INSTITUT DE STATISTIQUE

BIOSTATISTIQUE ET

SCIENCES ACTUARIELLES

(ISBA)

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

DISCUSSION PAPER

2013/05

Almost Expectation and Excess Dependence Notions

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March 1, 2013

Abstract

This paper weakens the expectation dependence concept due to Wright (1987) and its higher-order extensions proposed by Li (2011) to conform with the preferences generating the almost stochastic dominance rules introduced in Leshno and Levy (2002). A new dependence concept, called excess dependence is introduced, and studied in addition to expectation dependence. This new concept coincides with expectation dependence at first-degree but provides distinct higher-order extensions. Three applications, to portfolio diversification, to the determination of the sign of the equity premium in the consumption-based CAPM and to optimal investment in the presence of a background risk, illustrate the usefulness of the approach proposed in the present paper.

Keywords: almost stochastic dominance, portfolio theory, diversification, optimal investment, background risk.

JEL classification: D81

1 Introduction

Expectation dependence introduced by Wright (1987), called here first-degree expectation dependence and its higher-order extensions introduced by Li (2011) have been shown to play a key role in many financial problems, such as asset allocation (Wright, 1987; Hadar and Seo, 1988), demand for risky asset under background risk (Li, 2011) and asset pricing (Dionne et al., 2012). Although the literature has demonstrated fruitful applications of expectation dependence, the concept itself is quite strong as it must hold over the whole domain of a random variable. As pointed out by Leshno and Levy (2002) for stochastic dominance, it may be useful for applications to weaken this dependence concept, allowing for moderate departures from the initial definitions.

The following simple example illustrates the relevance of the approach proposed in the present paper. Assume for instance that the joint distribution of the end-of-period asset values is described by the next table, which displays the joint and marginal probabilities associated to the different cases (empty cells mean zero probability):

Value of Asset 1	Value of Asset 2			Marginal probabilities
	200	100	10	
1,000			0.01	0.01
-0.1		0.5		0.5
$-\frac{995}{49}$	0.49			0.49
Marginal probabilities	0.49	0.5	0.01	1

The mean value of asset 1 is 0. When asset 2 is known to underperform, i.e. asset 2 equals 10 or is less than, or equal to 100, then asset 1 equals 1,000 or 19.51 on average, respectively. Thus, a negative information about the performances of asset 2 increases the expected value of asset 1. In such a case, asset 1 is said to be negatively first-degree expectation dependent on asset 2 after Wright (1987)[1]. Including both assets in the same portfolio may be desirable as asset 1 mitigates possible losses on asset 2, being larger on average when asset 2 underperforms. In this case, all risk averse individuals would have non-negative demand for asset 1.

Let us now slightly modify the joint distribution of these two assets as follows:

Value of Asset 1	Value of Asset 2			Marginal probabilities
	200	100	10	
1,000		0.01		0.01
-0.1		0.49	0.01	0.5
$-\frac{995}{49}$	0.49			0.49
Marginal probabilities	0.49	0.5	0.01	1

Since the mean value of asset 1 is still zero whereas its expected value when asset 2 is 10 is only -0.1, these two assets are not in negative first-degree expectation dependence. However, for most individuals, these two assets are still "quite" negatively related, and the demand for asset 1 could be still positive.

There is thus a need for a more flexible version of expectation dependence, allowing for moderate departures from the original concept while retaining its main contents. This is

[1]See Section 2 for a precise definition.

precisely the approach leading to almost stochastic dominance proposed by Leshno and Levy (2002) and adapted here to define almost expectation dependence. The preceding example shows that such an almost expectation dependence concept could be useful to define the negative relationship existing between a pair of correlated assets. We come back to this example in Section 2 to show that asset 1 is indeed almost negatively first-degree expectation dependent on asset 2 in this case.

We know from Wright (1987) that negative first-degree expectation dependence ensures that the covariance of one random variable with any non-decreasing transformations of the other random variable is always negative. This fundamental result remains valid under almost negative first-degree expectation dependence provided the non-decreasing transformations share the property defining the utility functions generating almost first-order stochastic dominance. This result extends to higher-degree expectation dependence and higher-order stochastic dominance rules. This allows us to derive a condition ensuring diversification, extending the results obtained by Wright (1987) in portfolio theory, to consider the sign of equity premiums in the consumption-based CAPM, extending Dionne et al. (2012), and to study the demand for a risky asset in the presence of a background risk, extending previous works by Tsetlin and Winkler (2005) and Li (2011).

The remainder of the paper is organized as follows. In Section 2, we first recall the definition of first-degree expectation dependence defined by Wright (1987). Then, we introduce the concept of almost expectation dependence and we establish the counterpart of Theorem 3.1 in Wright (1987) for this new concept. In Section 3, we consider second-degree expectation dependence defined by Li (2011) and we extend this notion by allowing for moderate departures from the initial definition. In Section 4, we define a new dependence concept, called excess dependence, which can be seen as a dual version of expectation dependence. We also weaken this new concept to its almost version. Section 5 considers higher-degree extensions of both second-degree expectation and excess dependence concepts studied in Sections 3-4. Section 6 provides some applications. First, we derive restrictions on the stochastic structure of asset returns to ensure that all investors with a utility function in the class corresponding to almost second-order stochastic dominance hold a positive amount of each asset in their expected utility maximizing portfolio. Then, we consider the sign of equity premiums in the consumption-based CAPM. Finally, we apply the almost expectation dependence concept to the demand of a risky asset in the presence of a background risk. Section 7 briefly concludes the paper. The proofs of the main results are collected in an appendix.

Let us end this introductory section with some words about the notation adopted in the present paper. We denote as $u^{(k)}$ the kth derivative of the single-attribute utility function u, $k = 1, 2, \ldots$ Let $\mathcal{U}_{n,\mathrm{a}}$ be the set of all the utility functions exhibiting risk apportionment of orders 1 to n, as defined by Eeckhoudt and Schlesinger (2006) i.e.

$$\mathcal{U}_{n,\mathrm{a}} = \left\{ \text{utility functions } u | (-1)^{k+1} u^{(k)} \geq 0 \text{ for } k = 1, 2, \ldots, n \right\}.$$

For $n \geq 2$, the utility functions in $\mathcal{U}_{n,\mathrm{a}}$ are concave and, thus, exhibit risk aversion. We also consider the dual class

$$\mathcal{U}_{n,\ell} = \left\{ \text{utility functions } u | u^{(k)} \geq 0 \text{ for } k = 1, 2, \ldots, n \right\}$$

containing convex utility functions for $n \geq 2$, i.e. utilities exhibiting risk loving. For $n = 1$, $\mathcal{U}_{1,\mathrm{a}} = \mathcal{U}_{1,\ell}$ and this class is simply denoted as $\mathcal{U}_1$. The common preferences of the decision makers with utility in $\mathcal{U}_{n,\mathrm{a}}$ correspond to the nth-order stochastic dominance (or n-increasing concave order) whereas those of the decision makers with utility in $\mathcal{U}_{n,\ell}$ correspond to the nth-degree stop-loss order (or n-increasing convex order). See, e.g., Denuit, De Vylder and Lefevre (1999).

2 Almost first-degree expectation dependence

2.1 First-degree expectation dependence

Consider two random variables X_1 and X_2 valued in the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. In this paper, we are interested in the structure of dependence between these random variables, i.e. the way X_1 and X_2 interact. Several notions of dependence have been used in economics, including quadrant, expectation and regression dependence. See, e.g., Hong et al. (2011) for a brief description of these three concepts and Denuit et al. (2005) for a detailed account of dependence structures and their links with stochastic dominance rules. Here, we would like to assess the influence of X_2 on X_1 by specifying the impact of the information that X_2 is small (i.e. below some threshold x_2, say) on the expectation of X_1. This corresponds to the first-degree expectation dependence concept whose definition is recalled next.

Definition 1 (Wright, 1987) *The random variable X_1 is negatively first-degree expectation dependent on X_2 if*

$$E[X_1] \leq E[X_1|X_2 \leq x_2] \text{ for all } x_2. \tag{1}$$

Positive first-degree expectation dependence is defined by reversing the sign of the the inequality in (1).

Condition (1) means that the knowledge that X_2 is small, i.e. below the threshold x_2, increases X_1 on average. As

$$E[X_1] = E[X_1|X_2 \leq x_2]P[X_2 \leq x_2] + E[X_1|X_2 > x_2]P[X_2 > x_2]$$

we have

$$P[X_2 \leq x_2]\big(E[X_1] - E[X_1|X_2 \leq x_2]\big) = P[X_2 > x_2]\big(E[X_1|X_2 > x_2] - E[X_1]\big)$$

so that condition (1) defining negative first-degree expectation dependence can be equivalently rewritten as

$$E[X_1|X_2 > x_2] \leq E[X_1] \text{ for all } x_2. \tag{2}$$

The equivalent definition (2) corresponds to the negative expectation quadrant dependence introduced by Kowalczyk and Pleszczynska (1977). It states that the knowledge that X_2 is large (i.e. above the threshold x_2) decreases the expected value of X_1. Both inequalities (1) and (2) express some form of compensation between the variations in X_1 and X_2.

Let us now rewrite (2) as

$$P[X_2 > x_2]\big(E[X_1|X_2 > x_2] - E[X_1]\big) \leq 0 \text{ for all } x_2$$

$$\Leftrightarrow E\big[X_1 I[X_2 > x_2]\big] - E[X_1]E\big[I[X_2 > x_2]\big] \leq 0 \text{ for all } x_2$$

$$\Leftrightarrow Cov\big[X_1, I[X_2 > x_2]\big] \leq 0 \text{ for all } x_2$$

where $I[A]$ denotes the indicator function of the event A, equal to 1 if A is realized and to 0 otherwise. Thus, we see that negative first-degree expectation dependence is equivalent to the negative covariance of X_1 with the indicator of the event that X_2 exceeds some given threshold. As non-decreasing functions correspond to limits of sequences of non-decreasing step functions, we then expect that this covariance remains negative for any non-decreasing function of X_2. This is formally stated in Wright (1987, Theorem 3.1) who established that condition (1) ensures that $Cov[X_1, t(X_2)] \leq 0$ for all non-decreasing transformation $t(\cdot)$ of X_2, i.e. for all $t \in \mathcal{U}_1$.[2] This is easily seen from

$$\begin{aligned}
Cov[X_1, t(X_2)] &= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Big(P[X_1 \leq x_1, X_2 \leq x_2] - P[X_1 \leq x_1] P[X_2 \leq x_2] \Big) t^{(1)}(x_2) dx_1 dx_2 \\
&= \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \Big(P[X_1 \leq x_1 | X_2 \leq x_2] - P[X_1 \leq x_1] \Big) dx_1 \right) P[X_2 \leq x_2] t^{(1)}(x_2) dx_2 \\
&= \int_{a_2}^{b_2} \Big(E[X_1] - E[X_1 | X_2 \leq x_2] \Big) P[X_2 \leq x_2] t^{(1)}(x_2) dx_2. \qquad (3)
\end{aligned}$$

This also shows that positive first-degree expectation dependence guarantees that the covariance between X_1 and any non-decreasing transformation $t(\cdot)$ of X_2 is always positive.

2.2 Almost first-degree expectation dependence

Let us now consider transformations t corresponding to almost first-degree stochastic dominance. Specifically, Leshno and Levy (2002) defined the class $\mathcal{U}_1^\varepsilon$ of non-decreasing single-attribute utility functions u with first derivative assuming "acceptable" levels. Formally,

$$\mathcal{U}_1^\varepsilon = \left\{ u \in \mathcal{U}_1 \middle| u^{(1)}(x) \leq \inf \left\{ u^{(1)} \right\} \left(\frac{1}{\varepsilon} - 1 \right) \text{ for all } x \right\}, \qquad (4)$$

where $\varepsilon \in \left(0, \frac{1}{2}\right)$. The restriction of $\mathcal{U}_1$ to $\mathcal{U}_1^\varepsilon$ excludes some extreme forms of preferences, such as those of satisficers having a utility function equal to 0 up to some given wealth level and then jumping to, and staying constantly equal to 1, that is, $u(x) = I[x > w]$ for some fixed w.

Now, we would like to modify condition (1) to ensure that the covariance between X_1 and $t(X_2)$ is negative for any transformation $t \in \mathcal{U}_1^\varepsilon$. To this end, the first-degree expectation dependence concept of Wright (1987) is modified as follows, allowing for moderate violations of (1).

Definition 2 *Assume that X_1 is not negatively first-degree expectation dependent on X_2, i.e. condition (1) is violated for some x_2. Define the violation set Ω of condition (1) as the set of all x_2 such that*

$$E[X_1] > E[X_1 | X_2 \leq x_2].$$

Then, X_1 is ε-almost negatively first-degree expectation dependent on X_2 if

$$\begin{aligned}
&\int_\Omega \big(E[X_1] - E[X_1 | X_2 \leq x_2] \big) P[X_2 \leq x_2] dx_2 \\
\leq \;& \varepsilon \int_{a_2}^{b_2} \big| E[X_1] - E[X_1 | X_2 \leq x_2] \big| P[X_2 \leq x_2] dx_2. \qquad (5)
\end{aligned}$$

The condition defining ε-almost negative expectation dependence imposes that the violation set Ω and the extent to which (1) is violated on Ω must be moderate enough to ensure that the weighted average of the differences $E[X_1] - E[X_1 | X_2 \leq x_2]$ over Ω remains less than ε times the total weighted average of these differences, the weights being given by the distribution function of X_2.

[2] In general, $t(\cdot)$ is only defined as a non-decreasing transformation. Considering the applications discussed in this paper, we can see $t(\cdot)$ as a utility function.

Let us now come back to the motivating example given in the introductory Section 1. Denoting as X_i the end-of-period value of asset i, $i = 1, 2$, we have

$$\begin{aligned} E[X_1|X_2 \le x_2] &= E[X_1|X_2 = 10] = -0.1 \text{ for } 10 \le x_2 < 100 \\ E[X_1|X_2 \le x_2] &= E[X_1|X_2 \le 100] = 19.5098 \text{ for } 100 \le x_2 < 200 \\ E[X_1|X_2 \le x_2] &= E[X_1] = 0 \text{ for } x_2 \ge 200 \end{aligned}$$

under the slightly modified joint distribution. Therefore, $\Omega = [10, 100)$ and the left-hand side of (5) equals

$$\int_{10}^{100} (0 - (-0.1)) \times 0.01 dx_2 = 0.09$$

whereas the right-hand side of (5) equals

$$0.09 + \int_{100}^{200} (19.5098 - 0) \times 0.501 dx_2 = 995.09$$

so that X_1 is ε-almost negatively first-degree expectation dependent on X_2 for any $\varepsilon \ge \frac{0.09}{995.09} = 9.0444 \times 10^{-5}$.

Defining ε-almost negative expectation dependence cannot be done by just reversing the sign of X_1, contrary to the classical expectation dependence. The following definition provides the appropriate description of this dual concept.

Definition 3 *Assume that X_1 is not positively first-degree expectation dependent on X_2. Let $\Omega^c = [a_2, b_2] \setminus \Omega$ be the complement of the violation set Ω introduced in Definition 2. Then, X_1 is ε-almost positively first-degree expectation dependent on X_2 if*

$$\begin{aligned} &\int_{\Omega^c} \big(E[X_1|X_2 \le x_2] - E[X_1]\big) P[X_2 \le x_2] dx_2 \\ \le \quad &\varepsilon \int_{a_2}^{b_2} \big|E[X_1] - E[X_1|X_2 \le x_2]\big| P[X_2 \le x_2] dx_2. \end{aligned} \tag{6}$$

The intuitive meaning of ε-almost positive expectation dependence is similar to its negative counterpart discussed above.

Let us now extend Theorem 3.1 in Wright (1987) to almost first-degree expectation dependence.

Theorem 1 *(i) If X_1 is ε-almost negatively first-degree expectation dependent on X_2 then $Cov[X_1, t(X_2)] \le 0$ for every transformation t in $\mathcal{U}_1^\varepsilon$ for which the covariances exist. If the inequality in* (5) *is strict then $Cov[X_1, t(X_2)] < 0$.*

(ii) If X_1 is ε-almost positively first-degree expectation dependent on X_2 then $Cov[X_1, t(X_2)] \ge 0$ for every transformation t in $\mathcal{U}_1^\varepsilon$ for which the covariances exist. If the inequality in (6) *is strict then $Cov[X_1, t(X_2)] > 0$.*

The proof is given in appendix. As a direct consequence of Theorem 1, because the identity function $t(x) = x$ clearly belongs to $\mathcal{U}_1^\varepsilon$, if X_1 is ε-almost negatively first-degree expectation dependent on X_2, then $Cov[X_1, X_2] \le 0$. Similarly, if X_1 is ε-almost positively first-degree expectation dependent on X_2, then $Cov[X_1, X_2] \ge 0$. Thus, we see that negatively first-degree expectation dependent random variables are negatively correlated whereas positively first-degree expectation dependent random variables are positively correlated.

3 Almost second-degree expectation dependence

3.1 Second-degree expectation dependence

Coming back to inequality (1) defining negative first-degree expectation dependence, let us define

$$ED_1(X_1|x_2) = E[X_1] - E[X_1|X_2 \leq x_2].$$

Negative first-degree expectation dependence then corresponds to the condition $ED_1(X_1|x_2) \leq 0$ for all x_2, or equivalently to

$$P[X_2 \leq x_2]ED_1(X_1|x_2) \leq 0 \text{ for all } x_2.$$

Li (2011) suggested to extend this concept to second-degree expectation dependence by controlling the sign of $ED_2(X_1|x_2)$ corresponding to the integral of $ED_1(X_1|s)$ over the interval $[a_2, x_2]$, weighted by the distribution function $P[X_2 \leq s]$, i.e.

$$ED_2\left(X_1|\, x_2\right) = \int_{a_2}^{x_2} \big(E[X_1] - E[X_1|X_2 \leq s]\big)P[X_2 \leq s]ds.$$

Definition 4 (Li, 2011) *The random variable X_1 is negatively second-degree expectation dependent on X_2 if $ED_2\left(X_1|\, x_2\right) \leq 0$ for all x_2. Similarly, X_1 is positively second-degree expectation dependent on X_2 if $ED_2\left(X_1|\, x_2\right) \geq 0$ for all x_2.*

Notice that

$$\begin{aligned} ED_2\left(X_1|\, x_2\right) &= E[X_1]E[(x_2 - X_2)_+] - E[X_1(x_2 - X_2)_+] \\ &= -Cov[X_1, (x_2 - X_2)_+] \end{aligned} \tag{7}$$

where $(\cdot)_+$ returns to the positive part of its argument, i.e. $\xi_+ = \xi$ if $\xi \geq 0$ and $\xi_+ = 0$ otherwise. For $x_2 = b_2$, we recover $ED_2(X_1, b_2) = Cov[X_1, X_2]$. Thus, X_1 is negatively second-degree expectation dependent on X_2 if

$$Cov[X_1, (x_2 - X_2)_+] \geq 0 \text{ for all } x_2. \tag{8}$$

Similarly, X_1 is positively second-degree expectation dependent on X_2 if

$$Cov[X_1, (x_2 - X_2)_+] \leq 0 \text{ for all } x_2. \tag{9}$$

Conditions (8) and (9) show that second-degree expectation dependence controls the sign of the covariance between asset X_1 and the payoffs of all the put options written on asset X_2, whatever their exercise price x_2.

3.2 Almost second-degree expectation dependence

In this section, we aim to establish a result similar to Theorem 1 for transformations t in the subset of $\mathcal{U}_{2,\mathrm{a}}$ corresponding to almost second-order stochastic dominance. Specifically, we now consider transformations

$$t \in \mathcal{U}_{2,\mathrm{a}}^{\theta} = \left\{ u \in \mathcal{U}_{2,\mathrm{a}} \middle| -u^{(2)}(x) \leq \inf\left\{-u^{(2)}\right\}\left(\frac{1}{\theta} - 1\right) \text{ for all } x \right\}, \tag{10}$$

where $\theta \in \left(0, \frac{1}{2}\right)$. Utility functions in $\mathcal{U}_{2,\mathrm{a}}^{\theta}$ correspond to almost second-order stochastic dominance; see Leshno and Levy (2002) and Tzeng et al. (2013). Our aim is now to derive conditions on the joint distribution of X_1 and X_2 fixing the sign of $Cov[X_1, t(X_2)]$ for $t \in \mathcal{U}_{2,\mathrm{a}}^{\theta}$.

The following definition weakens the second-degree expectation dependence concept of Li (2011) by allowing for moderate violations of inequalities (8) and (9).

Definition 5 **(i)** *Assume that X_1 is not negatively second-degree expectation dependent on X_2, i.e. condition* (8) *is violated for some x_2. Define the violation set Φ of condition* (8) *as the set of all x_2 such that*

$$ED_2\left(X_1|\, x_2\right) > 0 \Leftrightarrow Cov[X_1, (x_2 - X_2)_+] < 0.$$

Then, X_1 is θ-almost negatively second-degree expectation dependent on X_2 if $ED_2(X_1|b_2) = Cov[X_1, X_2] \leq 0$ and

$$\int_{\Phi} ED_2\left(X_1|\, x_2\right) dx_2 \leq \theta \int_{a_2}^{b_2} \left|ED_2\left(X_1|\, x_2\right)\right| dx_2. \tag{11}$$

(ii) *Assume that X_1 is not positively second-degree expectation dependent on X_2. Let $\Phi^c = [a_2, b_2] \setminus \Phi$ be the complement of the violation set Φ defined in item (i). Then, X_1 is θ-almost positively second-degree expectation dependent on X_2 if $ED_2(X_1|b_2) = Cov[X_1, X_2] \geq 0$ and*

$$\int_{\Phi^c} ED_2\left(X_1|\, x_2\right) dx_2 \leq \theta \int_{a_2}^{b_2} \left|ED_2\left(X_1|\, x_2\right)\right| dx_2. \tag{12}$$

Notice that if X_1 is θ-almost negatively (or positively) first-degree expectation dependent on X_2, this does not always imply that X_1 is θ-almost negatively (or positively) second-degree expectation dependent on X_2. The value of θ may need to be adapted when switching from almost first-degree to almost second-degree expectation dependence.

Considering the motivating example given in Section 1, let us show that X_1 is negatively θ-almost second-degree expectation dependent on X_2. In this case, we have

$$\begin{aligned}
ED_2\left(X_1|\, x_2\right) &= (0 - (-0.1)) \times 0.01\,(x_2 - 10) \\
&= 0.001\,(x_2 - 10) \ \text{ for } 10 \leq x_2 < 100 \\
ED_2\left(X_1|\, x_2\right) &= (0 - 19.5098) \times 0.501\,(x_2 - 100) \\
&= -9.7744\,(x_2 - 100) \ \text{ for } 100 \leq x_2 < 200 \\
ED_2\left(X_1|\, x_2\right) &= 0 \text{ for } x_2 \geq 200.
\end{aligned}$$

Therefore, $\Phi = [10, 100)$ and the left-hand side of (11) equals

$$\int_{10}^{100} 0.001\,(x_2 - 10)\, dx_2 = 4.05$$

whereas the right-hand side of (11) equals

$$4.05 + \int_{100}^{200} 9.7744\,(x_2 - 100)\, dx_2 = 48,876.1088$$

so that X_1 is negatively θ-almost second-degree expectation dependent on X_2 for any $\theta \geq \frac{4.05}{48,876.1088} = 8.2863 \times 10^{-5}$.

The following result extends Theorem 1 to almost second-degree expectation dependence, restricting the class of transformations to $\mathcal{U}^{\theta}_{2,\mathrm{a}}$.

Theorem 2 *(i) If X_1 is θ-almost negatively second-degree expectation dependent on X_2 then $Cov[X_1, t(X_2)] \leq 0$ for every transformation t in $\mathcal{U}^{\theta}_{2,a}$ for which the covariances exist. If the inequality in* (11) *is strict then $Cov[X_1, t(X_2)] < 0$.*

(ii) If X_1 is θ-almost positively second-degree expectation dependent on X_2 then $Cov[X_1, t(X_2)] \geq 0$ for every transformation t in $\mathcal{U}_{2,a}^{\theta}$ for which the covariances exist. If the inequality in (12) *is strict then $Cov[X_1, t(X_2)] > 0$.*

The proof is given in appendix. As a direct consequence of Theorem 2, because the identity function $t(x) = x$ belongs to $\mathcal{U}_{2,\mathrm{a}}^{\theta}$, if X_1 is θ-almost negatively second-degree expectation dependent on X_2, then $Cov[X_1, X_2] \leq 0$. Also, if X_1 is θ-almost positively second-degree expectation dependent on X_2, then $Cov[X_1, X_2] \geq 0$.

4 Second-degree and almost second-degree excess dependence

4.1 Second-degree excess dependence

First-degree expectation dependence can be equivalently defined by means of conditionings of the form $X_2 \leq x_2$ or $X_2 > x_2$. However, this is not the case for second-degree expectation dependence where these two conditions lead to different concepts. This is why we introduce here a dual concept of dependence corresponding to the excess condition $X_2 > x_2$. Precisely, let us define

$$\overline{ED}_1(X_1|x_2) = E[X_1|X_2 > x_2] - E[X_1].$$

First-degree expectation dependence controls the sign of $\overline{ED}_1$. Now, let us integrate $\overline{ED}_1(X_1|s)$ over the interval $[x_2, b_2]$, weighted by the excess function $P[X_2 > s]$ of X_2 to get

$$\begin{aligned} \overline{ED}_2\left(X_1|\, x_2\right) &= \int_{x_2}^{b_2} \overline{ED}_1(X_1|x_2)P[X_2 > s]ds \qquad (13) \\ &= E[X_1(X_2 - x_2)_+] - E[X_1]E[(x_2 - X_2)_+] \\ &= Cov[X_1, (X_2 - x_2)_+]. \end{aligned}$$

For $x_2 = a_2$, we recover $\overline{ED}_2(X_1|a_2) = Cov[X_1, X_2]$.

We are now in a position to introduce the following new dependence concept.

Definition 6 *The random variable X_1 is positively second-degree excess dependent on X_2 if*

$$\begin{aligned} &\overline{ED}_2\left(X_1|\, x_2\right) \geq 0 \text{ for all } x_2 \qquad (14) \\ &\Leftrightarrow Cov\big[X_1, (X_2 - x_2)_+\big] \geq 0 \text{ for all } x_2. \end{aligned}$$

Similarly, X_1 is negatively second-degree excess dependent on X_2 is

$$\begin{aligned} &\overline{ED}_2\left(X_1|\, x_2\right) \leq 0 \text{ for all } x_2 \qquad (15) \\ &\Leftrightarrow Cov\big[X_1, (X_2 - x_2)_+\big] \leq 0 \text{ for all } x_2. \end{aligned}$$

Considering the second inequality in (14) and (15), we see that second-degree excess dependence controls the sign of the correlation between asset X_1 and the payoffs of all the call options written on asset X_2, whatever their exercise price x_2.

This new excess dependence concept possesses essentially the same properties as expectation dependence. In that respect, the main result is that it controls the sign of the covariance between X_1 and any non-decreasing and convex transformation of X_2, as stated next.

Theorem 3 *(i) If X_1 is negatively second-degree excess dependent on X_2 then $Cov[X_1, t(X_2)] \leq 0$ for every transformation t in $\mathcal{U}_{2,\ell}$ for which the covariances exist.*

(ii) If X_1 is positively second-degree excess dependent on X_2 then $Cov[X_1, t(X_2)] \geq 0$ for every transformation t in $\mathcal{U}_{2,\ell}$ for which the covariances exist.

The proof of this result is given in appendix.

4.2 Almost second-degree excess dependence

Let us now adapt the definition of second-degree excess dependence to control the sign of $Cov[X_1, t(X_2)]$ for any transformation

$$t \in \mathcal{U}_{2,\ell}^{\theta} = \left\{ u \in \mathcal{U}_{2,\ell} \middle| u^{(2)}(x) \leq \inf \left\{ u^{(2)} \right\} \left(\frac{1}{\theta} - 1 \right) \text{ for all } x \right\}, \tag{16}$$

where $\theta \in \left(0, \frac{1}{2}\right)$. The appropriate modifications to the conditions defining second-degree excess dependence are given next.

Definition 7 **(i)** *Assume that X_1 is not negatively second-degree excess dependent on X_2, i.e. condition* (15) *is violated for some x_2. Define the violation set Φ of condition* (**??**) *as the set of all x_2 such that*

$$\overline{ED}_2 \left(X_1 | \, x_2 \right) > 0.$$

Then, X_1 is θ-almost negatively second-degree excess dependent on X_2 if $\overline{ED}_2(X_1|a_2) = Cov[X_1, X_2] \leq 0$ and

$$\int_{\Phi} \overline{ED}_2 \left(X_1 | \, x_2 \right) dx_2 \leq \theta \int_{a_2}^{b_2} \left| \overline{ED}_2 \left(X_1 | \, x_2 \right) \right| dx_2. \tag{17}$$

(ii) *Assume that X_1 is not positively second-degree excess dependent on X_2. Let $\Phi^c = [a_2, b_2] \backslash \Phi$ be the complement of the violation set Φ defined in (i). Then, X_1 is θ-almost positively second-degree excess dependent on X_2 if $\overline{ED}_2(X_1|a_2) = Cov[X_1, X_2] \geq 0$ and*

$$\int_{\Phi^c} \overline{ED}_2 \left(X_1 | \, x_2 \right) dx_2 \leq \theta \int_{a_2}^{b_2} \left| \overline{ED}_2 \left(X_1 | \, x_2 \right) \right| dx_2. \tag{18}$$

The following result shows that conditions (17) and (18) defining almost second-degree excess dependence indeed determine the sign of $Cov[X_1, t(X_2)]$ when the transformation t belongs to the class (16).

Theorem 4 *(i) If X_1 is θ-almost negatively second-degree excess dependent on X_2 then $Cov[X_1, t(X_2)] \leq 0$ for every transformation t in $\mathcal{U}_{2,\ell}^{\theta}$ for which the covariances exist. If the inequality in* (17) *is strict then $Cov[X_1, t(X_2)] < 0$.*

(ii) If X_1 is θ-almost positively second-degree excess dependent on X_2 then $Cov[X_1, t(X_2)] \geq 0$ for every transformation t in $\mathcal{U}_{2,\ell}^{\theta}$ for which the covariances exist. If the inequality in (18) *is strict then $Cov[X_1, t(X_2)] > 0$.*

The proof is given in appendix.

5 Higher-order extensions

5.1 Higher-degree expectation and excess dependence notions

The extension from first-degree to second-degree expectation dependence by Li (2011) consists in integrating the difference $ED_1(X_1|\xi_2) = E[X_1] - E[X_1|X_2 \leq \xi_2]$ weighted by $P[X_2 \leq \xi_2]$ over the interval $[a_2, x_2]$ and by controlling the sign of the resulting $ED_2(X_1|x_2)$. This idea has been pursued by Li (2011) who defined higher-degree expectation dependence by imposing that iterated integrals of ED_2 over the interval $[a_2, x_2]$ have the same sign for all

x_2.This yields higher-degree expectation dependence. The present section briefly describes how the ideas developed earlier in this paper can be extended to higher-degree expectation dependence notions by appropriately constraining the higher derivatives of the utility function.

Starting from ED_1 and ED_2 introduced above, define iteratively

$$ED_{k+1}(X_1|x_2) = \int_{a_2}^{x_2} ED_k(X_1|s)ds \text{ for } k = 2, 3, \ldots$$

Definition 8 (Li, 2011) *The random variable X_1 is negatively nth-degree expectation dependent on X_2 if $ED_n(X_1|x_2) \leq 0$ for all x_2. Similarly, X_1 is positively nth-degree expectation dependent on X_2 if $ED_n(X_1|x_2) \geq 0$ for all x_2.*

It is shown in appendix that

$$ED_{k+1}(X_1|x_2) = -\frac{1}{k!}Cov[X_1, (x_2 - X_2)_+^k] \tag{19}$$

Then, X_1 is negatively (resp. positively) nth-degree expectation dependent on X_2 if $Cov[X_1, (x_2 - X_2)_+^{n-1}] \geq 0$ (resp. ≤ 0) for all x_2.

The same idea applies to the excess dependence notion introduced in the present paper. Specifically, starting from $\overline{ED}_1$ and $\overline{ED}_2$ introduced above, we can define iteratively

$$\overline{ED}_{k+1}(X_1|x_2) = \int_{x_2}^{b_2} \overline{ED}_k(X_1|s)ds \text{ for } k = 2, 3, \ldots$$

Definition 9 *The random variable X_1 is negatively nth-degree excess dependent on X_2 if $\overline{ED}_n(X_1|x_2) \leq 0$ for all x_2. Similarly, X_1 is positively nth-degree excess dependent on X_2 if $\overline{ED}_n(X_1|x_2) \geq 0$ for all x_2.*

It is shown in appendix that

$$\overline{ED}_{k+1}(X_1|x_2) = \frac{1}{k!}Cov[X_1, (X_2 - x_2)_+^k] \tag{20}$$

Then, X_1 is negatively (resp. positively) nth-degree excess dependent on X_2 if $Cov[X_1, (X_2 - x_2)_+^{n-1}] \leq 0$ (resp. ≥ 0) for all x_2.

These dependence notions allow to control the sign of $Cov[X_1, t(X_2)]$ for every transformation t in $\mathcal{U}_{n,\mathrm{a}}$ or in $\mathcal{U}_{n,\ell}$, as shown next.

Theorem 5 *(i) If X_1 is negatively nth-degree expectation dependent on X_2 and if $ED_k(X_1|b_2) \leq 0$ for $k = 2, \ldots, n-1$ then $Cov[X_1, t(X_2)] \leq 0$ for every transformation t in $\mathcal{U}_{n,a}$ for which the covariances exist.*

(ii) If X_1 is positively nth-degree expectation dependent on X_2 and if $ED_k(X_1|b_2) \geq 0$ for $k = 2, \ldots, n-1$ then $Cov[X_1, t(X_2)] \geq 0$ for every transformation t in $\mathcal{U}_{n,a}$ for which the covariances exist.

(iii) If X_1 is negatively nth-degree excess dependent on X_2 and if $\overline{ED}_k(X_1|a_2) \leq 0$ for $k = 2, \ldots, n-1$ then $Cov[X_1, t(X_2)] \leq 0$ for every transformation t in $\mathcal{U}_{n,\ell}$ for which the covariances exist.

(iv) If X_1 is positively nth-degree excess dependent on X_2 and if $\overline{ED}_k(X_1|a_2) \geq 0$ for $k = 2, \ldots, n-1$ then $Cov[X_1, t(X_2)] \geq 0$ for every transformation t in $\mathcal{U}_{n,\ell}$ for which the covariances exist.

The proof is given in appendix.

5.2 Higher-degree almost expectation and excess dependence notions

Almost expectation and excess dependence concepts can be defined by extending Definitions 5 and 7 to arbitrary n.

Definition 10 **(i)** *Assume that X_1 is not negatively nth-degree expectation dependent on X_2, i.e. condition $ED_n(X_1|x_2) \leq 0$ is violated for x_2 in a subset Φ of the interval $[a_2, b_2]$. Then, X_1 is θ-almost negatively nth-degree expectation dependent on X_2 if $ED_k(X_1|b_2) \leq 0$ for $k = 2, \ldots, n$ and*

$$\int_{\Phi} ED_n\left(X_1|\, x_2\right) dx_2 \leq \theta \int_{a_2}^{b_2} \left|ED_n\left(X_1|\, x_2\right)\right| dx_2. \tag{21}$$

(ii) *Assume that X_1 is not positively nth-degree expectation dependent on X_2. Let $\Phi^c = [a_2, b_2] \setminus \Phi$ be the complement of the violation set Φ defined in item (i). Then, X_1 is θ-almost positively nth-degree expectation dependent on X_2 if $ED_k(X_1|b_2) \geq 0$ for $k = 2, \ldots, n$ and*

$$\int_{\Phi^c} ED_n\left(X_1|\, x_2\right) dx_2 \leq \theta \int_{a_2}^{b_2} \left|ED_n\left(X_1|\, x_2\right)\right| dx_2. \tag{22}$$

Definition 11 **(i)** *Assume that X_1 is not negatively nth-degree excess dependent on X_2, i.e. condition $\overline{ED}_n(X_1|x_2) \leq 0$ is violated for x_2 in some subset Φ of $[a_2, b_2]$. Then, X_1 is θ-almost negatively nth-degree excess dependent on X_2 if $\overline{ED}_k(X_1|a_2) \leq 0$ for $k = 2, \ldots, n$ and*

$$\int_{\Phi} \overline{ED}_n\left(X_1|\, x_2\right) dx_2 \leq \theta \int_{a_2}^{b_2} \left|\overline{ED}_n\left(X_1|\, x_2\right)\right| dx_2. \tag{23}$$

(ii) *Assume that X_1 is not positively second-degree excess dependent on X_2. Let $\Phi^c = [a_2, b_2] \setminus \Phi$ be the complement of the violation set Φ defined in item (i). Then, X_1 is θ-almost positively nth-degree excess dependent on X_2 if $\overline{ED}_k(X_1|a_2) \geq 0$ for $k = 2, \ldots, n$ and*

$$\int_{\Phi^c} \overline{ED}_n\left(X_1|\, x_2\right) dx_2 \leq \theta \int_{a_2}^{b_2} \left|\overline{ED}_n\left(X_1|\, x_2\right)\right| dx_2. \tag{24}$$

This definition allows to extend the results in Theorems 2 and 4 to transformations

$$t \in \mathcal{U}_{n,\mathrm{a}}^{\theta} = \left\{ u \in \mathcal{U}_{n,\mathrm{a}} \middle| (-1)^{n+1} u^{(n)}(x) \leq \inf\left\{(-1)^{n+1} u^{(n)}\right\} \left(\frac{1}{\theta} - 1\right) \text{ for all } x \right\}$$

and

$$t \in \mathcal{U}_{n,\ell}^{\theta} = \left\{ u \in \mathcal{U}_{n,\ell} \middle| u^{(n)}(x) \leq \inf\left\{u^{(n)}\right\} \left(\frac{1}{\theta} - 1\right) \text{ for all } x \right\},$$

for some $0 < \theta < \frac{1}{2}$, respectively.

Theorem 6 *(i) If X_1 is θ-almost negatively nth-degree expectation dependent on X_2 then $Cov[X_1, t(X_2)] \leq 0$ for every transformation t in $\mathcal{U}_{n,a}^{\theta}$ for which the covariances exist. If the inequality in (21) is strict then $Cov[X_1, t(X_2)] < 0$.*

(ii) If X_1 is θ-almost positively nth-degree expectation dependent on X_2 then $Cov[X_1, t(X_2)] \geq 0$ for every transformation t in $\mathcal{U}_{n,a}^{\theta}$ for which the covariances exist. If the inequality in (22) *is strict then $Cov[X_1, t(X_2)] > 0$.*

Theorem 7 *(i) If X_1 is θ-almost negatively nth-degree excess dependent on X_2 then $Cov[X_1, t(X_2)] \leq 0$ for every transformation t in $\mathcal{U}_{n,\ell}^{\theta}$ for which the covariances exist. If the inequality in* (23) *is strict then $Cov[X_1, t(X_2)] < 0$.*

(ii) If X_1 is θ-almost positively nth-degree excess dependent on X_2 then $Cov[X_1, t(X_2)] \geq 0$ for every transformation t in $\mathcal{U}_{2,\ell}^{\theta}$ for which the covariances exist. If the inequality in (24) *is strict then $Cov[X_1, t(X_2)] > 0$.*

The proof of Theorems 6 and 7 are similar to those of Theorems 2 and 4 using the expansion formulas established in the proof of Theorem 5.

6 Applications

6.1 Diversification in the standard portfolio problem

Consider the following standard 2-asset portfolio problem. Let X_j, $j = 1, 2$, be the random return per monetary unit invested in risky asset j. Assume that the initial wealth is equal to unity and must be invested in one of these two assets by a risk-averse decision maker. This agent is assumed to maximize the expected utility of terminal wealth which is the end-of-period value $\lambda X_1 + (1 - \lambda)X_2$ of the portfolio, where λ represents the fraction of the initial wealth invested in asset 1. Denote as $\lambda^\star$ the solution to this equation, assumed to be unique.

We would like to find conditions on the joint distribution of X_1 and X_2 ensuring that the investor (i) has a positive demand for asset 1, i.e. $\lambda^\star > 0$, and (ii) diversifies his position, i.e. $1 > \lambda^\star > 0$ so that the optimal portfolio contains both assets. To this end, we derive the next result which extends Theorem 4.2 in Wright (1987) to our setting, where we say that X_1 is strictly ε-almost negatively nth-degree expectation dependent on X_2 if the inequality in (21) strictly holds.

Theorem 8 *Consider an investor with utility function in $\mathcal{U}_{n+1,a}^{\varepsilon}$ for some integer $n \geq 1$.*

(i) If $E[X_1] \geq E[X_2]$ then

$$X_1 \textit{ strictly } \varepsilon\textit{-almost negatively } n\textit{th-degree expectation dependent on } X_2 \Rightarrow \lambda^\star > 0$$

and

$$X_1 - X_2 \textit{ strictly } \varepsilon\textit{-almost negatively } n\textit{th-degree expectation dependent on } X_2 \Rightarrow \lambda^\star > 0.$$

(ii) If $E[X_1] = E[X_2]$ then

$$\left.\begin{array}{l} X_1 \textit{ strictly } \varepsilon\textit{-almost negatively } n\textit{th-degree expectation dependent on } X_2 \\ X_2 \textit{ strictly } \varepsilon\textit{-almost negatively } n\textit{th-degree expectation dependent on } X_1 \end{array}\right\} \Rightarrow 0 < \lambda^\star < 1$$

and

$$\left.\begin{array}{l} X_1 - X_2 \textit{ strictly } \varepsilon\textit{-almost negatively } n\textit{th-degree expectation dependent on } X_2 \\ X_2 - X_1 \textit{ strictly } \varepsilon\textit{-almost negatively } n\textit{th-degree expectation dependent on } X_1 \end{array}\right\} \Rightarrow 0 < \lambda^\star < 1.$$

The proof is given in appendix. Note that Theorem 8 does not specify the form of the utility function, except that it belongs to $\mathcal{U}_{n+1,\mathrm{a}}^{\varepsilon}$. Therefore, Theorem 8 provides a restriction on the joint distribution of the two assets that guarantees diversification.

6.2 Equity premium in consumption-based CAPM

In their extension of the consumption-based CAPM to representative agents with different risk attitudes (risk aversion and prudence), Dionne et al. (2012) used first- and second-degree expectation dependence to obtain the sign of the equity premium. This section extends their study to allow for moderate departures from these dependence structures.

Consider an investor who can buy an asset with random payoff X at a price p. Denoting ξ the quantity of the asset the investor chooses to buy, the decision problem consists in maximizing the utility of current c_0 and future C_1 consumption levels given by

$$u(c_0) + \beta E[u(C_1)]$$

under the constraints $c_0 = e_0 - p\xi$ and $C_1 = E_1 + X\xi$, where e_0 and E_1 represent the original consumption levels (if the investor bought none of the asset), and where β is the subjective discount factor. The first-order condition to this problem gives

$$p = \frac{E[X]}{r_f} + \beta \frac{Cov[X, u^{(1)}(C_1)]}{u^{(1)}(c_0)}$$

where r_f denotes the gross risk-free rate; see, e.g., Cochrane (2005, Section 1.4). Dionne et al. (2012) call the second term in the right-hand side of this expression the "first-degree expectation dependence effect". This term involves the subjective discount factor, the expectation dependence between the random payoff X and the consumption C_1, the Arrow-Pratt absolute risk aversion coefficient and the intertemporal marginal rate of substitution.

Dionne et al. (2012, Proposition 3.1) established that the equity premium for a risk-averse investor is positive (resp. negative) when the asset payoff is positively (resp. negatively) first-degree expectation dependent on the consumption. A similar result holds for prudent investors when asset payoff is second-degree expectation dependent on the consumption. Let us now extend this analysis to investors whose preferences agree with higher-order almost stochastic dominance rules.

Theorem 9 *Consider a representative agent with utility function in $\mathcal{U}^{\varepsilon}_{n+1,a}$ for some integer $n \geq 1$.*

(i) If X_1 is ε-almost negatively nth-degree expectation dependent on C then $p \geq \frac{E[X]}{r_f}$.

(ii) If X_1 is ε-almost positively nth-degree expectation dependent on C then $p \leq \frac{E[X]}{r_f}$.

As the sign of the equity premium $p - \frac{E[X]}{r_f}$ depends on the sign of $Cov[X, u^{(1)}(C_1)]$, the result is a direct consequence of Theorem 6 as $-u^{(1)}$ belongs to $\mathcal{U}^{\varepsilon}_{n,\mathrm{a}}$.

6.3 Demand of a risky asset in the presence of a background risk

In this section, we consider an investor who must determine the fraction of initial wealth to be invested in a risky asset, in presence of a background risk. We assume that the first attribute corresponds to the asset return and is assumed to be valued in the interval $[a_1, b_1]$. The second attribute corresponds to the background risk and is assumed to be valued in the interval $[a_2, b_2]$. As the background risk may not be of financial nature, we now consider that the decision maker uses a 2-attribute utility function. We use u to denote a utility function of two attributes x_1 and x_2 and we denote the cross derivatives of u as $u^{(i,j)} = \frac{\partial^{i+j} u(x_1,x_2)}{(\partial x_1)^i (\partial x_2)^j}$.

When faced with two attributes, the decision maker may mitigate the detrimental effect of a low outcome in one attribute with a high outcome of the other. This risk attitude is related to substitutability of goods shown by Eeckhoudt, Rey and Schlesinger (2007): the correlation averse[3] decision maker prefers a mix of a favorable together with an unfavorable case over two favorable or unfavorable outcomes arising simultaneously. In the expected utility setting, this means that the 2-attribute utility function u fulfills the condition $u^{(1,1)} \leq 0$. Henceforth, we denote as $\mathcal{U}_{\text{ca}}$ the class of all the correlation averse utility functions u, i.e. such that $u^{(1,0)} \geq 0$, $u^{(0,1)} \geq 0$ and $u^{(1,1)} \leq 0$. While conditions $u^{(1,0)} \geq 0$ and $u^{(0,1)} \geq 0$ simply mean that both attributes are goods, condition $u^{(1,1)} \leq 0$ expresses correlation averse behavior.

To rule out the extreme forms of preferences included in $\mathcal{U}_{\text{ca}}$, Denuit, Huang and Tzeng (2013) suggested to appropriately constrain the partial derivative $u^{(1,1)}$, following the approach of Leshno and Levy (2002) for univariate almost stochastic dominance. This leads to the concept of confined correlation aversion. Precisely, let ε be a real constant such that $0 < \varepsilon < \frac{1}{2}$. The class $\mathcal{U}_{\text{ca}}^{\varepsilon}$ contains all the elements of $\mathcal{U}_{\text{ca}}$ satisfying

$$0 \leq -u^{(1,1)}(x_1, x_2) \leq \inf\{-u^{(1,1)}\} \left(\frac{1}{\varepsilon} - 1\right) \text{ for all } x_1 \text{ and } x_2. \tag{25}$$

A utility function fulfilling (25) is said to express ε-confined correlation aversion.

Let us now apply almost expectation dependence to derive the conditions under which all decision makers exhibiting (ε)-confined correlation aversion have a positive demand for a risky asset when faced with a dependent background risk. This section extends previous works by Tsetlin and Winkler (2005) and by Li (2011). Specifically, consider a one period investment setting. The (ε)-confined correlation averse economic agent has a sure initial wealth of amount w and can invest it in a riskless asset (with return r_f) and in a single risky asset (with random return R). The decision maker must select the optimal amount α to be invested in the risky asset. The wealth at the end of the period is

$$(w - \alpha)(1 + r_f) + \alpha(1 + R) = w(1 + r_f) + \alpha(R - r_f).$$

Define $w_0 = w(1+r_f)$ and $X_1 = R - r_f$. The final wealth is then $w_0 + \alpha X_1$. Assume that the agent faces a correlated background risk X_2 and selects α to maximize expected utility $E[u(w_0 + \alpha X_1, X_2)]$ with $u \in \mathcal{U}_{\text{ca}}^{\varepsilon}$. The optimal $\alpha^\star$ solves the first-order condition

$$E[X_1 u^{(1,0)}(w_0 + \alpha X_1, X_2)] = 0.$$

We assume that the utility function u is such that the first-order condition uniquely determines $\alpha^\star$.[4] It is easily seen that

$$\alpha^\star \geq 0 \Leftrightarrow E[X_1 u^{(1,0)}(w_0, X_2)] \geq 0. \tag{26}$$

As pointed out by Li (2011), the expectation dependence concept due to Wright (1987) is the key to ensure that $\alpha^\star \geq 0$. Indeed, if $u \in \mathcal{U}_{\text{ca}}$ then $t(x_2) = -u^{(1,0)}(w_0, x_2)$ is non-decreasing and X_1 negatively first-degree expectation dependent on X_2 ensures

$$0 \geq Cov[X_1, t(X_2)] = E[X_1]E[u^{(1,0)}(w_0, X_2)] - E[X_1 u^{(1,0)}(w_0, X_2)] \tag{27}$$

$$\Rightarrow E[X_1 u^{(1,0)}(w_0, X_2)] \geq E[X_1]E[u^{(1,0)}(w_0, X_2)].$$

When $E[X_1] \geq 0$, we see that $E[X_1]E[u^{(1,0)}(w_0, X_2)] \geq 0$ which in turn ensures that $\alpha^\star \geq 0$ holds.

[3] This risk attitude was termed as risk aversion in Richard (1975). Correlation aversion can also be related to the correlation increasing transformations defined by Epstein and Tanny (1980).

[4] If $u^{(2,0)} < 0$, for instance, then the second-order condition holds and the optimal solution is uniquely determined.

Let us now switch to ε-confined correlation aversion, i.e. $u \in \mathcal{U}_{\text{ca}}^{\varepsilon}$ instead of $u \in \mathcal{U}_{\text{ca}}$. The same reasoning shows that Theorem 1 implies inequality (27). Again, when $E[X_1] \geq 0$, all investors with utility function u in $\mathcal{U}_{\text{ca}}^{\varepsilon}$ have a positive demand for the risky asset if X_1 is ε-almost negatively first-degree expectation dependent on X_2, whatever their initial wealth level. This is formally stated in the next result.

Theorem 10 *Assume that $u \in \mathcal{U}_{ca}^{\varepsilon}$ and $E[X_1] \geq 0$.*

(i) If X_1 is ε-almost negatively first-degree expectation dependent on X_2 then $\alpha^\star \geq 0$ for all w_0.

(ii) If X_1 is strictly ε-almost negatively first-degree expectation dependent on X_2 then $\alpha^\star > 0$ for all w_0.

Alternatively, the decision maker could prefer that the two favorable outcomes arise simultaneously. Those correlation loving decision makers have thus a preference for combining good with good and bad with bad. Denuit, Huang and Tzeng (2013) defined the concept of confined correlation loving as follows. Let ϕ be a real constant such that $0 < \phi < \frac{1}{2}$. The class $\mathcal{U}_{c\ell}^{\phi}$ contains all the 2-attribute utility functions u such that $u^{(1,0)} \geq 0$, $u^{(0,1)} \geq 0$ and

$$0 \leq u^{(1,1)}(x_1, x_2) \leq \inf\{u^{(1,1)}\} \left(\frac{1}{\phi} - 1\right) \text{ for all } x_1 \text{ and } x_2. \tag{28}$$

Such a utility function is said to express ϕ-confined correlation loving. Condition (28) requires that the ratio of the maximum value to the minimum value of $u^{(1,1)}$ be bounded and excludes some extreme preferences. The next result extends Theorem 10 to ϕ-confined correlation loving.

Theorem 11 *Assume that $u \in \mathcal{U}_{c\ell}^{\phi}$ and $E[X_1] \geq 0$.*

(i) If X_1 is ε-almost positively first-degree expectation dependent on X_2 then $\alpha^\star \geq 0$ for all w_0.

(ii) If X_1 is strictly ε-almost positively first-degree expectation dependent on X_2 then $\alpha^\star > 0$ for all w_0.

To demonstrate the higher-order extensions, let us assume that the agents are cross-prudent in wealth, i.e., $u^{(1,2)}(x_1, x_2) \geq 0$. let $\mathcal{U}_{\text{cap}}^{\varepsilon}$ be the class of all the 2-attribute utility functions u such that $u^{(1,0)} \geq 0$, $u^{(0,1)} \geq 0$, $u^{(1,1)} \leq 0$, and

$$0 \leq u^{(1,2)}(x_1, x_2) \leq \inf\{u^{(1,2)}\} \left(\frac{1}{\varepsilon} - 1\right) \text{ for all } x_1 \text{ and } x_2. \tag{29}$$

Similarly, let $\mathcal{U}_{c\ell\text{p}}^{\phi}$ be the class of all the 2-attribute utility functions u such that $u^{(1,0)} \geq 0$, $u^{(0,1)} \geq 0$, $u^{(1,1)} \geq 0$, and

$$0 \leq u^{(1,2)}(x_1, x_2) \leq \inf\{u^{(1,2)}\} \left(\frac{1}{\phi} - 1\right) \text{ for all } x_1 \text{ and } x_2. \tag{30}$$

Furthermore, Equation (26) can be rewritten as $\alpha^\star \geq 0$ if and only if

$$E\left[X_1\right] E\left[u^{(1,0)}(w_0, X_2)\right] + u^{(1,1)}(w_0, b_2) ED_2\left(X_1 | \, b_2\right)$$

$$- \int_{a_2}^{b_2} u^{(1,2)}(w_0, x_2) ED_2 \left(X_1 \middle| x_2 \right) dx_2 \geq 0. \tag{31}$$

If X_1 is ε-almost negatively second-degree expectation dependent on X_2 then, for $u^{(1,1)} \leq 0$ and $u^{(1,2)} \geq 0$, the second and the third terms of equation (31) are positive. On the other hand, equation (31) can be rewritten as

$$E[X_1] E\left[u^{(1,0)}(w_0, X_2)\right] + u^{(1,1)}(w_0, a_2)\overline{ED}_2 \left(X_1 \middle| a_2 \right)$$

$$+ \int_{a_2}^{b_2} u^{(1,2)}(w_0, x_2)\overline{ED}_2 \left(X_1 \middle| x_2 \right) dx_2 \geq 0. \tag{32}$$

If X_1 is ε-almost positively second-degree excess dependent on X_2 then, for $u^{(1,1)} \geq 0$ and $u^{(1,2)} \geq 0$, the second and the third terms of equation (31) are positive. The above findings can be summarized as follows.

Theorem 12 *Assume that* $E[X_1] \geq 0$.

(i) For $u \in \mathcal{U}^{\varepsilon}_{cap}$*, if* X_1 *is* ε*-almost negatively second-degree expectation dependent on* X_2 *then* $\alpha^\star \geq 0$ *for all* w_0.

(ii) For $u \in \mathcal{U}^{\varepsilon}_{c\ell p}$*, if* X_1 *is* ε*-almost positively second-degree excess dependent on* X_2 *then* $\alpha^\star \geq 0$ *for all* w_0.

7 Conclusion

In this paper, expectation dependence has been extended to almost(ε) expectation dependence allowing for small violations of the condition defining the original concept. Almost expectation dependence plays a similar role than the former expectation dependence notion due to Wright (1987). It ensures that investors hold diversified positions and have a positive demand for a risky asset in the presence of a correlated background risk.

The new excess dependence notion has been introduced and weakened in its almost form. This allowed us to extend to risk-seekers the results previously obtained in the literature for risk-averters. Also, higher-order extensions are discussed, allowing to sign $Cov[X_1, t(X_2)]$ in a wide range of situations. As such a covariance naturally arises in many applications, the results derived in the present paper help to draw useful conclusions in various economic contexts.

Acknowledgements

The financial support of PARC "Stochastic Modelling of Dependence" 2012-17 awarded by the Communauté française de Belgique is gratefully acknowledged by Michel Denuit.

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Appendix:
Proofs of the results

A Proof of Theorem 1

Let us start with statement (i). Consider a transformation t in $\mathcal{U}_1^\varepsilon$. Let Ω^c denote the complement of Ω in $[a_2, b_2]$ as introduced in Definition 3. It is easily seen that the condition (1) defining ε-almost negative expectation dependence can be equivalently written as

$$\begin{aligned} 0 \geq & (1-\varepsilon)\int_\Omega \big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ & +\varepsilon\int_{\Omega^c}\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2. \end{aligned} \tag{33}$$

Now, for any transformation $t\in\mathcal{U}_1^\varepsilon$ we have

$$\sup\{t^{(1)}\}\leq \inf\{t^{(1)}\}\left(\frac{1}{\varepsilon}-1\right)\Leftrightarrow \frac{\sup\{t^{(1)}\}}{\inf\{t^{(1)}\}}\leq\frac{1}{\varepsilon}-1.$$

Then, considering (3), we can write

$$\begin{aligned} Cov[X_1,t(X_2)] &= \int_{a_2}^{b_2} t^{(1)}(x_2)\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &= \int_\Omega t^{(1)}(x_2)\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &\quad +\int_{\Omega^c} t^{(1)}(x_2)\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &\leq \sup\{t^{(1)}\}\int_\Omega \big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &\quad +\inf\{t^{(1)}\}\int_{\Omega^c}\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &= \inf\{t^{(1)}\}\Bigg(\frac{\sup\{t^{(1)}\}}{\inf\{t^{(1)}\}}\int_\Omega \big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &\qquad +\int_{\Omega^c}\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2\Bigg) \\ &\leq \inf\{t^{(1)}\}\Bigg(\left(\frac{1}{\varepsilon}-1\right)\int_\Omega \big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &\qquad +\int_{\Omega^c}\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2\Bigg) \\ &= \frac{1}{\varepsilon}\inf\{t^{(1)}\}\Bigg((1-\varepsilon)\int_\Omega \big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2 \\ &\qquad +\varepsilon\int_{\Omega^c}\big(E[X_1]-E[X_1|X_2\leq x_2]\big)P[X_2\leq x_2]dx_2\Bigg) \end{aligned}$$

which is indeed negative if (33) holds, that is, if X_1 is ε-almost negatively first-degree expectation dependent on X_2. Note that $Cov[X_1,t(X_2)]<0$ if inequality (33) is strict. The proof for (ii) follows the same lines.

To end with, note that Wright (1987) established equivalence in his Theorem 3.1. The proof there is based on step functions (corresponding to satisficers' utility functions) which are precisely those excluded when reducing $\mathcal{U}_1$ to $\mathcal{U}_1^\varepsilon$.

B Proof of Theorem 2

Let us start with the statement (i). Consider a transformation t in $\mathcal{U}_{2,\mathrm{a}}^{\theta}$. It is easily seen that the condition (11) defining θ-almost negative second-degree expectation dependence can be equivalently written as

$$0 \geq (1-\theta) \int_{\Omega} ED_2\left(X_1 | \, x_2\right) dx_2 + \theta \int_{\Omega^c} ED_2\left(X_1 | \, x_2\right) dx_2. \tag{34}$$

Now, for any transformation $t \in \mathcal{U}_{2,\mathrm{a}}^{\theta}$ we have

$$\sup\{-t^{(2)}\} \leq \inf\{-t^{(2)}\} \left(\frac{1}{\theta} - 1\right) \Leftrightarrow \frac{\sup\{-t^{(2)}\}}{\inf\{-t^{(2)}\}} \leq \frac{1}{\theta} - 1.$$

Then, considering (3), we can write

$$\begin{aligned} Cov[X_1, t(X_2)] &= \int_{a_2}^{b_2} t^{(1)}(x_2) ED_1(X_1|x_2) P[X_2 \leq x_2] dx_2 \\ &= t^{(1)}(b_2) ED_2\left(X_1 | \, b_2\right) + \int_{a_2}^{b_2} \left(-t^{(2)}(x_2)\right) ED_2\left(X_1 | \, x_2\right) dx_2. \end{aligned} \tag{35}$$

If $ED_2\left(X_1 | \, b_2\right) \leq 0$, then the first term is negative since $t^{(1)}(b_2) \geq 0$. The second term can be rewritten as

$$\begin{aligned} & \int_{a_2}^{b_2} \left(-t^{(2)}(x_2)\right) ED_2\left(X_1 | \, x_2\right) dx_2 \\ = \; & \int_{\Phi} \left(-t^{(2)}(x_2)\right) ED_2\left(X_1 | \, x_2\right) dx_2 + \int_{\Phi^c} \left(-t^{(2)}(x_2)\right) ED_2\left(X_1 | \, x_2\right) dx_2 \\ \leq \; & \sup\{-t^{(2)}\} \int_{\Phi} ED_2\left(X_1 | \, x_2\right) dx_2 + \inf\{-t^{(2)}\} \int_{\Phi^c} ED_2\left(X_1 | \, x_2\right) dx_2 \\ = \; & \inf\{-t^{(2)}\} \left(\frac{\sup\{-t^{(2)}\}}{\inf\{-t^{(2)}\}} \int_{\Phi} ED_2\left(X_1 | \, x_2\right) dx_2 + \int_{\Phi^c} ED_2\left(X_1 | \, x_2\right) dx_2 \right) \\ \leq \; & \inf\{-t^{(2)}\} \left(\left(\frac{1}{\theta} - 1\right) \int_{\Phi} ED_2\left(X_1 | \, x_2\right) dx_2 + \int_{\Phi^c} ED_2\left(X_1 | \, x_2\right) dx_2 \right) \\ = \; & \frac{1}{\theta} \inf\{-t^{(2)}\} \left((1-\theta) \int_{\Phi} ED_2\left(X_1 | \, x_2\right) dx_2 + \theta \int_{\Phi^c} ED_2\left(X_1 | \, x_2\right) dx_2 \right) \end{aligned}$$

which is indeed negative if (11) holds. Thus, if X_1 is θ-almost negatively second-degree expectation dependent on X_2, then $Cov[X_1, t(X_2)] \leq 0$. Note that $Cov[X_1, t(X_2)] < 0$ if inequality (11) is strict. The proof for (ii) follows the same lines.

C Proof of Theorem 3

Considering a transformation $t \in \mathcal{U}_{2,\ell}$, we can rewrite (3) as

$$\begin{aligned} Cov[X_1, t(X_2)] &= \int_{a_2}^{b_2} t^{(1)}(x_2) \overline{ED}_1(X_1|x_2) P[X_2 > x_2] dx_2 \\ &= t^{(1)}(a_2) \overline{ED}_2\left(X_1 | \, a_2\right) + \int_{a_2}^{b_2} t^{(2)}(x_2) \overline{ED}_2\left(X_1 | \, x_2\right) dx_2. \end{aligned} \tag{36}$$

Thus, $Cov[X_1, t(X_2)] \geq 0$ for all $t \in \mathcal{U}_{2,\ell}$ when X_1 is positively second-degree excess dependent on X_2 and $Cov[X_1, t(X_2)] \leq 0$ for all $t \in \mathcal{U}_{2,\ell}$ when X_1 is negatively second-degree excess dependent on X_2

D Proof of Theorem 4

Considering (36), we see that we have to establish the validity of the inequality

$$\int_{a_2}^{b_2} \overline{ED}_2(X_1|x_2)t^{(2)}(x_2)dx_2 \leq 0$$

when X_1 is θ-almost negatively second-degree excess dependent on X_2. This integral can be rewritten as

$$\begin{aligned}
& \int_{\Phi} \overline{ED}_2(X_1|x_2)t^{(2)}(x_2)dx_2 + \int_{\Phi^c} \overline{ED}_2(X_1|x_2)t^{(2)}(x_2)dx_2 \\
\leq & \sup\{t^{(2)}\} \int_{\Phi} \overline{ED}_2(X_1|x_2)dx_2 + \inf\{t^{(2)}\} \int_{\Phi^c} \overline{ED}_2(X_1|x_2)dx_2 \\
= & \inf\{t^{(2)}\} \left(\frac{\sup\{t^{(2)}\}}{\inf\{t^{(2)}\}} \int_{\Phi} \overline{ED}_2(X_1|x_2)dx_2 + \int_{\Phi^c} \overline{ED}_2(X_1|x_2)dx_2 \right) \\
\leq & \inf\{t^{(2)}\} \left(\left(\frac{1}{\theta} - 1 \right) \int_{\Phi} \overline{ED}_2(X_1|x_2)dx_2 + \int_{\Phi^c} \overline{ED}_2(X_1|x_2)dx_2 \right) \\
\leq & \frac{1}{\theta} \inf\{t^{(2)}\} \left((1-\theta) \int_{\Phi} \overline{ED}_2(X_1|x_2)dx_2 + \theta \int_{\Phi^c} \overline{ED}_2(X_1|x_2)dx_2 \right)
\end{aligned}$$

which is indeed negative when X_1 is θ-almost negatively second-degree excess dependent on X_2.

E Proof of Equations (19) and (20)

Let us first establish that the announced representation (19) holds for $k = 2$, i.e. for ED_3. To this end, it suffices to use (7) to write

$$\begin{aligned}
ED_3(X_1|x_2) &= \int_{a_2}^{x_2} ED_2(X_1|s)ds \\
&= \int_{a_2}^{x_2} \big(E[X_1]E[(s-X_2)_+] - E[X_1(s-X_2)_+] \big) ds \\
&= E[X_1] \frac{E[(s-X_2)_+^2]}{2} - E\left[X_1 \frac{(s-X_2)_+^2}{2} \right] \\
&= -\frac{1}{2} Cov[X_1, (x_2 - X_2)_+^2].
\end{aligned}$$

Now, assume that the representation (19) holds for $ED_2, ED_3, \ldots, ED_k$ and let us establish its validity for ED_{k+1}:

$$\begin{aligned}
ED_{k+1}(X_1|x_2) &= \int_{a_2}^{x_2} ED_k(X_1|s)ds \\
&= \int_{a_2}^{x_2} \left(E[X_1] \frac{E[(s-X_2)_+^{k-1}]}{(k-1)!} - E\left[X_1 \frac{(s-X_2)_+^{k-1}}{(k-1)!} \right] \right) ds \\
&= E[X_1] \frac{E[(s-X_2)_+^k]}{k!} - E\left[X_1 \frac{(s-X_2)_+^k}{k!} \right] \\
&= -\frac{1}{k!} Cov[X_1, (x_2 - X_2)_+^k],
\end{aligned}$$

as announced in (19). The proof of the validity of (20) follows the same lines.

F Proof of Theorem 5

Starting from (35), we get

$$\begin{aligned} Cov[X_1, t(X_2)] &= t^{(1)}(b_2)ED_2\left(X_1|\, b_2\right) - t^{(2)}(b_2)ED_3\left(X_1|\, b_2\right) \\ &\quad + \int_{a_2}^{b_2} t^{(3)}(x_2)ED_3\left(X_1|\, x_2\right) dx_2. \end{aligned}$$

Continuing in this way gives

$$\begin{aligned} Cov[X_1, t(X_2)] &= t^{(1)}(b_2)ED_2\left(X_1|\, b_2\right) - t^{(2)}(b_2)ED_3\left(X_1|\, b_2\right) \\ &\quad + \ldots + (-1)^n t^{(n-1)}(b_2)ED_n\left(X_1|\, b_2\right) \\ &\quad + \int_{a_2}^{b_2} (-1)^{n+1} t^{(n)}(x_2)ED_n\left(X_1|\, x_2\right) dx_2. \end{aligned}$$

Similarly, considering (36), we can write

$$\begin{aligned} Cov[X_1, t(X_2)] &= t^{(1)}(a_2)\overline{ED}_2\left(X_1|\, a_2\right) + t^{(2)}(a_2)\overline{ED}_3\left(X_1|\, a_2\right) \\ &\quad + \int_{a_2}^{b_2} t^{(3)}(x_2)\overline{ED}_3\left(X_1|\, x_2\right) dx_2 \end{aligned}$$

which finally gives

$$\begin{aligned} Cov[X_1, t(X_2)] &= t^{(1)}(a_2)\overline{ED}_2\left(X_1|\, a_2\right) + t^{(2)}(a_2)\overline{ED}_3\left(X_1|\, a_2\right) \\ &\quad + \ldots + t^{(n-1)}(a_2)\overline{ED}_n\left(X_1|\, a_2\right) \\ &\quad + \int_{a_2}^{b_2} t^{(n)}(x_2)\overline{ED}_n\left(X_1|\, x_2\right) dx_2. \end{aligned}$$

G Proof of Theorem 8

Let us consider the result stated under (i). The investor with utility function u selects λ in order to maximize the objective function

$$\mathcal{O}(\lambda) = E[u(\lambda X_1 + (1-\lambda)X_2)].$$

The first-order condition is

$$\frac{d}{d\lambda}\mathcal{O}(\lambda) = 0 \Leftrightarrow E[(X_1 - X_2)u^{(1)}(\lambda X_1 + (1-\lambda)X_2)] = 0.$$

Denote as $\lambda^\star$ the solution to this equation, assumed to be unique. Clearly,

$$\begin{aligned} \lambda^\star > 0 &\Leftrightarrow \left.\frac{d}{d\lambda}\mathcal{O}(\lambda)\right|_{\lambda=0} > 0 \\ &\Leftrightarrow E[(X_1 - X_2)u^{(1)}(X_2)] > 0 \\ &\Leftrightarrow E[X_1 u^{(1)}(X_2)] > E[X_2 u^{(1)}(X_2)]. \end{aligned}$$

Define the non-decreasing transformation t by $t(\xi) = -u^{(1)}(\xi)$. We see that $t \in \mathcal{U}_n^\varepsilon$ and $t \leq 0$. Then,

$$\lambda^\star > 0 \Leftrightarrow E[X_1 t(X_2)] < E[X_2 t(X_2)]. \tag{37}$$

Let us now rewrite condition (37) as

$$Cov[X_1, t(X_2)] + E[X_1]E[t(X_2)] < Cov[X_2, t(X_2)] + E[X_2]E[t(X_2)]$$

$$\Leftrightarrow Cov[X_1, t(X_2)] < Cov[X_2, t(X_2)] + (E[X_2] - E[X_1])E[t(X_2)].$$

If X_1 is strictly ε-almost negatively nth-degree expectation dependent on X_2 then we know from Theorem 6 that $Cov[X_1, t(X_2)] < 0$ since the transformation t belongs to $\mathcal{U}_n^{\varepsilon}$. As t is non-decreasing, we have $Cov[X_2, t(X_2)] \geq 0$. Moreover, $t \leq 0$ guarantees that the second term in the right-hand side is also non-negative when $E[X_1] \geq E[X_2]$. This shows that the inequality (37) is indeed satisfied under the retained assumptions and ends the proof of (i).

To get the second part of (i), let us start again from the equivalence

$$\lambda^{\star} > 0 \Leftrightarrow E[(X_1 - X_2)u^{(1)}(X_2)] > 0.$$

Now, using the same transformation $t = -u^{(1)}$, we get

$$\begin{aligned} E[(X_1 - X_2)u^{(1)}(X_2)] &= -E[(X_1 - X_2)t(X_2)] \\ &= -Cov[X_1 - X_2, t(X_2)] - \big(E[X_1] - E[X_2]\big)E[t(X_2)] \end{aligned}$$

which is strictly positive if $X_1 - X_2$ is strictly ε-almost negatively first-degree expectation dependent on X_2 so that the first term in the right-hand side is strictly positive whereas the second one is non-negative.

Considering (ii), we are in a position to apply (i) twice: first to the proportion λ of initial wealth invested in asset 1 and then to the proportion $1 - \lambda$ invested in asset 2. We then get that $\lambda^{\star} > 0$ and $1 - \lambda^{\star} > 0$ which is the result announced in (ii).