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Bivariate Almost Stochastic Dominance

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Abstract

Univariate almost stochastic dominance has been widely studied and applied since introduced by Leshno and Levy (2002). This paper extends the rule to the bivariate case. We generalize the setting by adopting two-attribute utility functions. This paper first confines correlation aversion and correlation loving to some acceptable levels. We respectively investigate bivariate almost stochastic dominance for the preferences exhibiting confined correlation aversion and confined correlation loving. The impact of a change in risk in terms of bivariate almost stochastic dominance on optimal saving is also analyzed as an application.

Keywords: almost stochastic dominance, correlation aversion, correlation loving, optimal saving.

JEL classification: D81

1 Introduction

Stochastic dominance has been widely studied in the literature for decades. One famous drawback of the rule is that it can only partially order distributions. One of the main reasons for this partial ordering comes from the fact that the rule seeks all decision makers in a given class, such as increasing and concave preferences. To overcome the drawback, Leshno and Levy (2002) proposed excluding some extreme preferences in the given class since these extreme preferences presumably rarely represent real-world decision makers' preferences. They further established a concept of "almost stochastic dominance" for most decision makers to rank distributions. Almost stochastic dominance allows for some violation of stochastic dominance and thus substantially increases the applicability in practice. Bali, Ozgur Demirtas, Levy, and Wolf (2009), Bali, Brown and Ozgur Demirtas (2011) and Levy (2011) have provided evidence to show that almost stochastic dominance rules can support several popular investment practices which cannot be supported by stochastic dominance rules.

The purpose of this paper is to extend this line of research by analyzing bivariate almost stochastic dominance. Our paper generalizes the univariate almost stochastic dominance rule constructed by Leshno and Levy (2002) and corrected by Tzeng et al. (2012) by adopting two-attribute utility functions which are commonly studied in the literature. We provide the integral condition for bivariate almost stochastic dominance which can be related to two-attribute utility functions excluding extreme preferences.

When faced with two attributes, the decision maker may mitigate the detrimental effect of a low outcome for one attribute with a high outcome for the other. This risk attitude is related to the substitutability of goods shown by Eeckhoudt, Rey and Schlesinger (2007): the correlation averse[1] decision maker prefers a mix of a favorable together with an unfavorable case over two favorable or unfavorable outcomes arising simultaneously. On the other hand, Eeckhoudt, Rey and Schlesinger (2007) also show that, for a correlation lover, the preference ordering for the above lotteries is always reversed. Multivariate stochastic dominance rules expressing the common preferences of correlation averse decision makers (those having a preference for combining good with bad) and of correlation loving decision makers (those having a preference for combining good with good and bad with bad) have been studied by

[1]This risk attitude was termed risk aversion in Richard (1975). Correlation aversion can also be related to the correlation increasing transformations defined by Epstein and Tanny (1980).

Denuit et al. (2013).

As noticed by Leshno and Levy (2002) for one-attribute utility, for the preferences with two attributes, there are some extreme correlation averse (loving) utility functions which presumably rarely represent real-world decision makers' preferences, and thus should be excluded. For example, assume that the two attributes in an individual's utility function are consumption and leisure hours. Let lottery (X_1, X_2) be the one with an equal probability of obtaining ($1, 0.999 hours) and ($0.999, 1 hour), and let lottery (Y_1, Y_2) be the one with a 0.1% probability of obtaining ($0.999, 0.999 hours) and a 99.9% probability of obtaining ($10, 10 hours). It is obvious that most of the correlation averse decision makers would prefer the latter distribution to the former. However, the bivariate stochastic dominance rule cannot rank these two distributions.[2]

Thus, in the above example, the bivariate stochastic dominance rules may fail to rank some obvious choices since the rules are sought for all correlation averse decision makers. To reveal a preference for most decision makers, but not for all of them, we restrict the class of correlation averse (resp. loving) utility functions to a subset of it by confining correlation aversion (resp. loving) to some acceptable levels. We then study the new stochastic dominance rule corresponding to the common preferences of the decision makers exhibiting confined correlation aversion (resp. loving).

The remainder of the paper is organized as follows. In Section 2, we introduce the concept of confined correlation aversion and examine the stochastic dominance rule expressing the common preferences of all the decision makers exhibiting confined correlation aversion. Section 3 turns the focus to confined correlation loving and the corresponding bivariate almost stochastic dominance rule. Section 4 provides an application of our findings to the saving decision. Section 5 briefly concludes the paper. The proofs are collected in an appendix.

Let us end this introductory section with some words on the notation adopted in the present paper. We assume that the first attribute is valued in the interval $[a_1, b_1]$ and the second one is valued in the interval $[a_2, b_2]$. We use u to denote a utility function of two attributes x_1 and x_2 and we denote the cross derivatives as $u^{(i,j)} = \frac{\partial^{i+j} u(x_1, x_2)}{(\partial x_1)^i (\partial x_2)^j}$. We assume that $u^{(1,0)} \geq 0$ and $u^{(0,1)} \geq 0$, which simply means that both attributes are goods. When the attributes are of a financial nature, for example, and are aggregated in a single position, we

[2]We will discuss this example in more detail in the next section.

denote as v a utility function of the single resulting attribute. Henceforth, we denote as $\preceq_{\text{fsd}}$ (resp. $\preceq_{\text{ssd}}$) the univariate first (resp. second) order stochastic dominance. Recall that in the expected utility paradigm, $\preceq_{\text{fsd}}$ expresses the common preferences of all the profit-seeking decision makers and $\preceq_{\text{ssd}}$ reflects the common preferences of all the profit-seeking risk-averse decision makers, i.e., those preferring $E[X]$ over X, whatever the random variables X are. To be precise, by denoting as $v^{(k)}$ the kth derivative of the single-attribute utility function v and defining

$$\begin{aligned} \mathcal{U}_{\text{fsd}} &= \left\{\text{single-attribute utility functions } v | v^{(1)} \geq 0\right\} \\ \mathcal{U}_{\text{ssd}} &= \left\{\text{single-attribute utility functions } v | v^{(1)} \geq 0 \text{ and } v^{(2)} \leq 0\right\}, \end{aligned}$$

we have $X \preceq_{\text{fsd}} Y$ (resp. $X \preceq_{\text{ssd}} Y$) if, and only if, $E[v(X)] \leq E[v(Y)]$ for all $u \in \mathcal{U}_{\text{fsd}}$ (resp. $u \in \mathcal{U}_{\text{ssd}}$).

2 Correlation aversion and confined correlation aversion

Correlation aversion as defined by Eeckhoudt, Rey and Schlesinger (2007) is related to the substitutability of goods first addressed by Richard (1975). More precisely, let us define the lotteries L offering either (x_1, x_2) or $(x_1 + h_1, x_2 + h_2)$ with equal probability 0.5 and M offering either $(x_1, x_2 + h_2)$ or $(x_1 + h_1, x_2)$ with equal probability 0.5. Eeckhoudt, Rey and Schlesinger (2007) require the preference of M over L regardless of whether x_1, x_2, $h_1 \geq 0$ and $h_2 \geq 0$ for correlation aversion. In the expected utility setting, this means that the utility function has to fulfill

$$u(x_1, x_2) + u(x_1 + h_1, x_2 + h_2) \leq u(x_1, x_2 + h_2) + u(x_1 + h_1, y_2) \tag{1}$$

for any x_1, x_2, $h_1 \geq 0$ and $h_2 \geq 0$. If u is twice differentiable, (1) is equivalent to $u^{(1,1)} \leq 0$.

Note that this kind of risk aversion is defined by comparing two lotteries that are marginally equivalent, in the sense that under both L and M, the individual obtains x_1 or $x_1 + h_1$ with the same probability, and x_2 or $x_2 + h_2$ with the same probability. The difference between L and M lies in the combination of the two goods: the correlation averse decision maker prefers a mix of a favorable together with an unfavorable case over two favorable or unfavorable outcomes arising simultaneously.

Henceforth, we denote as $\mathcal{U}_{\text{ca}}$ the class of all the correlation averse utility functions, i.e., non-decreasing functions u satisfying (1). To be precise,

$$u \in \mathcal{U}_{\text{ca}} \Leftrightarrow \begin{cases} u^{(1,0)} \geq 0 \\ u^{(0,1)} \geq 0 \\ u^{(1,1)} \leq 0. \end{cases}$$

While conditions $u^{(1,0)} \geq 0$ and $u^{(0,1)} \geq 0$ simply mean that both attributes are goods, condition $u^{(1,1)} \leq 0$ expresses correlation averse behavior.

2.1 Confined correlation aversion

The class $\mathcal{U}_{\text{ca}}$ of correlation averse utilities includes some extreme forms of preferences, i.e., the preferences cannot rank some lotteries that most of the correlation averse individuals can easily choose. Specifically, let us consider the example in the Introduction. In the example, lottery (X_1, X_2) is the one with an equal probability of obtaining ($1, 0.999 hours) and ($0.999, 1 hour), whereas lottery (Y_1, Y_2) is the one with a 0.1% probability of obtaining ($0.999, 0.999 hours) and a 99.9% probability of obtaining ($10, 10 hours). It is not surprising that most of the correlation averse decision makers would prefer lottery (Y_1, Y_2) to (X_1, X_2). However, the decision maker with the following utility function would prefer lottery (X_1, X_2) to (Y_1, Y_2):

$$u(x_1, x_2) = 1 - \exp\left(-20x_1 - 20x_2\right). \tag{2}$$

Decision makers with such a utility function should be considered extreme because the preference assigns relatively small $u^{(1,1)}$ when both x_1 and x_2 are 0.999, but relatively large $u^{(1,1)}$ when both x_1 and x_2 are 10. Specifically, the maximum value of $u^{(1,1)}$ is about 2.3088×10^{156} times the minimum value of $u^{(1,1)}$ in this problem.

To rule out this kind of preference, we can appropriately constrain the partial derivative $u^{(1,1)}$, in the spirit of the univariate almost stochastic dominance introduced by Leshno and Levy (2002). This leads to the concept of confined correlation aversion which is precisely defined as follows.

Definition 1 *Let ε be a real constant such that $0 < \varepsilon < \frac{1}{2}$. The class $\mathcal{U}_{ca}^{\varepsilon}$ contains all the elements of $\mathcal{U}_{ca}$ satisfying*

$$0 \leq -u^{(1,1)}(x_1, x_2) \leq \inf\{-u^{(1,1)}\} \left(\frac{1}{\varepsilon} - 1\right) \text{ for all } x_1 \text{ and } x_2. \tag{3}$$

Such a utility function is said to express ε-confined correlation aversion.

2.2 Almost correlation averse stochastic dominance

In this subsection, we intend to provide the condition for all decision makers exhibiting confined correlation aversion to rank distributions. From Epstein and Tanny (1980), we know that $E[u(X_1, X_2)] \leq E[u(Y_1, Y_2)]$ for all $u \in \mathcal{U}_{\text{ca}}$ if, and only if,

$$\Pr[X_1 \leq t_1, X_2 \leq t_2] \geq \Pr[Y_1 \leq t_1, Y_2 \leq t_2] \text{ for all } t_1 \text{ and } t_2. \tag{4}$$

We write $(X_1, X_2) \preceq_{\text{ca}} (Y_1, Y_2)$ if (4) holds. The bivariate stochastic dominance $\preceq_{\text{ca}}$ expresses the common preferences of all the correlation averse decision makers.

Note that (4) ensures that $\Pr[X_1 \leq t_1] \geq \Pr[Y_1 \leq t_1]$ for all t_1 and $\Pr[X_2 \leq t_2] \geq \Pr[Y_2 \leq t_2]$ for all t_2, that is, $X_i \preceq_{\text{fsd}} Y_i$ for $i = 1, 2$.

Let us now consider the common preferences of all the ε-confined correlation averse decision makers. To this end, let us define the following new bivariate stochastic dominance rule which allows for some moderate violations of condition (4) corresponding to $\preceq_{\text{ca}}$, and is referred as "almost correlation averse stochastic dominance".

Definition 2 ***(Almost Correlation Averse Stochastic Dominance)*** *Assume that $(X_1, X_2) \preceq_{ca} (Y_1, Y_2)$ does not hold and define the violation set S of condition (4) as the set of all (x_1, x_2) such that $\Pr[X_1 \leq x_1, X_2 \leq x_2] < \Pr[Y_1 \leq x_1, Y_2 \leq x_2]$. Then, $(X_1, X_2) \preceq_{ca}^{\varepsilon} (Y_1, Y_2)$ holds if $X_1 \preceq_{fsd} Y_1$, $X_2 \preceq_{fsd} Y_2$ and*

$$\int\int_S \Big(\Pr[Y_1 \leq x_1, Y_2 \leq x_2] - \Pr[X_1 \leq x_1, X_2 \leq x_2]\Big) dx_1 dx_2$$

$$\leq \varepsilon \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Big|\Pr[X_1 \leq x_1, X_2 \leq x_2] - \Pr[Y_1 \leq x_1, Y_2 \leq x_2]\Big| dx_1 dx_2. \tag{5}$$

In words, condition (5) means that the violation set S and the extent to which (4) is violated on S must be moderate enough to ensure that the integral over S of the difference in the respective joint distribution functions is smaller than ε times the total volume confined between these distribution functions. Henceforth, let S^c denote the complement of S in $[a_1, b_1] \times [a_2, b_2]$.

Note that $\preceq_{\text{ca}}^{\varepsilon}$ still induces a preference for low correlation as $(X_1, X_2) \preceq_{\text{ca}}^{\varepsilon} (Y_1, Y_2)$ implies that the attributes (X_1, X_2) are more correlated than the attributes (Y_1, Y_2) when the univariate marginals are identical, i.e.,

$$Pr[X_1 \leq t] = Pr\,[Y_1 \leq t] \text{ and } Pr[X_2 \leq t] = Pr\,[Y_2 \leq t] \text{ hold for all } t. \quad (6)$$

This is formally stated next.

Property 1 *If (X_1, X_2) and (Y_1, Y_2) have the same univariate marginals, that is, condition (6) holds true, then*

$$(X_1, X_2) \preceq_{ca}^{\varepsilon} (Y_1, Y_2) \Rightarrow Cov[X_1, X_2] \geq Cov[Y_1, Y_2].$$

The proof of this result can be found in Appendix A. Property 1 shows that any $\preceq_{\text{ca}}^{\varepsilon}$ ranking implies a preference for less correlated attributes when the marginals are identical.

The following result connects $\preceq_{\text{ca}}^{\varepsilon}$ to the utility functions in $\mathcal{U}_{\text{ca}}^{\varepsilon}$. It states that $\preceq_{\text{ca}}^{\varepsilon}$ expresses the common preferences of all the decision makers with an ε-confined correlation averse utility function.

Theorem 1 $(X_1, X_2) \preceq_{ca}^{\varepsilon} (Y_1, Y_2) \Leftrightarrow E[u(X_1, X_2)] \leq E[u(Y_1, Y_2)]$ *for all u in $\mathcal{U}_{ca}^{\varepsilon}$.*

The proof of this result can be found in Appendix B.

We know from Epstein and Tanny (1980) that the ranking $(X_1, X_2) \preceq_{\text{ca}} (Y_1, Y_2)$ allows us to compare linear combinations $w_1X_1 + w_2X_2$ and $w_1Y_1 + w_2Y_2$ made of (X_1, X_2) and (Y_1, Y_2) with non-negative coefficients w_1 and w_2 in second order stochastic dominance. To be precise, we have

$$(X_1, X_2) \preceq_{\text{ca}} (Y_1, Y_2) \Rightarrow w_1X_1 + w_2X_2 \preceq_{\text{ssd}} w_1Y_1 + w_2Y_2, \text{ for all } w_1 \text{ and } w_2 \geq 0.$$

This result shows that when all the correlation averse decision makers prefer (Y_1, Y_2) over

(X_1, X_2), any linear combination of Y_1 and Y_2 dominates in second-order stochastic dominance the corresponding linear combination of X_1 and X_2.

Let us now establish that a similar result holds for $\preceq^{\varepsilon}_{\text{ca}}$ and ε-almost second-order stochastic dominance defined by Leshno and Levy (2002) and suitably corrected by Tzeng et al. (2012). As explained in Leshno and Levy (2002), $\mathcal{U}_{\text{ssd}}$ contains some extreme utility functions which presumably rarely represent real-world investors' preferences. The prototype is $v(x) = \min\{x, r\}$ for some constant r. To reveal a preference for most investors, but not for all of them, Leshno and Levy (2002) further impose restrictions on the utility function and define

$$\mathcal{U}^{\varepsilon}_{\text{ssd}} = \left\{ v \in \mathcal{U}_{\text{ssd}} \middle| -v^{(2)}(x) \leq \inf\left\{-v^{(2)}\right\}\left(\frac{1}{\varepsilon} - 1\right) \text{ for all } x \right\}, \tag{7}$$

where $\varepsilon \in \left(0, \frac{1}{2}\right)$. Now, $X \preceq^{\varepsilon}_{\text{ssd}} Y \Leftrightarrow E[v(X)] \leq E[v(Y)]$ for every single-attribute utility function v in $\mathcal{U}^{\varepsilon}_{\text{ssd}}$.

Property 2 *$(X_1, X_2) \preceq^{\varepsilon}_{ca} (Y_1, Y_2) \Rightarrow w_1X_1 + w_2X_2 \preceq^{\varepsilon}_{ssd} w_1Y_1 + w_2Y_2$ for all w_1 and $w_2 \geq 0$.*

The proof of this result can be found in Appendix C.

3 Correlation loving and confined correlation loving

Decision makers may not be correlation averse. For example, an individual might think that money is less valuable when leisure time is limited. These types of individuals are called correlation loving. According to Eeckhoudt, Rey and Schlesinger (2007), a correlation lover would prefer lottery L to M which are mentioned in the previous section, i.e.,

$$u(x_1, x_2) + u(x_1 + h_1, x_2 + h_2) \geq u(x_1, x_2 + h_2) + u(x_1 + h_1, y_2) \tag{8}$$

for any x_1, x_2, $h_1 \geq 0$ and $h_2 \geq 0$. In other words, if u is twice differentiable, (8) is equivalent to $u^{(1,1)} \geq 0$. Obviously, as we demonstrated in the case of $u^{(1,1)} \leq 0$, some preferences in $u^{(1,1)} \geq 0$ could be too extreme and could be excluded.

Let us first denote $\mathcal{U}_{\text{cl}}$ as the class of all the correlation loving utility functions, i.e., non-decreasing functions u satisfying (8). Precisely,

$$u \in \mathcal{U}_{\rm cl} \Leftrightarrow \begin{cases} u^{(1,0)} \geq 0 \\ u^{(0,1)} \geq 0 \\ u^{(1,1)} \geq 0. \end{cases}$$

3.1 Confined correlation loving

Let us then define the concept of confined correlation loving as follows:

Definition 3 *Let ϕ be a real constant such that $0 < \phi < \frac{1}{2}$. The class $\mathcal{U}_{cl}^{\phi}$ contains all the elements of $\mathcal{U}_{cl}$ satisfying*

$$0 \leq u^{(1,1)}(x_1, x_2) \leq \inf\{u^{(1,1)}\} \left(\frac{1}{\phi} - 1\right) \text{ for all } x_1 \text{ and } x_2. \tag{9}$$

Such a utility function is said to express ϕ-confined correlation aversion.

Condition (9) requires that the ratio of the maximum value to the minimum value of $u^{(1,1)}$ be bounded. Thus, it can exclude some extreme preferences.

3.2 Almost correlation loving stochastic dominance

Let $\preceq_{\rm cl}$ express the preferences of all the correlation loving decision makers. Denuit, Lefevre and Mesfioui (1999) indicate that (Y_1, Y_2) dominates (X_1, X_2) in the sense of bivariate (1,1) convex stochastic dominance if $E[u(X_1, X_2)] \leq E[u(Y_1, Y_2)]$ for all $u \in \mathcal{U}_{\rm cl}$. It is easy to find that the necessary and sufficient distribution condition satisfies

$$\Pr[X_1 > t_1, X_2 > t_2] \leq \Pr[Y_1 > t_1, Y_2 > t_2] \text{ for all } t_1 \text{ and } t_2. \tag{10}$$

The joint survival, or excess function of (Y_1, Y_2) is always larger than the corresponding function for (X_1, X_2), indicating that it is more likely that Y_1 and Y_2 simultaneously assume large values compared to X_1 and X_2.[3] The following new stochastic dominance rule, which is referred to as "almost correlation loving stochastic dominance", allows for some moderate violations of the condition (10) corresponding to $\preceq_{\rm cl}$.

[3]The terminology survival function refers to survival analysis in biostatistics. Here, we prefer the name excess function since the probabilities of exceeding the levels t_1 and t_2 are involved.

Definition 4 ***(Almost Correlation Loving Stochastic Dominance)*** *Assume that* $(X_1, X_2) \preceq_{cl}$ (Y_1, Y_2) *does not hold and define the violation set* $\hat{S}$ *of condition (10) as the set of all* (x_1, x_2) *such that* $\Pr[X_1 > x_1, X_2 > x_2] > \Pr[Y_1 > x_1, Y_2 > x_2]$. *Then,* $(X_1, X_2) \preceq_{cl}^{\phi} (Y_1, Y_2)$ *holds if* $X_1 \preceq_{fsd} Y_1$ *and* $X_2 \preceq_{fsd} Y_2$ *and*

$$\int\int_{\hat{S}} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2] \Big) dx_1 dx_2$$

$$\leq \phi \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Big| \Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2] \Big| dx_1 dx_2. \tag{11}$$

In words, condition (11) still means that the violation set $\hat{S}$ must be moderate enough to ensure that the integral over $\hat{S}$ of the difference in the respective joint excess functions is smaller than ϕ times the total volume confined between these excess functions.

Similar to Property 1, $(X_1, X_2) \preceq_{\text{cl}}^{\phi} (Y_1, Y_2)$ implies that the attributes (X_1, X_2) are less correlated than the attributes (Y_1, Y_2) when the univariate marginals are identical as shown in the following property:

Property 3 *If* (X_1, X_2) *and* (Y_1, Y_2) *have the same univariate marginals, that is, condition (6) holds true, then*

$$(X_1, X_2) \preceq_{cl}^{\phi} (Y_1, Y_2) \Rightarrow Cov[X_1, X_2] \leq Cov[Y_1, Y_2].$$

The proof of this result can be found in Appendix D. The following Theorem further connects $\preceq_{\text{cl}}^{\phi}$ to the utility functions in $\mathcal{U}_{\text{cl}}^{\phi}$.

Theorem 2 $(X_1, X_2) \preceq_{cl}^{\phi} (Y_1, Y_2) \Leftrightarrow E[u(X_1, X_2)] \leq E[u(Y_1, Y_2)]$ *for all* u *in* $\mathcal{U}_{cl}^{\phi}$.

The proof of this result can be found in Appendix E. Theorem 2 states that $\preceq_{\text{cl}}^{\phi}$ expresses the preferences of all the decision makers with an ϕ-confined correlation loving utility function.

Now, let us further examine whether the ranking $(X_1, X_2) \preceq_{\text{cl}}^{\phi} (Y_1, Y_2)$ can help us to compare linear combinations $w_1 X_1 + w_2 X_2$ and $w_1 Y_1 + w_2 Y_2$. Following Tzeng et al. (2012), we first derive the condition to rank distributions for most risk lovers which is defined as follows

$$U_{\mathrm{rl}}^{\phi}=\left\{v\in\mathcal{U}_{\mathrm{fsd}}\middle|0\le v^{(2)}(x)\le\inf\left\{v^{(2)}\right\}\left(\frac{1}{\phi}-1\right)\text{ for all }x\right\},$$

where $\phi\in\left(0,\frac{1}{2}\right)$. Furthermore, define $X\preceq_{\mathrm{rl}}^{\phi}Y$ as $E[X]\le E[Y]$, and

$$\int_{\Omega}\int_{x}^{b}\Big(\Pr[X>t]-\Pr[Y>t]\Big)dtdx\le\phi\int_{a}^{b}\Big|\int_{x}^{b}\Big(\Pr[X>t]-\Pr[Y>t]\Big)dt\Big|dx,\quad(12)$$

where Ω denotes the set of x such that $\int_x^b\Pr(X>t)dt>\int_x^b\Pr(Y>t)dt$. The following result connects $\preceq_{\mathrm{rl}}^{\phi}$ with v in $\mathcal{U}_{\mathrm{rl}}^{\phi}$.

Theorem 3 $X\preceq_{rl}^{\phi}Y\Leftrightarrow E[v(X)]\le E[v(Y)]$ *for all* v *in* $\mathcal{U}_{rl}^{\phi}$.

The proof of this result can be found in Appendix F. Note that when ϕ approaches zero, Equation (12) holds if and only if Ω is an empty set, i.e., $\int_x^b\Pr(X>t)dt\le\int_x^b\Pr(Y>t)dt$ for all x. This inequality defines the increasing convex order (see Shaked and Shanthikumar, 2007), also called the stop-loss order in actuarial sciences (see Denuit et al., 2005). Furthermore, the set of $\mathcal{U}_{\mathrm{rl}}^{\phi}$ is equivalent to $\mathcal{U}_{\mathrm{rl}}$ when ϕ approaches zero. Therefore, Theorem 3 predicts that $E[v(X)]\le E[v(Y)]$ for all v in $\mathcal{U}_{\mathrm{rl}}$ if and only if

$$\int_x^b\Pr[X>t]dt\le\int_x^b\Pr[Y>t]dt\text{ for all }x,$$

or, equivalently, if and only if

$$\int_x^b\Pr[X\le t]dt\ge\int_x^b\Pr[Y\le t]dt\text{ for all }x.$$

This is consistent with the second degree stochastic dominance rule for risk lovers in Levy and Wiener (1998).

The following property relates the univariate and bivariate almost stochastic dominance rules for correlation and risk lovers.

Property 4 $(X_1,X_2)\preceq_{cl}^{\phi}(Y_1,Y_2)\Rightarrow w_1X_1+w_2X_2\preceq_{rl}^{\phi}w_1Y_1+w_2Y_2$ *for all* w_1 *and* $w_2\ge0$.

The proof of this result can be found in Appendix G.

4 An application to the saving problem

In this section, we illustrate an application of the results derived in the preceding sections by analyzing the saving problem. The effect of risk on the saving decision is essential to understanding the intertemporal behavior of consumption. Recently, Eeckhoudt and Schlesinger (2008) have adopted a univariate model and have examined how a deterioration in the sense of nth-degree stochastic dominance in the background risk changes the optimal saving. Denuit et al. (2011) use a bivariate model to examine the effect of an increase in the joint distribution of the background financial and non-financial risks in the sense of the effect of bivariate higher order increasing concave stochastic dominance on the optimal saving. Our findings can extend the literature to understand the relationship between a change in risk in terms of almost bivariate stochastic dominance and the optimal saving.

In a two-period model, assume that the decision maker with initial wealth w_0 and health condition h_0 can decide to save α in a risk-free asset, which will generate a positive return r in the second period. In the second period, the decision maker faces a financial background risk Y_1 and a health background risk Y_2. Let u_0 and u_1 denote respectively the utility function in the first and second periods with $u_i^{(1,0)} \geq 0$, $i = 0, 1$. Thus, the objective function is

$$\max_{\alpha} \; u_0 \left(w_0 - \alpha, h_0 \right) + E\big[u_1 \left(Y_1 + \alpha \left(1 + r \right), Y_2 \right) \big].$$

The first-order condition of the problem is

$$-u_0^{(1,0)} \left(w_0 - \alpha, h_0 \right) + \left(1 + r \right) E \left[u_1^{(1,0)} \left(Y_1 + \alpha \left(1 + r \right), Y_2 \right) \right] = 0. \tag{13}$$

Assume that $u_i^{(2,0)} \leq 0$, $i = 0, 1$, so that the second-order condition holds. Let α_Y denote the optimal saving.

Assume that the pair of background risks (Y_1, Y_2) experience a deterioration in terms of almost correlation averse stochastic dominance and become (X_1, X_2). Let α_X denote the optimal saving under (X_1, X_2). Since the second-order condition holds, we have $\alpha_X \geq \alpha_Y$ if, and only if,

$$-u_0^{(1,0)} \left(w_0 - \alpha_Y, h_0 \right) + \left(1 + r \right) E \left[u_1^{(1,0)} \left(X_1 + \alpha_Y \left(1 + r \right), X_2 \right) \right] \geq 0.$$

From Equation (13), the above equation can be written as

$$E\left[-u_1^{(1,0)}\left(X_1+\alpha_Y\left(1+r\right),X_2\right)\right]\leq E\left[-u_1^{(1,0)}\left(Y_1+\alpha_Y\left(1+r\right),Y_2\right)\right].$$

According to Theorem 1, we know that $(X_1, X_2) \preceq_{\text{ca}}^{\varepsilon} (Y_1, Y_2)$ ensures

$$E\left[-u_1^{(1,0)}(X_1+\alpha_Y\left(1+r\right),X_2)\right]\leq E\left[-u_1^{(1,0)}(Y_1+\alpha_Y\left(1+r\right),Y_2)\right]$$

for all utilities u_1 such that $-u_1^{(1,0)} \in \mathcal{U}_{\text{ca}}^{\varepsilon}$.

Note that $-u_1^{(1,0)} \in \mathcal{U}_{\text{ca}}^{\varepsilon}$ means that

$$u_1^{(2,0)} \leq 0, u_1^{(1,1)} \leq 0, u_1^{(2,1)} \geq 0 \text{ and}$$

$$u_1^{(2,1)}(x_1,x_2) \leq \inf\{u_1^{(2,1)}\}\left(\frac{1}{\varepsilon}-1\right) \text{ for all } x_1 \text{ and } x_2. \tag{14}$$

Thus, Theorem 1 helps us to conclude that $\alpha_X \geq \alpha_Y$ for all u_1 satisfying conditions in (14) if, and only if, $(X_1, X_2) \preceq_{\text{ca}}^{\varepsilon} (Y_1, Y_2)$.

By the same token, Theorem 2 indicates that $\alpha_X \geq \alpha_Y$ for all u_1 satisfying the following conditions

$$u_1^{(2,0)} \leq 0, u_1^{(1,1)} \leq 0, u_1^{(2,1)} \leq 0 \text{ and}$$

$$-u_1^{(2,1)}(x_1,x_2) \leq \inf\{-u_1^{(2,1)}\}\left(\frac{1}{\phi}-1\right) \text{ for all } x_1 \text{ and } x_2, \tag{15}$$

if, and only if, $(X_1, X_2) \preceq_{\text{cl}}^{\phi} (Y_1, Y_2)$.

5 Conclusion

In this paper, the concepts of correlation aversion and correlation loving have been extended to ε-confined correlation aversion and ε-confined correlation loving, respectively. We have defined bivariate almost stochastic dominance to allow for small violations of the condition defining the original concepts for confined correlation averse and confined correlation loving preferences. The impact of a change in risk in terms of bivariate almost stochastic dominance on optimal saving has been analyzed as an application of our findings. The extension to confine higher-order cross derivatives and to seek bivariate almost higher-degree stochastic dominance may be fruitful for future studies.

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Appendix:

Proofs of the results

A Proof of Property 1

Under (6) integration by parts gives

$$E[X_1X_2] - E[Y_1Y_2] = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2.$$

Now,

$$\begin{aligned}
0 &\le \int\int_S \Big(\Pr[Y_1 \le x_1, Y_2 \le x_2] - \Pr[X_1 \le x_1, X_2 \le x_2]\Big)dx_1dx_2 \\
&\le \varepsilon\Big(\int\int_{S^c} \Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2 \\
&\quad - \int\int_S \Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2\Big)
\end{aligned}$$

so that

$$\begin{aligned}
0 &\le \int\int_{S^c} \Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2 \\
&\quad + \frac{1-\varepsilon}{\varepsilon}\int\int_S \Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2.
\end{aligned}$$

As $\varepsilon < 0.5 \Rightarrow \frac{1-\varepsilon}{\varepsilon} > 1$ and $\int\int_S \Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2 \le 0$ we have

$$\begin{aligned}
0 &\le \int\int_{S^c} \Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2 \\
&\quad + \int\int_S \Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2 \\
&= E[X_1X_2] - E[Y_1Y_2].
\end{aligned}$$

This ends the proof since

$$Cov[X_1, X_2] - Cov[Y_1, Y_2] = E[X_1X_2] - E[Y_1Y_2]$$

under (6)

B Proof of Theorem 1

Let us start with the "⇒" part. Integration by parts shows that

$$\begin{aligned} E[u(X_1,X_2)] &= u(b_1,b_2) - \int_{a_1}^{b_1} u^{(1,0)}(x_1,b_2)\Pr[X_1\le x_1]dx_1 \\ &\quad - \int_{a_2}^{b_2} u^{(0,1)}(b_1,x_2)\Pr[X_2\le x_2]dx_2 \\ &\quad + \int_{a_1}^{b_1}\int_{a_2}^{b_2} u^{(1,1)}(x_1,x_2)\Pr[X_1\le x_1, X_2\le x_2]dx_1dx_2. \end{aligned} \tag{16}$$

Hence,

$$\begin{aligned} & E[u(X_1,X_2)] - E[u(Y_1,Y_2)] \\ = & \int_{a_1}^{b_1} u^{(1,0)}(x_1,b_2)\Big(\Pr[Y_1\le x_1] - \Pr[X_1\le x_1]\Big)dx_1 \\ & + \int_{a_2}^{b_2} u^{(0,1)}(b_1,x_2)\Big(\Pr[Y_2\le x_2] - \Pr[X_2\le x_2]\Big)dx_2 \\ & + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big(\Pr[X_1\le x_1, X_2\le x_2] - \Pr[Y_1\le x_1, Y_2\le x_2]\Big)u^{(1,1)}(x_1,x_2)dx_1dx_2. \end{aligned}$$

The first two terms appearing in the expansion of $E[u(X_1,X_2)] - E[u(Y_1,Y_2)]$ are clearly negative as u is non-decreasing and $X_1 \preceq_{\text{fsd}} Y_1$, $X_2 \preceq_{\text{fsd}} Y_2$. Let us now show that the last one is also negative. To this end, consider u in $\mathcal{U}_{\text{ca}}^{\varepsilon}$ such that $u^{(1,1)} \neq 0$ and denote

$$\begin{aligned} \gamma &= \inf\{-u^{(1,1)}\} > 0 \\ \delta &= \sup\{-u^{(1,1)}\} > 0. \end{aligned}$$

Note that (3) ensures that

$$\delta \le \gamma\left(\frac{1}{\varepsilon} - 1\right) \Leftrightarrow \frac{\gamma}{\gamma+\delta} \ge \varepsilon. \tag{17}$$

Then,

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big(\Pr[X_1\le x_1,X_2\le x_2]-\Pr[Y_1\le x_1,Y_2\le x_2]\Big)u^{(1,1)}(x_1,x_2)dx_1dx_2\\
=\;& \int\int_S\Big(\Pr[Y_1\le x_1,Y_2\le x_2]-\Pr[X_1\le x_1,X_2\le x_2]\Big)\big(-u^{(1,1)}(x_1,x_2)\big)dx_1dx_2\\
&+\int\int_{S^c}\Big(\Pr[Y_1\le x_1,Y_2\le x_2]-\Pr[X_1\le x_1,X_2\le x_2]\Big)\big(-u^{(1,1)}(x_1,x_2)\big)dx_1dx_2\\
\le\;& \delta\int\int_S\Big(\Pr[Y_1\le x_1,Y_2\le x_2]-\Pr[X_1\le x_1,X_2\le x_2]\Big)dx_1dx_2\\
&+\gamma\int\int_{S^c}\Big(\Pr[Y_1\le x_1,Y_2\le x_2]-\Pr[X_1\le x_1,X_2\le x_2]\Big)dx_1dx_2\\
=\;& -\gamma\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big|\Pr[X_1\le x_1,X_2\le x_2]-\Pr[Y_1\le x_1,Y_2\le x_2]\Big|dx_1dx_2\\
&+(\gamma+\delta)\int\int_S\Big(\Pr[Y_1\le x_1,Y_2\le x_2]-\Pr[X_1\le x_1,X_2\le x_2]\Big)dx_1dx_2\\
\le\;& 0 \text{ by (17)},
\end{aligned}$$

which ends the proof of the "$\Rightarrow$" part.

Let us now turn to the "$\Leftarrow$" part. We assume that $E[u(X_1,X_2)]\le E[u(Y_1,Y_2)]$ holds for all u in $\mathcal{U}^{\varepsilon}_{\text{ca}}$ and we have to show that (5) holds. Let us proceed by contradiction and assume that

$$\begin{aligned}
&\int\int_S\Big(\Pr[Y_1\le x_1,Y_2\le x_2]-\Pr[X_1\le x_1,X_2\le x_2]\Big)dx_1dx_2\\
&>\varepsilon\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big|\Pr[X_1\le x_1,X_2\le x_2]-\Pr[Y_1\le x_1,Y_2\le x_2]\Big|dx_1dx_2. \qquad (18)
\end{aligned}$$

Let us show that we can then construct a utility function u in $\mathcal{U}^{\varepsilon}_{\text{ca}}$ such that $E[u(X_1,X_2)]>E[u(Y_1,Y_2)]$. Let γ and δ be positive real numbers such that $\varepsilon=\frac{\gamma}{\gamma+\delta}$. Now consider a utility function u such that $u^{(1,0)}(x_1,b_2)=0$, $u^{(0,1)}(b_1,x_2)=0$, $u^{(1,1)}=-\gamma$ on S^c and $u^{(1,1)}=-\delta$ on S, that is, a utility u proportional to $(x_1-b_1)(x_2-b_2)$ up to additive constants. We then see that

$$
\begin{aligned}
& E[u(X_1,X_2)] - E[u(Y_1,Y_2)] \\
= \;& \delta \int\int_S \Big(\Pr[Y_1 \le x_1, Y_2 \le x_2] - \Pr[X_1 \le x_1, X_2 \le x_2] \Big) dx_1 dx_2 \\
& +\gamma \int\int_{S^c} \Big(\Pr[Y_1 \le x_1, Y_2 \le x_2] - \Pr[X_1 \le x_1, X_2 \le x_2] \Big) dx_1 dx_2 \\
= \;& -\gamma \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Big| \Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2] \Big| dx_1 dx_2 \\
& +(\gamma+\delta) \int\int_S \Big(\Pr[Y_1 \le x_1, Y_2 \le x_2] - \Pr[X_1 \le x_1, X_2 \le x_2] \Big) dx_1 dx_2 \\
> \;& 0,
\end{aligned}
$$

which ends the proof.

C Proof of Property 2

Consider $v \in \mathcal{U}^{\varepsilon}_{\text{ssd}}$. The announced result is valid if we can prove that the bivariate utility function u defined as

$$u(x_1,x_2) = v(w_1x_1 + w_2x_2)$$

belongs to $\mathcal{U}^{\varepsilon}_{\text{ca}}$. To this end, notice that

$$u^{(1,1)}(x_1,x_2) = w_1w_2v^{(2)}(w_1x_1 + w_2x_2)$$

so that

$$
\begin{aligned}
-u^{(1,1)}(x_1,x_2) &= -w_1w_2v^{(2)}(w_1x_1 + w_2x_2) \\
&\le w_1w_2 \inf\{-v^{(2)}(w_1x_1 + w_2x_2)\} \left(\frac{1}{\varepsilon} - 1\right) \\
&= \inf\{-u^{(1,1)}\} \left(\frac{1}{\varepsilon} - 1\right).
\end{aligned}
$$

This ends the proof.

D Proof of Property 3

Under (6) integration by parts gives

$$E[X_1X_2] - E[Y_1Y_2] = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big(\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2]\Big)dx_1dx_2.$$

Note that

$$\Pr[X_1 \le x_1, X_2 \le x_2] + 1 - \Pr[X_1 > x_1, X_2 > x_2] = \Pr[X_1 \le x_1] + \Pr[X_2 \le x_2].$$

Under (6), we have

$$\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[X_1 > x_1, X_2 > x_2] = \Pr[Y_1 \le x_1, Y_2 \le x_2] - \Pr[Y_1 > x_1, Y_2 > x_2].$$

Or,

$$\Pr[X_1 \le x_1, X_2 \le x_2] - \Pr[Y_1 \le x_1, Y_2 \le x_2] = \Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2].$$

Therefore,

$$E[X_1X_2] - E[Y_1Y_2] = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big)dx_1dx_2.$$

Now,

$$\begin{aligned}
0 &\le \int\int_{\hat{S}}\Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big)dx_1dx_2 \\
&\le \phi\Big(-\int\int_{\hat{S}^c}\Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big)dx_1dx_2 \\
&\qquad +\int\int_{\hat{S}}\Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big)dx_1dx_2\Big)
\end{aligned}$$

so that

$$\begin{aligned}
0 &\ge \int\int_{\hat{S}^c}\Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big)dx_1dx_2 \\
&\quad +\frac{1-\phi}{\phi}\int\int_{\hat{S}}\Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big)dx_1dx_2.
\end{aligned}$$

As $\phi < 0.5 \Rightarrow \frac{1-\phi}{\phi} > 1$ and $\int\int_{\hat{S}} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2] \Big) dx_1 dx_2 \geq 0$ we have

$$
\begin{aligned}
0 \quad &\geq \int\int_{\hat{S}^c} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2] \Big) dx_1 dx_2 \\
&\quad + \int\int_{\hat{S}} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2] \Big) dx_1 dx_2 \\
&= E[X_1 X_2] - E[Y_1 Y_2].
\end{aligned}
$$

This ends the proof since

$$
Cov[X_1, X_2] - Cov[Y_1, Y_2] = E[X_1 X_2] - E[Y_1 Y_2]
$$

under (6).

E Proof of Theorem 2

Let us start with the "⇒" part. By Corollary 1.6.12 in Denuit et al. (2005) applied to $X_i - a_i$ and $Y_i - a_i$, we have

$$
\begin{aligned}
&E[u(X_1, X_2)] - E[u(Y_1, Y_2)] \\
= &\int_{a_1}^{b_1} u^{(1,0)}(x_1, a_2) \Big(\Pr[X_1 > x_1] - \Pr[Y_1 > x_1] \Big) dx_1 \\
&+ \int_{a_2}^{b_2} u^{(0,1)}(a_1, x_2) \Big(\Pr[X_2 > x_2] - \Pr[Y_2 > x_2] \Big) dx_2 \\
&+ \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2] \Big) u^{(1,1)}(x_1, x_2) dx_1 dx_2.
\end{aligned}
\tag{19}
$$

The first two terms appearing in the expansion of $E[u(X_1, X_2)] - E[u(Y_1, Y_2)]$ are negative based on our assumptions. Let us now show that the last one is also negative. To this end, consider u in $\mathcal{U}_{\mathrm{cl}}^{\phi}$ such that $u^{(1,1)} \neq 0$ and denote

$$
\begin{aligned}
\underline{\theta} &= \inf\{u^{(1,1)}\} > 0 \\
\bar{\theta} &= \sup\{u^{(1,1)}\} > 0.
\end{aligned}
$$

Note that (3) ensures that

$$\bar{\theta} \le \underline{\theta}\left(\frac{1}{\phi} - 1\right) \Leftrightarrow \frac{\underline{\theta}}{\bar{\theta} + \underline{\theta}} \ge \phi. \tag{20}$$

Then,

$$\begin{aligned}
& \int_{a_1}^{b_1}\int_{a_2}^{b_2} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big) u^{(1,1)}(x_1,x_2)dx_1dx_2 \\
= \; & \int\int_{\hat{S}} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big) u^{(1,1)}(x_1,x_2)dx_1dx_2 \\
& + \int\int_{\hat{S}^c} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big) u^{(1,1)}(x_1,x_2)dx_1dx_2 \\
\le \; & \bar{\theta}\int\int_{S} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big) dx_1dx_2 \\
& + \underline{\theta}\int\int_{S^c} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big) dx_1dx_2 \\
= \; & -\underline{\theta}\int_{a_1}^{b_1}\int_{a_2}^{b_2} \Big|\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big| dx_1dx_2 \\
& + (\bar{\theta} + \underline{\theta})\int\int_{S} \Big(\Pr[X_1 > x_1, X_2 > x_2] - \Pr[Y_1 > x_1, Y_2 > x_2]\Big) dx_1dx_2 \\
\le \; & 0 \text{ by (17)},
\end{aligned}$$

which ends the proof of the "$\Rightarrow$" part.

With Equation (19), the "$\Leftarrow$" part is similar to the "$\Leftarrow$" part in the proof for Theorem 1. Thus, it is omitted.

F Proof of Theorem 3

Notice that

$$\int_{x}^{b} \Pr[X > t]dt = E[(X - x)_+]$$

where ξ_+ denotes the positive part of the real ξ (equal to 0 if ξ is negative and to ξ otherwise). According to Corollary 1.6.10 in Denuit et al. (2005) applied to $X - a$ and $Y - a$, we have

$$\begin{aligned}
& E[v(X)] - E[v(Y)] \\
= \; & v'(a)\big(E[X] - E[Y]\big) + \int_a^b v^{(2)}(x)\int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big)dtdx.
\end{aligned}$$

The first term is negative based on our assumptions. Let us now show that the second term of the above equation is also negative. Consider v in $\mathcal{U}_{\text{rl}}^{\phi}$ such that $v^{(2)} \neq 0$ and denote

$$\begin{aligned}
\underline{\lambda} &= \inf\{v^{(2)}\} > 0 \\
\bar{\lambda} &= \sup\{v^{(2)}\} > 0.
\end{aligned}$$

Note that (12) ensures that

$$\bar{\lambda} \leq \underline{\lambda}\left(\frac{1}{\phi} - 1\right) \Leftrightarrow \frac{\underline{\lambda}}{\bar{\lambda} + \underline{\lambda}} \geq \phi. \tag{21}$$

Then,

$$\begin{aligned}
& \int_a^b v^{(2)}(x) \int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big) dt dx \\
= \; & \int_\Omega v^{(2)}(x) \int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big) dt dx + \int_{\Omega^c} v^{(2)}(x) \int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big) dt dx \\
\leq \; & \bar{\lambda} \int_\Omega \int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big) dt dx + \underline{\lambda} \int_{\Omega^c} \int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big) dt dx \\
= \; & \big(\underline{\lambda} + \bar{\lambda}\big) \int_\Omega \int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big) dt dx \\
& - \bar{\lambda} \int_a^b \Big| \int_x^b \Big(\Pr[X > t] - \Pr[Y > t]\Big) dt \Big| dx \\
\leq \; & 0 \text{ by (12)},
\end{aligned}$$

where Ω^c denotes the complement of Ω. It ends the proof of the "$\Rightarrow$" part.

With Equation (19), the "$\Leftarrow$" part is similar to the "$\Leftarrow$" part in the proof for Theorem 1. Thus, it is omitted.

G Proof of Property 4

Consider $v \in \mathcal{U}_{\text{cl}}^{\phi}$. The announced result is valid if we can prove that the bivariate utility function u defined as

$$u(x_1, x_2) = v(w_1 x_1 + w_2 x_2)$$

belongs to $\mathcal{U}_{\mathrm{cl}}^{\phi}$. To this end, notice that

$$u^{(1,1)}(x_1, x_2) = w_1 w_2 v^{(2)}(w_1 x_1 + w_2 x_2)$$

so that

$$\begin{aligned} u^{(1,1)}(x_1, x_2) &= w_1 w_2 v^{(2)}(w_1 x_1 + w_2 x_2) \\ &\leq w_1 w_2 \inf\{v^{(2)}(w_1 x_1 + w_2 x_2)\} \left(\frac{1}{\phi} - 1\right) \\ &= \inf\{u^{(1,1)}\} \left(\frac{1}{\phi} - 1\right). \end{aligned}$$

This ends the proof.