Time-varying copulas: a survey

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Motivation example 1

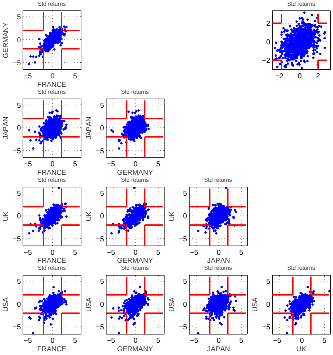

Figure: Scatter plots of standardized returns of G5 countries, weekly observations from 11 October 1989 till 31 May 2006

Motivation example 2

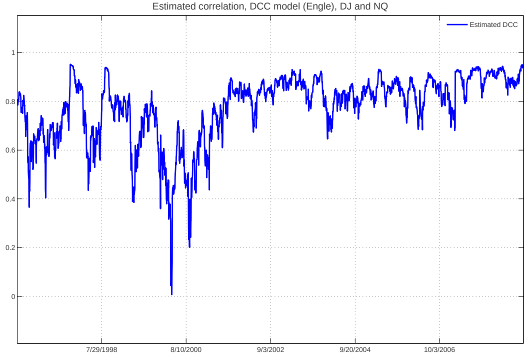

Figure: Correlation estimated with DCC model (Engle), DJ and NQ, daily observations 17 July 1996 till 21 October 2008

Estimating a copula model

The Copula model $F(X_{1t}, X_{2t}) = C\{F_1(X_{1t}), F_2(X_{2t})\}$

The joint *pdf*

$$f(X_{1t}, X_{2t}) = c(F_1(X_{1t};\phi_1), F_2(X_{2t};\phi_2);\theta)\prod_{i=1}^{2} f_i(X_{it},;\phi_i)$$

The joint log-likelihood

$$\begin{aligned}\mathbb{L}(\theta,\phi) &= \sum_{t=1}^{T}\ln c(F_1(X_{1t};\phi_1), F_2(X_{2t};\phi_2);\theta) \\ &\quad + \sum_{t=1}^{T}\ln f_1(X_{1t};\phi_1) + \sum_{t=1}^{T}\ln f_2(z_{2t};\phi_2)\end{aligned}$$

$$\mathbb{L}(\theta,\phi) = \mathbb{L}_C(\theta,\phi) + \mathbb{L}_V(\phi)$$

$(\phi,\theta) = [\phi_1', \phi_2', \theta]'$ is the parameter vector to be estimated

$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$

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Estimating a copula model

Two-step Maximum likelihood

First step

$$\widetilde{\phi} = \arg\max_{\phi \in \Phi} \mathbb{L}_V(\phi)$$

Second step

$$\widetilde{\theta} = \arg\max \mathbb{L}_C(\theta, \widetilde{\phi})$$

Drawback *loss in efficiency*

Solution *apply Newton-Rhapson algorithm*

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- Copula estimation

Time-varying copula models
- Parametric models
- Stochastic and semiparametric models
- LCP and RSC

Simulation study and model selection

Empirical example

Conclusion

For Further Reading

Parametric models

Patton

Patton (2006): θ is a function of lagged past observations and autoregressive term

$$\rho_t = \Lambda_1\left(\omega + \beta\Lambda_1^{-1}(\rho_{t-1}) + \alpha\frac{1}{m}\sum_{i=1}^{m}\Phi^{-1}(U_{1,t-i})\Phi^{-1}(U_{2,t-i})\right)$$
$$\theta_t = \Lambda_2\left(\omega + \beta\theta_{t-1} + \alpha\frac{1}{m}\sum_{j=0}^{m-1}|u_{t-j} - v_{t-j}|\right)$$

Dynamic conditional correlation (DCC)

Heinen, Valdesogo (2008): The correlation is driven by the crossproduct of lagged standardized residuals and autoregressive term

$$R_t = \mathrm{diag}\{Q\}^{-1/2} Q_t \,\mathrm{diag}\{Q\}^{-1/2}$$
$$Q_t = \Omega(1-\alpha-\beta) + \alpha Y_{t-1} Y'_{t-1} + \beta Q_{t-1}$$
$$\tau_t = \frac{2}{\pi} arcsin(\rho_t),\ \theta_t = \gamma(\tau_t)$$

where $Y_{it} = \Phi^{-1}(U_{i,t}),\ Y_t = (Y_{1t}, Y_{2t})'$

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Stochastic and semiparametric models

Stochastic autoregressive copula (SCAR)

Hafner, Manner (2009): θ is driven by an independent stochastic process

$$\begin{aligned} \lambda_t &= \omega + \beta\lambda_{t-1} + \sigma_\eta \eta_t \\ \eta_t &\sim \textit{iid } N(0,1) \\ \theta &= \Lambda(\lambda_t) \end{aligned}$$

Semiparametric dynamic copula (SDC)

Hafner, Reznikova (2009): θ a smooth function of time

$$\begin{aligned} L_C(\theta; h, \tau) &= \sum_{t=1}^{T} \ell(U_{1t}, U_{2t}; \theta) K_h(t/T - \tau) \\ \widehat{\theta}(\tau) &= \mathit{arg}\max_{\theta} L(\theta; h, \tau) \end{aligned}$$

where $K(\cdot)$ is a kernel and h is a bandwidth

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Local parametric fitting

Local change point (LCP)

Giacomini et al. (2009): θ is approximated by a constant on a time invariant interval

$$I_t = [t - m_t, t[, t = 1, \ldots, T$$

Idea: test sequentially the nested intervals from I_t on the presence of the break point.

Regime switching copula (RSC)

Pelletier(2006), Garcia, Tsafack (2008), Chollete et al.(2008): allow for two regimes, characterized by different levels of dependence

$$
\begin{aligned}
\mathcal{L}(\theta) &= \sum_{t=1}^{T} \log(1'(\widehat{\xi}_{t|t-1} \odot \eta_t)) \\
\eta_t &= \begin{pmatrix} c_1(U_{1t}, U_{2t}; \theta_1) \\ c_2(U_{1t}, U_{2t}; \theta_2) \end{pmatrix}
\end{aligned}
$$

where

- $\widehat{\xi}_{t|t-1}$ is the vector of estimated transition probabilities using information until $(t-1)$
- $\odot$ is the Hadamard product

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Simulation study and model selection

Simulation design: Simulate 1000 observations from

Gaussian copula with ρ_t

Step: $\rho_t = 0.2 + 0.6 I_{t>500}$
Sine: $\rho_t = 0.5 + 0.4 \cos(2\pi t/400)$
AR(1):

$$\begin{aligned} \rho_t &= \frac{\exp(2\lambda_t) - 1}{exp(2\lambda_t) + 1} \\ \lambda_t &= 0.02 + 0.97\lambda_{t-1} + 0.1\epsilon_t \end{aligned}$$

Measures: MSE, Log-likelihood, Anderson-Darling test

Simulation study: MSE

$$MSE = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\rho}_t^k - \rho_t^{0k}\right)^2$$

MSE	Const	DCC	PATT	SDC	LCP	SCAR	RSC
Step	0.092	0.016	0.053	0.007	0.017	0.008	**0.004**
Sine	0.082	0.021	0.048	**0.006**	0.047	0.010	0.020
AR(1)	0.076	0.040	0.052	0.035	0.063	**0.025**	0.036

Model selection by log-likelihood

The fraction of times each copula is selected as the best fitting.

	Sine						
	Const	**DCC**	**PATT**	**SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	0.212	**0.981**	0.007	**1.000**	0.350	**1.000**	**0.999**
Clayton	0.008	0.002	0.002	0.000	0.010	0.000	0.000
Frank	**0.697**	0.006	0.488	0.000	0.260	0.000	0.000
Gumbel	0.083	0.011	**0.503**	0.000	**0.380**	0.000	0.001

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Model selection by Anderson-Darling test

Anderson-Darling test: Is the data generated by a C_i?

$$H_0 : C_i(u_t, v_t; \hat{\theta}_{it}) = C_0(u_t, v_t; \theta_t^0)$$

$$\hat{z}_t = C_i(u_t | v_t; \hat{\theta}_{it}) = \frac{\partial C_i(u_t, v_t; \hat{\theta}_{it})}{\partial v_t} \sim U(0,1)$$

The size and power for the AD test at 5% nominal level (the fraction of times the H_0 is rejected)

	Const	**DCC**	**PATT**	**Sine SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	0.352	**0.129**	0.324	**0.068**	0.260	**0.060**	**0.041**
Clayton	0.643	0.898	0.635	0.640	0.770	0.790	0.762
Frank	**0.051**	0.142	**0.134**	0.212	**0.110**	0.329	0.130
Gumbel	0.539	0.625	0.561	0.552	0.520	0.671	0.595

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Empirical example

Data set:

- exchange rates Euro-USD and Yen-USD
- from 31 December 1999 till 30 December 2005
- daily returns, T = 1564

Data is corrected for autocorrelation

$$
\begin{aligned}
X_t^E &= \underset{(1.7E-04)}{-9.7E-05} - \underset{(0.03)}{0.06}\, X_{t-1}^E + \epsilon_t^E \\
X_t^Y &= \underset{(1.5E-04)}{9.8E-05} - \underset{(0.03)}{0.04} + \epsilon_t^Y
\end{aligned}
$$

and conditional heteroscedasticity

$$
\begin{aligned}
h_t^E &= \underset{(1.3E-07)}{3.5E-07} + \underset{(0.01)}{0.02}\, \epsilon_{t-1}^E + \underset{(0.01)}{0.97}\, h_{t-1}^E, \;\; \nu^E = \underset{(12.03)}{28.83} \\
h_t^Y &= \underset{(1.5E-07)}{5.3E-07} + \underset{(0.01)}{0.02}\, \epsilon_{t-1}^Y + \underset{(0.01)}{0.96}\, h_{t-1}^E, \;\; \nu^Y = \underset{(1.15)}{7.11}
\end{aligned}
$$

Empirical example: Log-likelihood

(a) Log-likelihood							
	Const	**DCC**	**PATT**	**SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	132.6	**194.3**	170.3	**228.9**	151.9	**202.2**	**207.63**
Gumbel	123.7	176.5	161.0	200.6	169.9	173.7	178.53
Clayton	113.4	145.2	142.9	161.9	135.3	149.5	151.86
Frank	**146.5**	**194.2**	**194.9**	**226.8**	**183.1**	**201.8**	**205.32**
rot Gumbel	134.4	182.9	169.5	198.3	169.3	177.6	169.04
rot Clayton	95.3	131.1	128.4	161.2	140.7	110.6	144.10

Empirical example: Anderson-Darling test

$$H_0 : C_i(u_t, v_t; \hat{\theta}_{it}) = C_0(u_t, v_t; \theta_t^0)$$

(b) AD test (Pvalues)							
	Const	**DCC**	**PATT**	**SDC**	**LCP**	**SCAR**	**RSC**
Gaussian	0.00	0.00	0.00	0.00	0.00	0.03	0.03
Gumbel	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Clayton	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Frank	**0.14**	**0.16**	**0.51**	**0.48**	**0.17**	**0.32**	**0.25**
rot Gumbel	0.00	0.00	0.00	0.00	0.00	0.00	0.00
rot Clayton	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Empirical example: Dynamic Quantile (DQ) test

DQ test Engle and Manganelli (2004): is the model correctly specified?

- $VaR_t(\alpha) = F_{t+1}^{-1}(\alpha)$
- $hit_t^\alpha = \mathbb{I}(X_t \leq VaR_t(\alpha))$

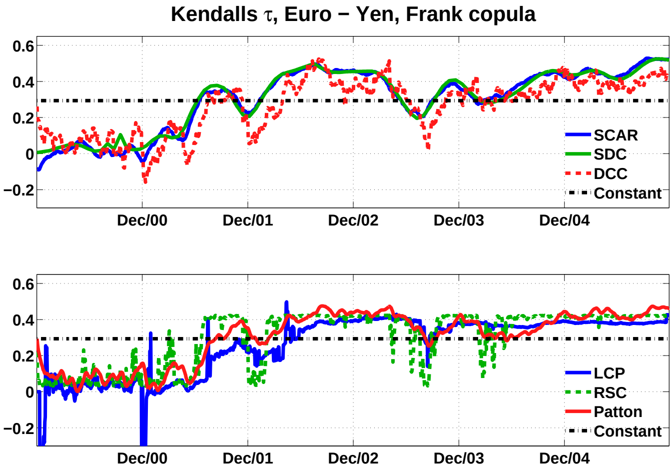
$$hit_t^\alpha - \alpha = \delta_0 + \delta_1 hit_{t-1}^\alpha + \ldots + \delta_5 hit_{t-5}^\alpha + \delta_6 VaR_t(\alpha) + \nu_t$$

- $H_0 : \delta_0 = \ldots = \delta_6 = 0$

(c) DQ test (Pvalues)

	Const	DCC	PATT	SDC	LCP	SCAR	RSC
Gaussian	0.04	**0.07**	**0.17**	**0.07**	0.04	**0.64**	**0.07**
Gumbel	0.03	0.03	0.00	**0.29**	0.02	**0.10**	0.03
Clayton	**0.19**	**0.44**	0.00	**0.23**	0.00	0.02	**0.26**
Frank	**0.27**	0.04	**0.05**	**0.17**	0.03	**0.16**	0.03
rot Gumbel	**0.08**	0.00	0.02	0.00	0.03	0.00	0.02
rot Clayton	0.02	0.01	**0.05**	0.04	0.03	**0.11**	**0.07**

Empirical example: estimated dependence



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Conclusion

Results

- log-likelihood is a strong model selection criterion, when variation of the dependence parameter is taken into account
- Anderson-Darling test has acceptable size and power properties
- DQ test of Engle and Manganelli (2004) only shows if the model fits the data

Recommendations

- RSC model showed good performance in the simulation study, is easy to program and is not computationally tedious

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For Further Reading

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