$X_t = T_1(X_{t-1}) + \dots + T_n(X_{t-n}) + Z_t$$

are linear processes.

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■ Main novelty:

Allow *integral operators* aimed at modelling spatio-temporal transformation, such as

$$T_i(f) = x \mapsto \int_{y \in [0,1]^d} K_i(x, y) f(y) dy.$$

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■ Are linear processes *jointly regularly varying*?

→ *Regular variation* in $C([0, 1]^d, \mathbb{R})$?

Answer: *spectral measure*.

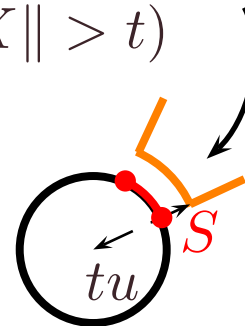
→ *Joint regular variation* for the series ?

Answer: *spectral process*.

Multivariate case

- Let X be a random vector in \mathbb{R}^d .
- Denote $\mathbb{S} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ the unit ball of \mathbb{R}^d .
- If, given $S \in \mathcal{B}_{\mathbb{S}}$ with $\lambda(\partial S) = 0$, as $t \rightarrow \infty$,

$$\frac{P(\|X\| > tu, X/\|X\| \in S)}{P(\|X\| > t)} \rightarrow \frac{1}{u^\alpha} \lambda(S)$$



with a probability measure λ , then λ is the *spectral measure* of X .

- This define the *regular variation* of X in \mathbb{R}^d :

Functional case

- The construction with spheres makes sense in $C([0, 1]^d, \mathbb{R})$.
- Put $\mathbb{S} = \{x \in C([0, 1]^d, \mathbb{R}) : \|x\| = 1\}$.
- **Definition:** A random continuous function X is *regularly varying* if it possesses a *spectral measure* in the sense

$$\frac{P(\|X\| > tu, X/\|X\| \in \cdot)}{P(\|X\| > t)} \xrightarrow{w} \frac{1}{u^\alpha} \lambda(\cdot)$$

as $t \rightarrow \infty$, for a probability measure λ on \mathbb{S} .

- For *linear processes* $X_t = \sum_{i \in \mathbb{Z}} T_i(Z_{t-i})$ we have

$$\frac{P(\|X_0\| > tu, X_0/\|X_0\| \in \cdot)}{P(\|X_0\| > t)} \xrightarrow{w} \frac{1}{u^\alpha} \lambda(\cdot),$$

where

$$\lambda(f) = \int f d\lambda = \frac{\sum_{i \in \mathbb{Z}} E \left[f \left(\frac{T_i(\Theta^Z)}{\|T_i(\Theta^Z)\|} \right) \|T_i(\Theta^Z)\|^\alpha \right]}{\sum_{i \in \mathbb{Z}} E[\|T_i(\Theta^Z)\|^\alpha]},$$

with Θ^Z distributed according to the spectral measure of Z .

- After transformation, the spectral measure λ also can be given by the limit

$$\mathcal{L} (X_0 / \|X_0\| \mid \|X_0\| \geq x) \xrightarrow{d} \lambda(\cdot).$$

- Can we replace X_0 by $(X_t)_{t \in \mathbb{Z}}$ to obtain a *limit process* $(\Theta_t)_{t \in \mathbb{Z}}$ such that

$$\mathcal{L} ((X_t / \|X_0\|)_{t \in \mathbb{Z}} \mid \|X_0\| \geq x) \xrightarrow{d} \mathcal{L} ((\Theta_t)_{t \in \mathbb{Z}}) ?$$

- Then the limit distribution $(\Theta_t)_{t \in \mathbb{Z}}$ would include all features about the joint tail distribution of $(X_t)_{t \in \mathbb{Z}}$.
- With this definition $\mathcal{L}(\Theta_0)$ is still equal to λ .

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- This construction makes sense for special cases.

- *Independent time series* $(X_t)_{t \in \mathbb{Z}}$:

$$\mathcal{L}(\Theta_t) = 0 \quad \text{if } t \neq 0.$$

- *Fully dependent series* $(X_t)_{t \in \mathbb{Z}}$:

$$\mathcal{L}(\Theta_t) = \lambda \quad \forall t \in \mathbb{Z}.$$

- For *linear processes*,

$$\mathcal{L} \left((X_t / \|X_0\|)_{t \in \mathbb{Z}} \mid \|X_0\| \geq x \right) \xrightarrow{d} \mathcal{L} \left((\Theta_t)_{t \in \mathbb{Z}} \right)$$

where

$$E \left[f \left(\Theta_{-s}, \dots, \Theta_t \right) \right] = \frac{\sum_{i \in \mathbb{Z}} E \left[f \left(\frac{T_{-s+i}(\Theta^Z)}{\|T_i(\Theta^Z)\|}, \dots, \frac{T_{t+i}(\Theta^Z)}{\|T_i(\Theta^Z)\|} \right) \|T_i(\Theta^Z)\|^\alpha \right]}{\sum_{i \in \mathbb{Z}} E[\|T_i(\Theta^Z)\|^\alpha]},$$

with Θ^Z distributed according to the spectral measure of Z .

- Given a stationary process $(X_t)_{t \in \mathbb{Z}}$, if there exists $(\Theta_t)_{t \in \mathbb{Z}}$ such that

$$\mathcal{L} \left((X_t / \|X_0\|)_{t \in \mathbb{Z}} \mid \|X_0\| \geq x \right) \xrightarrow{d} \mathcal{L} \left((\Theta_t)_{t \in \mathbb{Z}} \right),$$

we call $(\Theta_t)_{t \in \mathbb{Z}}$ the *spectral process* of $(X_t)_{t \in \mathbb{Z}}$.

- **General theorem:** A stationary process $(X_t)_{t \in \mathbb{Z}}$ in $C([0, 1]^d, \mathbb{R})$ admits a *spectral process* $(\Theta_t)_{t \in \mathbb{Z}}$ if and only if $(X_t)_{t \in \mathbb{Z}}$ is *jointly regularly varying* in the sense of, for every k ,

$$(X_0, \dots, X_k)$$

is regularly varying with the same index.

- **Extremal index** θ of the series $(\|X_t\|)_{t \in \mathbb{Z}}$:

$$\theta = \lim_{t \rightarrow \infty} \lim_{x \rightarrow \infty} P\left(\max_{i=1, \dots, t} \|X_i\| \leq x \mid \|X_0\| > x\right)$$

(this equality holds under additional conditions).

- Explicit expression thanks to the spectral process:

$$\theta = E\left[\sup_{i \geq 0} \|\Theta_i\|^\alpha - \sup_{i \geq 1} \|\Theta_i\|^\alpha\right].$$

- Causal linear process ($i \geq 0$):

$$\theta = \frac{E[\sup_{t \geq 0} \|T_t(\Theta^Z)\|^\alpha]}{\sum_{t \geq 0} E[\|T_t(\Theta^Z)\|^\alpha]}.$$

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- Theory of joint regular variation for spatial time series
- Example: Functional linear processes
- Other implications for functional processes:
 - Point processes limit for spatial extremes
 - Tail dependence quantities / extremograms
- All is actually valid in separable Banach spaces
- Other ideas: semi-continuous functions...