Topics on semi-Markov processes and their applications

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Young Researchers Day - 5/2/10

Outline of the talk

- Introduction/motivation
- Marked point processes
- Markov processes
- Semi-Markov processes
- Conclusions

A framework

- Imagine a system in a given state
- It spends some time there (the duration) and then switches to another state
- And so on....

A common misbelief

- Many articles and textbooks mention something like " If the duration distribution is not exponential, the process can't satisfy the Markov property"
- I used to believe this.....
- The only way around this exponential distribution seemed to be semi-Markov processes (for which I know the duration time can be arbitrary).
- Until.....a conversation with Jean-Marie Rolin

An interesting discussion

- He told me that non-homogeneous Markov processes have duration times that are not necesseraly exponential.
- He asked if what I called semi-Markov processes was what he called non-homogeneous Markov? What are the differences (if any)? What is more general?
- To be honest I was not sure.
- This is why I looked things up and I want to present what I ”found out” .

Some definitions

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $T = \mathbb{R}^+$ and $E = \{1, ..., m\}$ with m finite. Let $\mathcal{E}$ be the sigma algebra of all parts of E.
- A marked point process is a process $(T_n, X_n)_{n \geq 1}$ with $T_n \in T$ and $X_n \in E$
- Interpretation: T_n and X_n for the time and state of a process.
- The jump measure is the random integer-valued measure μ on $T \times E$ defined by

$$\mu = \sum_{n=1}^{\infty} \delta_{(T_n, X_n)}$$

- The jump measure gives a complete description of the MPP.

The compensator

- To such a measure μ, one can always associate a "compensator"
- The compensator is the random positive measure $\nu(\omega, dt, dx)$ defined on TxE such that $\mu((0,t], B) - \nu((0,t], B)$ is a martingale. for every $t \in T$ and $B \in \mathcal{E}$.
- Again, it gives a **complete** characterization of the MPP.
- If we can write $\nu(\omega, dt, dx) = \mathbb{1}_{\{T_n < t \leq T_{n+1}\}} \lambda_t(\omega, dx) dt$ then λ is called the intensity.

The compensator/the intensity

- Roughly speaking, the intensity models the propensity of the process having a jump at time t **given the whole history up to** t.
- Let $S_n = T_n - T_{n-1}$ be the distribution of the duration time. Let $G_n(dt, dx)$ be the distribution of (S_{n+1}, X_{n+1}) conditional on $\mathcal{F}_{T_N}$.
- Then the compensator is given by

$$\nu(\omega, dt, dx) = \sum_{n \geq 0} \frac{G_n(dt - T_n, dx)}{G_n([t - T_n, \infty], E)} \mathbb{1}_{\{T_n < t \leq T_{n+1}\}}$$

- Idea: look at the compensator and the intensity to answer our questions.

Some definitions

- Markov property: $X_{t+s} \perp\!\!\!\perp \mathcal{F}_t \mid X_t$.
- Classic approach (if $X_t = i$):
  $\mathbb{P}(X_{t+s} = j|\mathcal{F}_t) = \mathbb{P}(X_{t+s} = j|X_t = i) := p_{t,s}(ij)$
- $p_{s,t}(ij)$ is known as the transition function of the Markov process
- Markov process is homogeneous if
  $\mathbb{P}(X_{t+s} = j|X_t = i) = \mathbb{P}(X_s = j|X_0 = i) := p_s(ij)$ independent of t.

Infinitesimal transition rates

- Infinitesimal transition rates: describes the evolution of probability of a transition as time goes from t to $t + dt$.
- $q_t(ij) = \left[\frac{dp_{t,\,s}(i,j)}{ds}\right]_{s=t} \quad j \neq i$
- $q_t(ii) = -\sum_{j\neq i} q_t(ij) := q_t(i)$
- In the case of homogeneous Markov process: $q_t(ij) = q(ij)$ and $q_t(i) = q_i$ independent of t.
- The infinitesimal changes in the process depend **at most** on the current time t (and the different states)

Some properties

- It can be shown that $\mathbb{P}(T_{n+1} - T_n \leq t | \mathcal{F}_{T_n}) = 1 - \exp\left(-\int_{T_n}^{T_n+t} q_s(X_n) ds\right)$
- It is an exponential distribution only if $q_s(X_n) = q(X_n)$ i.e. if the process is homogeneous.

Some properties

- Let $\Pi_t(ij)$ be a probability distribution for fixed t and i.
- The compensator of a Markov process is given by

$$\nu(\omega, dt, j) = \sum_{n\geq 0} \mathbb{1}_{\{T_n<t\leq T_{n+1}\}}\Pi_t(X_n, j)q_t(X_n)dt$$

- Or in the homogeneous case:

$$\nu(\omega, dt, j) = \sum_{n\geq 0} \mathbb{1}_{\{T_n<t\leq T_{n+1}\}}\Pi(X_n, j)q(X_n)dt$$

Intensity

- So in case of non-homogeneous Markov process, the intensity depends solely on the current time t(and the states considered for transition).
- For homogeneous Markov processes, it is time independent.
- This means that the propensity of the process to jump in a very small time interval depends at most on the present time t. The past has no further influence and this effect of the Markov property translates directly in the intensity of the associated MPP.

Definitions

- The stochastic process $\{X_n, T_n; n \geq 0\}$ is a time-homogeneous Markov renewal process if

$$\begin{aligned} \mathbb{P}[X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, ..., X_n; T_0, ... T_n] &= \\ \mathbb{P}[X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i] &= Q_{ij}(t) \end{aligned}$$

  for all $n \geq 0$, $i, j \in E$ and $t \in \mathbb{R}^+$.
- Let ν_t be given by $\nu_t = \sup(n \geq 0 : T_n \leq t)$
- Define Y as $Y_t = X_{\nu_t}$
- The process Y is called a semi-markov process with kernel Q.

Some properties

- X_n is a markov chain whose transition matrix is $P_{ij} = Q_{ij}(\infty)$
- Let $F_{ij}(t) = \frac{Q_{ij}(t)}{P_{ij}}$
- Then, $F_{ij}(t) = \mathbb{P}(T_{n+1} - T_n \leq t | X_n = i, X_{n+1} = j)$
- This can be very general and depend on both the present and future state....
- .....but if we make it independent of the future state, we can show that homogeneous Markov processes are just a special case of semi-Markov processes.

Markov property and compensator

- Property 1: The semi-Markov process Y doesn't -in general- satisfy the Markov property except at times of jump.
- Property 2: The process $(Y_t, t - T_n)$ is a Markov process.
- Can we see this in the compensator?
- $$\nu(\omega, dt, j) = \sum_{n \geq 0} \mathbb{1}_{\{T_n < t \leq T_{n+1}\}} \frac{P_{X_n, j} f_{X_n, j}(t - T_n)}{1 - \sum_j Q_{X_n, j}(t - T_n)} dt$$
- Clear dependence on the past!!!

- This marked point process approach allows us to answer the questions.
- Markov processes don't necesseraly have exponentially distributed durations.
- Non-homogeneous Markov processes are not equivalent to semi-Markov processes.
- Homogeneous Markov processes are a special case of semi-Markov processes.

What to use and when?

- There is of course no universal answer........
- ....but when it comes to using these processes, one should really have the application in mind and use the process adapted to the situation (Markov property or not?,homogeneity?, duration distribution?)
- In my situation, it is interesting to use semi-Markov processes, not only for its mathematical interest but because many authors tend to reject the Markov property (in interest rate models for example).

Thank you. Questions?