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RISK ATTITUDES AND THE VALUE OF RISK TRANSFORMATIONS

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Abstract

An increase in risk aversion, defined by a concavification of the utility function, does not always increase the willingness-to-pay for a mean preserving reduction in risk. This is why Ross (1981) proposed a stronger measure of increased risk aversion that maintains for mean preserving changes in risk the result obtained by Arrow (1965) and Pratt (1964) for risk elimination. Ross (1981)'s contribution was later on extended to higher orders using Ekern (1980)'s notion of an higher degree increase in risk. In this paper, we show that these measures remain valid under less restrictive assumptions than those implied by Ekern (1980)'s approach and we refer to the concept of mean preserving stochastic dominance. Finally, we also extend the analysis conducted for the willingness-to-pay to the willingness-to-accept.

Key words and phrases: Ekern increase in risk, stochastic dominance, risk aversion, comparative risk attitude, willingness-to-pay (WTP), willingness-to-accept (WTA).

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1 Introduction

Almost all human activities induce risk transformations which are viewed as beneficial or detrimental by the decision-maker. As a result, economists have developed the concepts of willingness-to-pay (WTP) and willingness-to-accept (WTA) in order to monetarize such transformations.

Properties of WTP or WTA have been fully characterized by Pratt (1964) and La Vallée (1968)[1] when the risk transformation consists in the elimination of an existing risk (WTP) or in its introduction (WTA). One of the results in this framework is that, in accordance with intuition, both WTP and WTA increase when the decision-maker becomes more risk averse, i.e. when his utility function is concavified.

This very intuitive result was used many times both in theoretical models and in empirical or experimental studies and it has become a cornerstone in the economics of risk. However, the attractiveness of the result was to some extent reduced when Ross (1981) observed that the risk premium[2] is not necessarily monotone in the degree of (absolute) risk aversion. As a result, Ross (1981) proposed a stronger notion of risk aversion that preserves for risk reductions the properties obtained for risk eliminations.

While Ross (1981) considered only mean preserving changes in risk as defined by Rothschild and Stiglitz (1970) other risk transformations were subsequently discussed in the literature; see, e.g., Modica and Scarsini (2005), Jindapon and Neilson (2007), Crainich and Eeckhoudt (2008), Li (2009), and Denuit and Eeckhoudt (2010). These other transformations all belong to Ekern (1980)'s definition of an sth degree increase in risk. Although this notion has many interesting features, it implies a comparison between lotteries of which the first $s-1$ moments are equal. When s increases this restriction severely limits the number of lotteries that can be compared.

Given this environment, the purpose of the present paper is twofold. First, we will extend the number of risk transformations that can be compared by giving up Ekern (1980)'s notion and replacing it by a more general one that we term mean preserving stochastic dominance (MPSD). We will characterize the restriction to be put on the utility function so that the WTP for a risk improvement satisfying MPSD increases when the decision-maker becomes more risk averse. Because the set of distributions that satisfy MPSD is larger than the one which statisfies sth degree increase in risk, more restrictions will have to be imposed on the utility function in order to obtain the intuitive result recalled above.

Our second goal will be to also analyze the willingness-to-accept (WTA) risk deteriorations. While Pratt discussed the link between increased risk aversion and both WTP and WTA, Ross (1981)'s contribution and subsequent ones limited themselves to WTP. This restriction is unsatisfactory because quite often decision-makers have to monetarize risk deteriorations.

Our paper is organized as follows. In Section 2, we briefly recall the definitions of the stochastic dominance rules that will be used in Section 3 and we define the notion of MPSD involved in the main results of the paper. Section 3 contains the results about WTP and WTA. Specifically, we establish there that both WTP and WTA related to a MPSD shift in

[1] They use the equivalent terms of ask and bid prices instead of those of WTA and WTP.

[2] For a zero-mean risk, WTP and the risk premium are equal.

risk increase when the utility function becomes more risk averse in the sense of Ross (1981) generalized by Denuit and Eeckhoudt (2010). The proofs are gathered in appendices. We briefly conclude in Section 4.

2 Mean preserving stochastic dominance

We consider random variables valued in some interval $[a, b]$ of the real line and utility functions defined on the same interval. Let us define the class $\mathcal{U}_{s-\text{icv}}$, $s = 1, 2, \ldots$, of the regular s-increasing concave functions as the class containing all the functions u with derivatives $u^{(1)}, u^{(2)}, \ldots, u^{(s)}$ of degrees 1 to s such that $(-1)^{k+1}u^{(k)} \geq 0$ on $[a, b]$ for $k = 1, 2, \ldots, s$. Letting s tend to $+\infty$ gives utilities with all odd derivatives positive and all even derivatives negative. In this case, utility functions express mixed risk aversion, as studied in Caballe and Pomansky (1996).

The common preferences of all the decision-makers with s-increasing concave utility functions generate the sth degree stochastic dominance rule, called here the s-increasing concave order. More precisely, given two random variables X and Y, X is said to be smaller than Y in the s-increasing concave order, denoted by $X \preceq_{s-\text{icv}} Y$ when $E[u(X)] \leq E[u(Y)]$ holds for all the utility functions u in $\mathcal{U}_{s-\text{icv}}$, provided the expectations exist.

Let F_X be the distribution function for such a random variable X. Starting from $F_X^{[1]} = F_X$, we define $F_X^{[2]}, F_X^{[3]}, \ldots$ recursively from repeated integrals:

$$F_X^{[k+1]}(x) = \int_a^x F_X^{[k]}(y)dy, \ \ k = 1, 2, \ldots$$

Using integration by parts, it is easily seen that

$$F_X^{[k]}(x) = \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1}dF_X(t) = \frac{\mathbb{E}[(x-X)_+^{k-1}]}{(k-1)!}.$$

The s-increasing concave orders can then be characterized as follows:

$$X \preceq_{s-\text{icv}} Y \quad \Leftrightarrow \quad \begin{cases} F_X^{[k]}(b) \geq F_Y^{[k]}(b) \text{ for } k = 1, 2, \ldots, s-1 \\ F_X^{[s]}(t) \geq F_Y^{[s]}(t) \text{ for } t \in [a, b]. \end{cases}$$

These $\preceq_{s-\text{icv}}$ order relations are closely related to the increasing sth degree risk of Ekern (1980) which restricts $\preceq_{s-\text{icv}}$ to pairs X and Y such that $E[X^k] = E[Y^k]$ for $k = 1, 2, \ldots, s-1$. By formula (A.1) in appendix, we see that X is an sth degree increase in risk of Y if, and only if, the inequality $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ holds for all the utility functions u such that $(-1)^{s+1}u^{(s)} \geq 0$. Such utility functions are called s-concave, hence the notation $\preceq_{s-\text{cv}}$ for Ekern's relation. Let us define the class $\mathcal{U}_{s-\text{cv}}$ as the class containing all the functions u with $(-1)^{s+1}u^{(s)} \geq 0$. We can then define the s-concave orders $\preceq_{s-\text{cv}}$ as

$$\begin{aligned} X \preceq_{s-\text{cv}} Y \quad &\Leftrightarrow \quad \begin{cases} F_X^{[k]}(b) = F_Y^{[k]}(b) \text{ for } k = 1, 2, \ldots, s-1 \\ F_X^{[s]}(t) \geq F_Y^{[s]}(t) \text{ for } t \in [a, b] \end{cases} \\ &\Leftrightarrow \quad E[u(X)] \leq E[u(Y)] \text{ for all } u \text{ in } \mathcal{U}_{s-\text{cv}} \\ &\Leftrightarrow \quad X \preceq_{s-\text{icv}} Y \text{ and } E[X^k] = E[Y^k] \text{ for } k = 1, 2, \ldots, s-1. \end{aligned}$$

Ekern (1980)'s definition includes well-known special cases. Famous examples are the mean preserving increase in risk of Rothschild and Stiglitz (1970) or the increase in downside risk defined by Menezes, Geiss and Tressler (1980) in which mean and variance are kept constant while there is a dispersion transfer from low to high wealth levels. More recently, Menezes and Wang (2005) defined an increase in outer risk, i.e. a 4th order increase in risk. All these contributions as well as their extensions to higher orders are unified by Eeckhoudt and Schlesinger (2006) who give both an intuitive interpretation of the signs of successive derivatives of a utility function and their link with sth degree increases in risk. For more details about the $\preceq_{s-\text{cv}}$ orders, we refer the interested readers e.g. to Denuit, Lefevre and Shaked (1998).

The notion of sth degree increase in risk is globally elegant and it is efficient at low values for s. However when s increases Ekern (1980)'s notion becomes little operational. Indeed at the sth degree it requires that the first $s-1$ moments of the risks that are compared be identical. Once s exceeds the values used in the above mentioned papers (i.e., $s = 2$, 3 or 4), the equality of moments imposes very strong restrictions on the risks that can be compared so that the notion loses empirical relevance. This is why in this paper we relax $\preceq_{s-\text{cv}}$ into mean preserving stochastic dominance (MPSD) allowing for unequal moments of degrees 2 and higher. More precisely, the concept of MPSD is defined as $X \preceq_{s-\text{icv}} Y$ and $E[X] = E[Y]$. The next result identifies the set of utility functions corresponding to this stochastic dominance rule. The proof is given in appendix A.

Property 2.1. *Consider two random variables X and Y valued in $[a, b]$. Then, for $s \geq 3$,*

$$
\begin{aligned}
& X \preceq_{s-icv} Y \text{ and } E[X] = E[Y] \\
\Leftrightarrow\ & E[u(X)] \leq E[u(Y)] \text{ for all } u \text{ such that } (-1)^{k+1} u^{(k)} \geq 0 \text{ for } k = 2, \ldots, s \\
\Leftrightarrow\ & \begin{cases} E[X] = E[Y] \\ E[(b-X)^k] \geq E[(b-Y)^k] \text{ for } k = 2, \ldots, s-1 \\ E[(t-X)_+^{s-1}] \geq E[(t-Y)_+^{s-1}] \text{ for all } t \in [a, b]. \end{cases}
\end{aligned}
$$

As in the seminal contribution by Ross (1981), the restriction to risks with the same expected value is needed to establish the comparison results involving WTP and WTA. This condition in fact means that the risks in presence have the same "size" but differ in their "variability".

3 Value of risk transformations

As in Jindapon and Neilson (2007), we use the following definition: u is more sth degree Ross risk averse than v if the inequality

$$
(-1)^{s+1} \frac{u^{(s)}(x)}{u^{(1)}(y)} \geq (-1)^{s+1} \frac{v^{(s)}(x)}{v^{(1)}(y)} \tag{3.1}
$$

holds for all x and y. This condition has been imposed by Ross (1981) for $s = 2$, by Modica and Scarsini (2005) for $s = 3$, and by Li (2009) as well as Denuit and Eeckhoudt (2010) for arbitrary $s \geq 4$. Note that this is equivalent to requiring that

$$
\frac{u^{(s)}}{v^{(s)}} \geq \lambda \geq \frac{u^{(1)}}{v^{(1)}}
$$

holds for some $\lambda > 0$.

Let us now extend Proposition 3.1 in Denuit and Eeckhoudt (2010) in two directions. First, we show in Proposition 3.1 below that this result also holds for WTA, whereas - as their predecessors - Denuit and Eeckhoudt (2010) restricted their analysis to WTP. Second, we allow not only for Ekern (1980)'s sth degree increase in risk but also for MPSD shifts in Proposition 3.2.

Let us start with the following result. Items (i)-(iii) obtained by Denuit and Eeckhoudt (2010) are supplemented with item (iv) concerning WTA.

Proposition 3.1. *Statements (i)-(iii) below are equivalent for non-decreasing u, v in $\mathcal{U}_{s-cv}$, $s \geq 2$:*

(i) u is more sth degree Ross risk averse than v;

(ii) there exists a function ϕ such that $\phi^{(1)} \leq 0$, $(-1)^{s+1}\phi^{(s)} \geq 0$, and a constant $\lambda > 0$ such that $u = \lambda v + \phi$;

(iii) given X and Y such that $Y \preceq_{s-cv} X$,

$$\left.\begin{array}{r} E[u(Y)] = E[u(X-\pi_u)] \\ E[v(Y)] = E[v(X-\pi_v)] \end{array}\right\} \Rightarrow \pi_u \geq \pi_v.$$

(iv) given X and Y such that $Y \preceq_{s-cv} X$,

$$\left.\begin{array}{r} \mathbb{E}[u(m_u+Y)] = \mathbb{E}[u(X)] \\ \mathbb{E}[v(m_v+Y)] = \mathbb{E}[v(X)] \end{array}\right\} \Rightarrow m_u \geq m_v.$$

The proof of the equivalence between (i)-(iii) and (iv) is given in appendix B.

Let us now extend this result to stochastic dominance shifts, keeping only the means unchanged but allowing for different higher moments (contrarily to Ekern (1980)'s increase in risk which imposes equal moments). We state the result for $s \geq 3$; for $s = 2$, the equivalence between statements (i)-(ii)-(iii) is the result obtained by Ross (1981).

Proposition 3.2. *Statements (i)-(iv) below are equivalent for u, v in $\mathcal{U}_{s-icv}$, $s \geq 3$:*

(i) u is more kth degree Ross risk averse than v for $k = 2, \ldots, s$;

(ii) there exists a function ϕ such that $\phi^{(1)} \leq 0$, $(-1)^{k+1}\phi^{(k)} \geq 0$ for $k = 2, \ldots, s$, and a constant $\lambda > 0$ such that $u = \lambda v + \phi$;

(iii) given X and Y such that $Y \preceq_{s-icv} X$ and $E[X] = E[Y]$,

$$\left.\begin{array}{r} E[u(Y)] = E[u(X-\pi_u)] \\ E[v(Y)] = E[v(X-\pi_v)] \end{array}\right\} \Rightarrow \pi_u \geq \pi_v.$$

(iv) given X and Y such that $Y \preceq_{s-icv} X$ and $E[X] = E[Y]$,

$$\left.\begin{array}{r} E[u(Y+m_u)] = E[u(X)] \\ E[v(Y+m_v)] = E[v(X)] \end{array}\right\} \Rightarrow m_u \geq m_v.$$

The proof of this result is given in appendix C. An example of function ϕ satisfying the assumptions of Proposition 3.2(ii) is $\phi(x) = -x - \exp(-x)$ defined on a subset $[a, b]$ of $(0, +\infty)$ for which $\phi^{(1)}(x) = \exp(-x) - 1 < 0$ and $\phi^{(k)}(x) = (-1)^{k+1} \exp(-x)$ for $k \geq 2$.

Comparing the statements in Propositions 3.1 and 3.2, we see that the more general shifts in risk allowed in Proposition 3.2 (MPSD instead of Ekern (1980)'s increase in risk) require more conditions on the utility functions (more precisely, on the function ϕ making u more risk averse than v). The need for equal means comes from the decreasingness of ϕ (as it can be seen from the proof in appendix and from Property 2.1).

4 Conclusion

Many economic or financial activities lead to the reduction of existing risks while not eliminating them. To express the wilingness-to-pay for such reductions, the concept of a stronger measure of risk aversion and its extension to any order is indispensable. In this note, we have shown that these stronger measures are appropriate for many more risk comparisons than it was admitted so far.

For instance, while recent papers have applied the stronger measures to risk comparisons involving an sth degree risk reduction *à la* Ekern (1980), we have shown here that the measures are also useful for a much wider class of risk comparisons involving pairs of risks with equal means ordered by sth degree stochastic dominance. In this way, one avoids for large s the severe limitations imposed by the equality of the $s - 1$ first moments in Ekern (1980)'s ranking.

Also, results obtained for the willingness-to-pay extend to the willingness-to-accept, under both Ekern and mean preserving stochastic dominance shifts.

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A Proof of Property 2.1

As u is s times continuously differentiable, the following expansion formula is easily obtained using integration by parts:

$$\begin{aligned} E[u(X)] &= \sum_{k=0}^{s-1}(-1)^k u^{(k)}(b)F_X^{[k+1]}(b) + (-1)^s \int_a^b u^{(s)}(x)F_X^{[s]}(x)dx \\ &= \sum_{k=0}^{s-1}(-1)^k u^{(k)}(b)\frac{E[(b-X)^k]}{k!} + (-1)^s \int_a^b u^{(s)}(x)\frac{E[(x-X)_+^{s-1}]}{(s-1)!}dx. \quad \text{(A.1)} \end{aligned}$$

Hence,

$$\begin{aligned} E[u(Y)] - E[u(X)] &= \sum_{k=2}^{s-1}(-1)^k u^{(k)}(b)\frac{E[(b-Y)^k]-E[(b-X)^k]}{k!} \\ &\quad +(-1)^s \int_a^b u^{(s)}(x)\frac{E[(x-Y)_+^{s-1}]-E[(x-X)_+^{s-1}]}{(s-1)!}dx, \end{aligned}$$

which ends the proof.

B Proof of Proposition 3.1

The equivalence of statements (i)-(iii) has been established by Denuit and Eeckhoudt (2010). The proof of (ii)⇒(iv) is as follows:

$$\begin{aligned}
E[u(m_u+Y)] &= E[u(X)] \\
&= \lambda E[v(X)] + E[\phi(X)] \\
&\geq \lambda E[v(X)] + E[\phi(Y)] \text{ as } Y \preceq_{s-\text{cv}} X \text{ and } \phi \in \mathcal{U}_{s-\text{cv}} \\
&= \lambda E[v(m_v+Y)] + E[\phi(Y)] \\
&\geq \lambda E[v(m_v+Y)] + E[\phi(m_v+Y)] \text{ as } \phi \text{ is non-increasing} \\
&= E[u(m_v+Y)]
\end{aligned}$$

which implies $m_u \geq m_v$ since u is non-decreasing.

Let us now prove (iv)⇒(i). To this end, we use the lotteries $X_s(x,h)$ and $Y_s(x,h)$ introduced in Denuit and Eeckhoudt (2010). Recall that for s odd, these lotteries are defined by

$$X_s(x,h) = \begin{cases} x \text{ with the probability } \frac{1}{2^{s-1}} \\ x+2h \text{ with the probability } \frac{\binom{s}{s-2}}{2^{s-1}} \\ \vdots \\ x+(s-1)h \text{ with the probability } \frac{s}{2^{s-1}} \end{cases}$$

and

$$Y_s(x,h) = \begin{cases} x+h \text{ with the probability } \frac{s}{2^{s-1}} \\ x+3h \text{ with the probability } \frac{\binom{s}{s-3}}{2^{s-1}} \\ \vdots \\ x+sh \text{ with the probability } \frac{1}{2^{s-1}} \end{cases}$$

whereas for s even,

$$X_s(x,h) = \begin{cases} x+h \text{ with the probability } \frac{s}{2^{s-1}} \\ x+3h \text{ with the probability } \frac{\binom{s}{s-3}}{2^{s-1}} \\ \vdots \\ x+(s-1)h \text{ with the probability } \frac{s}{2^{s-1}} \end{cases}$$

and

$$Y_s(x,h) = \begin{cases} x \text{ with the probability } \frac{1}{2^{s-1}} \\ x+2h \text{ with the probability } \frac{\binom{s}{s-2}}{2^{s-1}} \\ \vdots \\ x+sh \text{ with the probability } \frac{1}{2^{s-1}}. \end{cases}$$

Now, for s even, define

$$X = \begin{cases} X_s(x,h) \text{ with probability } p \\ y \text{ with probability } 1-p \end{cases}$$
$$Y = \begin{cases} Y_s(x,h) \text{ with probability } p \\ y \text{ with probability } 1-p. \end{cases}$$

For s odd, define

$$X = \begin{cases} Y_s(x,h) \text{ with probability } p \\ y \text{ with probability } 1-p \end{cases}$$
$$Y = \begin{cases} X_s(x,h) \text{ with probability } p \\ y \text{ with probability } 1-p. \end{cases}$$

We know from Denuit and Eeckhoudt (2010) that $Y \preceq_{s-\mathrm{cv}} X$ holds for all s. Clearly, both m_u and m_v defined by $E[u(m_u + Y)] = E[u(X)]$ and $E[v(m_v + Y)] = E[v(X)]$ are functions of h, that is, $m_u = m_u(h)$ and $m_v = m_v(h)$. Letting h tend to 0 makes X and Y identically distributed so that $m_u(0) = m_v(0) = 0$. Let us give the proof of the implication (iv)⇒(i) for s even (the reasoning is similar for s odd). The equality $E[u(m_u + Y)] = E[u(X)]$ can be expanded as

$$(1-p)u(m_u+y) + p \sum_{k\in\{0,2,\ldots,s\}} \frac{\binom{s}{k}}{2^{s-1}} u\big(m_u + x + (s-k)h\big)$$

$$= (1-p)u(y) + p \sum_{k\in\{1,3,\ldots,s-1\}} \frac{\binom{s}{k}}{2^{s-1}} u\big(x + (s-k)h\big). \tag{B.1}$$

Taking the first derivative of both sides with respect to h gives

$$(1-p)u^{(1)}(m_u+y)m_u^{(1)} + p \sum_{k\in\{0,2,\ldots,s\}} \frac{\binom{s}{k}}{2^{s-1}} (m_u^{(1)} + s - k) u^{(1)}\big(m_u + x + (s-k)h\big)$$

$$= p \sum_{k\in\{1,3,\ldots,s-1\}} \frac{\binom{s}{k}}{2^{s-1}} (s-k) u^{(1)}\big(x + (s-k)h\big).$$

Letting h tend to 0, m_u tends to 0 as well, and we get

$$(1-p)u^{(1)}(y)m_u^{(1)}(0) + pm_u^{(1)}(0)u^{(1)}\big(x\big) \underbrace{\sum_{k\in\{0,2,\ldots,s\}} \frac{\binom{s}{k}}{2^{s-1}}}_{=1} = pu^{(1)}(x) \sum_{k=0}^{s} \frac{\binom{s}{k}}{2^{s-1}} (-1)^{k+1}(s-k).$$

Recall that for any polynomial g of degree at most k, we have

$$\Delta_h^s g(x) = \sum_{k=0}^{s} \binom{s}{k} (-1)^k g\big(x + (s-k)h\big) = 0$$

whatever $h > 0$ for $s \geq k+1$. Defining ψ_j as $\psi_j(x) = x^j$, $j = 1, 2, \ldots$, we see that $m_u^{(1)}(0)$ is proportional to

$$\Delta_1^s \psi_1(0) = \sum_{k=0}^{s} \binom{s}{k} (-1)^k (s-k)$$

which is equal to 0 for $s \geq 2$. Taking successive derivatives of both sides of equality (B.1) with respect to h shows that $m_u^{(j)}(0) = 0$ for $j = 1, 2, \ldots, s-1$ since it can be checked that $m_u^{(j)}(0)$ is proportional to $\Delta_1^s \psi_j(0)$.

Let us now consider the sth derivative of both sides of (B.1). Focusing on the terms involving $m_u^{(s)}$ on the left-hand side, we get

$$\begin{aligned}
& (1-p)u^{(1)}(m_u + y)m_u^{(s)} + p \sum_{k \in \{0,2,\ldots,s\}} \frac{\binom{s}{k}}{2^{s-1}} m_u^{(s)} u^{(1)}\big(m_u + x + (s-k)h\big) \\
& + p \sum_{k \in \{0,2,\ldots,s\}} \frac{\binom{s}{k}}{2^{s-1}} (m_u^{(1)} + s - k)^s u^{(s)}\big(m_u + x + (s-k)h\big) \\
& + \text{terms proportional to } m_u^{(j)},\ j = 1, 2, \ldots, s-1 \\
= \;& p \sum_{k \in \{1,3,\ldots,s-1\}} \frac{\binom{s}{k}}{2^{s-1}} \big(s-k\big)^s u^{(s)}\big(x + (s-k)h\big).
\end{aligned}$$

Letting h tend to 0 gives

$$\begin{aligned}
& (1-p)u^{(1)}(y)m_u^{(s)}(0) + p \sum_{k \in \{0,2,\ldots,s\}} \frac{\binom{s}{k}}{2^{s-1}} m_u^{(s)}(0) u^{(1)}\big(x\big) \\
& + p \sum_{k \in \{0,2,\ldots,s\}} \frac{\binom{s}{k}}{2^{s-1}} (s-k)^s u^{(s)}\big(x\big) \\
= \;& p u^{(s)}(x) \sum_{k \in \{1,3,\ldots,s-1\}} \frac{\binom{s}{k}}{2^{s-1}} (s-k)^s.
\end{aligned}$$

which in turn gives

$$
\begin{aligned}
& m_u^{(s)}(0)\left((1-p)u^{(1)}(y)+pu^{(1)}(x)\underbrace{\sum_{k\in\{0,2,\ldots,s\}}\frac{\binom{s}{k}}{2^{s-1}}}_{=1}\right) \\
= \quad & pu^{(s)}(x)\sum_{k=0}^{s}(-1)^{k+1}\frac{\binom{s}{k}}{2^{s-1}}(s-k)^s \\
= \quad & -p\Delta_1^s\psi_s(0)\frac{u^{(s)}(x)}{2^{s-1}}.
\end{aligned}
$$

A similar expression can be established for the utility function v. Now, $m_u \geq m_v$ implies that $m_u^{(s)}(0) \geq m_v^{(s)}(0)$, that is,

$$
\frac{pu^{(s)}(x)}{pu^{(1)}(x)+(1-p)u^{(1)}(y)} \leq \frac{pv^{(s)}(x)}{pv^{(1)}(x)+(1-p)v^{(1)}(y)}
$$

$$
\Leftrightarrow \frac{u^{(s)}(x)}{v^{(s)}(x)} \geq \frac{pu^{(1)}(x)+(1-p)u^{(1)}(y)}{pv^{(1)}(x)+(1-p)v^{(1)}(y)}
$$

which is true for all x, y and p if, and only if, we have for all x, y

$$
\frac{u^{(s)}(x)}{v^{(s)}(x)} \geq \frac{u^{(1)}(y)}{v^{(1)}(y)}
$$

which is indeed equivalent to (i).

C Proof of Proposition 3.2

To see that (i)⇒(ii), it suffices to note that

$$
\phi^{(1)} \leq 0 \Leftrightarrow u^{(1)} - \lambda v^{(1)} \leq 0 \Leftrightarrow \frac{u^{(1)}}{v^{(1)}} \leq \lambda
$$

and that for $k = 2, \ldots, s$,

$$
(-1)^{k+1}\phi^{(k)} = (-1)^{k+1}u^{(k)} - (-1)^{k+1}\lambda v^{(k)} \geq 0 \Leftrightarrow \frac{u^{(k)}}{v^{(k)}} \geq \lambda.
$$

The proof of (ii)⇒(iii) is as follows:

$$
\begin{aligned}
E[u(X-\pi_u)] &= \lambda E[v(Y)] + E[\phi(Y)] \\
&\leq \lambda E[v(Y)] + E[\phi(X)] \text{ by Property 2.1} \\
&= \lambda E[v(X-\pi_v)] + E[\phi(X)] \\
&\leq \lambda E[v(X-\pi_v)] + E[\phi(X-\pi_v)] \text{ as } \phi \text{ is non-increasing} \\
&= E[u(X-\pi_v)]
\end{aligned}
$$

which implies $\pi_u \geq \pi_v$ since u is non-decreasing. To prove (iii)⇒(i), we know that for $k = 2, \ldots, s$, it is possible to find lotteries $X_k(x, h)$ and $Y_k(x, h)$ ordered in the $\preceq_{k-\mathrm{cv}}$-sense (and,thus, fulfilling the requirements of (iii)). Then, proceeding as in Denuit and Eeckhoudt (2010), we can show that this implies that u is more kth degree Ross risk averse than v.

The proof of (ii)⇒(iv) is as follows:

$$
\begin{aligned}
E[u(Y + m_u)] &= E[u(X)] \\
&= \lambda E[v(X)] + E[\phi(X)] \\
&\geq \lambda E[v(X)] + E[\phi(Y)] \text{ by Property 2.1} \\
&= \lambda E[v(Y + m_v)] + E[\phi(Y)] \\
&\geq \lambda E[v(Y + m_v)] + E[\phi(m_v + Y)] \text{ as } \phi \text{ is non-increasing} \\
&= E[u(Y + m_v)]
\end{aligned}
$$

which implies $m_u \geq m_v$ since u is non-decreasing. To prove (iv)⇒(i), we can use for $k = 2, \ldots, s$ the lotteries $X_k(x, h)$ and $Y_k(x, h)$ ordered in the $\preceq_{k-\mathrm{cv}}$-sense (and, thus, fulfilling the requirements of (iv)). Then, proceeding as in the proof of Proposition 3.1 above, we can show that this implies that u is more kth degree Ross risk averse than v. This ends the proof.