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# Comonotonicity, orthant convex order and sums of random variables

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# COMONOTONICITY, ORTHANT CONVEX ORDER AND SUMS OF RANDOM VARIABLES

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**Abstract**

This paper extends a useful property of the increasing convex order to the multivariate orthant convex order. Specifically, it is shown that vectors of sums of comonotonic random variables dominate in the orthant convex order vectors of sums of random variables that are smaller in the increasing convex sense, whatever their dependence structure. This result is then used to derive orthant convex order bounds on random vectors of sums of random variables. Extensions to vectors of compound sums are also discussed.

*Keywords:* stochastic order relation, orthant convex order, stochastic bounds, comonotonicity, convolution.

Subject classification: 60E15

# 1 Introduction and motivation

Stochastic orderings have been used successfully to solve various problems in applied probability and risk management. Once the validity of such a relation is established, it can be exploited to derive a host of inequalities among various quantities. Multivariate stochastic order relations allow to simultaneously compare the marginal behavior and the underlying dependence structure. For more details about stochastic orders and their applications, we refer the reader, e.g., to Shaked and Shanthikumar (2007).

Among possible stochastic order relations, the increasing convex order appears to be particularly useful in applications. Recall that given two real-valued random variables $X$ and $Y$ with finite means, $X$ is smaller than $Y$ in the increasing convex order if the inequality $E[(X-r)_+] \leq E[(Y-r)_+]$ holds for any real $r$, where $(\xi)_+$ denotes the positive part of the real $\xi$. This is henceforth denoted as $X \preceq_{\text{icx}} Y$. See Shaked and Shanthikumar (2007) for a presentation of $\preceq_{\text{icx}}$ and Denuit et al. (2005) for an overview of the applications of this stochastic dominance rule in actuarial science and risk management. In these fields, the increasing convex order is known as the stop-loss order because $E[(X-r)_+]$ is the expected reinsurance payment under a stop-loss reinsurance treaty with retention $r$, where the reinsurer pays for the losses $X$ incurred by the direct insurer above $r$. It is well known that $X \preceq_{\text{icx}} Y$ holds if, and only if, $E[g(X)] \leq E[g(Y)]$ for any non-decreasing and convex function $g$ such that the expectations exist. Intuitively speaking, $X$ being smaller than $Y$ in the increasing convex order means that $X$ is simultaneously "smaller" and "less variable" than $Y$.

The increasing convex order is known to be closed under convolution. Specifically, given two sets $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ of independent random variables such that $X_i \preceq_{\text{icx}} Y_i$ holds for every $i$, the stochastic inequality $\sum_{i=1}^n X_i \preceq_{\text{icx}} \sum_{i=1}^n Y_i$ holds true. See, e.g., Theorem 4.A.8(d) in Shaked and Shanthikumar (2007). Another, maybe less standard result involving sums of ordered random variables is as follows; see, e.g. Denuit et al. (2005, Proposition 3.4.29) for a proof. Recall that the random variables $Y_1, \ldots, Y_n$ are comonotonic if they are all non-decreasing functions of the same underlying random variable $Z$, i.e. the identity $Y_i = g_i(Z)$ holds for some non-decreasing transformation $g_i$, $i = 1, \ldots, n$. In particular, we may choose $Z$ to be uniformly distributed over the unit interval $[0,1]$ and $g_i$ to be the quantile function of $Y_i$, i.e. the left-continuous inverse of the distribution function of $Y_i$. We refer the interested reader to the review paper by Dhaene et al. (2002) for a detailed introduction to comonotonicity. Now, if $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are two sets of random variables such that the stochastic inequality $X_i \preceq_{\text{icx}} Y_i$ holds for $i = 1, ..., n$, with $Y_1, Y_{2,...,} Y_n$ comonotonic, then the stochastic inequality $\sum_{i=1}^n X_i \preceq_{\text{icx}} \sum_{i=1}^n Y_i$ holds true. Note that we make no assumption concerning the dependency among the random variables $X_1, \ldots, X_n$ so that the result is valid whatever the dependence structure among $X_1, \ldots, X_n$ provided $Y_1, Y_{2,...,} Y_n$ are comonotonic.

Let us now switch to the multivariate case and consider random variables $X_{1j}, X_{2j}, \ldots, X_{nj}$ and $Y_{1j}, Y_{2j}, \ldots, Y_{nj}$, $j = 1, \ldots, d$. If $Y_{1j}, Y_{2j}, \ldots, Y_{nj}$ for $j = 1, \ldots, d$ form $d$ sets of comonotonic random variables and if $X_{ij} \preceq_{\text{icx}} Y_{ij}$ for all $i$ and $j$ then we can invoke the result mentioned before and the stochastic inequalities

$$\sum_{i=1}^{n} X_{i1} \preceq_{\text{icx}} \sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \preceq_{\text{icx}} \sum_{i=1}^{n} Y_{id}$$

all hold true. But what can we say about the ordering of the random vectors

$$\left(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id}\right) \text{ and } \left(\sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id}\right).$$

This is the problem investigated in the present paper. To answer this question, the orthant convex order will be used. This multivariate stochastic order relation extends the increasing convex order to several dimensions. It will be shown that provided the random vectors $(X_{i_1 1}, \ldots, X_{i_d d})$ and $(Y_{i_1 1}, \ldots, Y_{i_d d})$ are ordered in the orthant convex order for every choice of indices $i_1, \ldots, i_d$ in $\{1, \ldots, n\}$, the same order relation holds between $(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id})$ and $(\sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id})$.

Such multivariate stochastic inequalities appear to be particularly useful is many applied probability problems. For instance, considering $d$ devices subjected to shocks and denoting as $X_{1j}, X_{2j}, X_{3j}, \ldots$ the inter-times between consecutive shocks, the random vector $(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id})$ collects the random times elapsed until the occurrence of the $n$th shock affecting these devices. In actuarial science, $\sum_{i=1}^{n} X_{ij}$ typically represents the total amount paid by an insurer for the $n$ contracts comprised in the $j$th class of business in a given time period. In queueing theory, this sum corresponds to the service time for $n$ clients lining up in the $j$th queue.

The remainder of our work is organized as follows. Section 2 contains the main result of this paper. It starts by recalling some definitions and then provides the condition ensuring that $(\sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id})$ and $(\sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id})$ are ordered in the orthant convex order. Then, we extend this result to random vectors of compound sums. In Section 3, the main result of Section 2 is used to derive bounds on random vectors made of sums of correlated random variables, extending to the multivariate case previous results appeared in the literature. All the random variables in this paper are assumed to be non-negative.

# 2 Main result

## 2.1 Preliminaries

If the random variables $Y_1, Y_2, \ldots, Y_n$ are comonotonic then, for any real $r$, there exist $r_1, r_2, \ldots, r_n$, satisfying $\sum_{i=1}^{n} r_i = r$, such that

$$E\left[\left(\sum_{i=1}^{n} Y_i - r\right)_+\right] = \sum_{i=1}^{n} E\left[(Y_i - r_i)_+\right]. \tag{2.1}$$

See Theorem 6 in Dhaene et al. (2002). It can even be seen from the proof of this theorem that for any $r$, there exist $r_1, r_2, \ldots, r_n$ such that the equality

$$\left(\sum_{i=1}^{n} Y_i - r\right)_+ = \sum_{i=1}^{n} (Y_i - r_i)_+ \tag{2.2}$$

holds with probability 1. If the $Y_i$'s have increasing continuous distribution functions then we can just take $r_i = F_{Y_i}^{-1}\left(F_{\sum_{i=1}^n Y_i}(r)\right)$. More care is needed in the general case for selecting the appropriate $r_i$.

Our main result uses the orthant convex order for comparing random vectors. Recall that the random vector $(X_1, X_2, \ldots, X_n)$ is smaller than $(Y_1, Y_2, \ldots, Y_n)$ in the upper orthant convex order if the inequality

$$E\Big[\prod_{i=1}^n g_i(X_i)\Big] \leq E\Big[\prod_{i=1}^n g_i(Y_i)\Big]$$

holds for every univariate non-negative increasing convex functions $g_1, g_2, \ldots, g_n$. See, e.g., Theorem 7.A.40 in Shaked and Shanthikumar (2007). This is henceforth denoted as $(X_1, X_2, \ldots, X_n) \preceq_{\text{uo-cx}} (Y_1, Y_2, \ldots, Y_n)$. Equivalently, the stochastic inequality $(X_1, X_2, \ldots, X_n) \preceq_{\text{uo-cx}} (Y_1, Y_2, \ldots, Y_n)$ holds if, and only if,

$$E\left[\prod_{i=1}^n (X_i - r_i)_+\right] \leq E\left[\prod_{i=1}^n (Y_i - r_i)_+\right] \text{ for all } r_1, r_2, \ldots, r_n, \tag{2.3}$$

and for any $k = 1, \ldots, n-1$ and $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$

$$E\left[\prod_{j=1}^k (X_{i_j} - r_{i_j})_+\right] \leq E\left[\prod_{j=1}^k (Y_{i_j} - r_{i_j})_+\right] \text{ for all } r_{i_1}, r_{i_2}, \ldots, r_{i_k}. \tag{2.4}$$

In particular, $(X_1, X_2, \ldots, X_n) \preceq_{\text{uo-cx}} (Y_1, Y_2, \ldots, Y_n) \Rightarrow X_i \preceq_{\text{icx}} Y_i$ for $i = 1, \ldots, n$ so that $\preceq_{\text{uo-cx}}$ can be seen as one of the possible extensions of $\preceq_{\text{icx}}$ to higher dimensions. Let us also mention that the $\preceq_{\text{uo-cx}}$ order corresponds to the multivariate 2-increasing convex order. We refer the reader to Denuit and Mesfioui (2010) for a general study of the multivariate $s$-increasing convex order relations. The characterization (2.3)-(2.4) is a direct consequence of Proposition 3.1 in that paper. Denuit and Mesfioui (2010) also establish various stochastic inequalities for functions of the components of $\preceq_{\text{uo-cx}}$-ordered random vectors.

## 2.2 Orthant convex comparisons

We are now ready to state the main result of this section. Recall from Shaked and Shanthikumar (2007) that $\preceq_{\text{uo-cx}}$ is stable under convolution, that is, componentwise sums of independent random vectors ordered in the $\preceq_{\text{uo-cx}}$-sense remain ordered in the same way. This property is extended here to sums of correlated random vectors by means of comonotonicity.

**Proposition 2.1.** *Let $X_{ij}$ and $Y_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n$, be two finite arrays of random variables with respective marginal distribution functions $F_{ij}$ and $G_{ij}$. Assume that*

*A1 the random variables $Y_{1j}, \ldots, Y_{nj}$ are comonotonic for every $j = 1, \ldots, d$, i.e. there exist unit uniform random variables $U_1, \ldots, U_d$ such that $Y_{ij} = G_{ij}^{-1}(U_j)$.*

*A2 the stochastic inequality*

$$(X_{i_1 1}, \ldots, X_{i_d d}) \preceq_{uo-cx} (Y_{i_1 1}, \ldots, Y_{i_d d}) \Leftrightarrow (X_{i_1 1}, \ldots, X_{i_d d}) \preceq_{uo-cx} (G^{-1}_{i_1 1}(U_1), \ldots, G^{-1}_{i_d d}(U_d))$$

*is valid for all* $i_j \in \{1, \ldots, n\}$, $j = 1, \ldots, d$.

*Denoting as* $S_j = \sum_{i=1}^n X_{ij}$ *and* $T_j = \sum_{i=1}^n Y_{ij}$, $j = 1, \ldots, d$, *we then have*

$$(S_1, \ldots, S_d) \preceq_{uo-cx} (T_1, \ldots, T_d).$$

*Proof.* We know from (2.2) that for any real $r_j$, there exist $r_{1j}, \ldots, r_{nj}$ such that

$$r_j = \sum_{i=1}^n r_{ij} \text{ and } (T_j - r_j)_+ = \sum_{i=1}^n (Y_{ij} - r_{ij})_+ \text{ almost surely.}$$

Moreover, one has

$$(S_j - r_j)_+ \leq \sum_{i=1}^n (X_{ij} - r_{ij})_+ \text{ with probability 1.}$$

Therefore, assumption A2 ensures that

$$\begin{aligned} E\left[\prod_{j=1}^d (S_j - r_j)_+\right] &\leq \sum_{i_1=1}^n \cdots \sum_{i_d=1}^n E\left[(X_{i_1 1} - r_{i_1 1})_+ \cdots (X_{i_d d} - r_{i_d d})_+\right] \\ &\leq \sum_{i_1=1}^n \cdots \sum_{i_d=1}^n E\left[(Y_{i_1 1} - r_{i_1 1})_+ \cdots (Y_{i_d d} - r_{i_d d})_+\right] \\ &= E\left[\sum_{i_1=1}^n (Y_{i_1 1} - r_{i_1 1})_+ \cdots \sum_{i_d=1}^n (Y_{i_d d} - r_{i_d d})_+\right] \\ &= E\left[\prod_{j=1}^d (T_j - r_j)_+\right]. \end{aligned}$$

Similarly, one obtains for any $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$,

$$E\left[\prod_{j=1}^k (S_{i_j} - r_{i_j})_+\right] \leq E\left[\prod_{j=1}^k (T_{i_j} - r_{i_j})_+\right] \text{ for all } r_{i_1}, r_{i_2}, \ldots, r_{i_k},$$

whence the announced result follows. □

Let us mention that assumption A2 in Proposition 2.1 may appear to be quite restrictive at first sight, as the stochastic inequality has to hold between any pair of random vectors constructed by picking one element inside each $(X_{11}, \ldots, X_{n1})$, $(X_{12}, \ldots, X_{n2})$, …, $(X_{1d}, \ldots, X_{nd})$ to form $(X_{i_1 1}, \ldots, X_{i_d d})$, and the corresponding elements inside each $(Y_{11}, \ldots, Y_{n1})$, $(Y_{12}, \ldots, Y_{n2})$, …, $(Y_{1d}, \ldots, Y_{nd})$ to form $(Y_{i_1 1}, \ldots, Y_{i_d d})$. Such a condition is nevertheless needed, as explained next. Let us associate to any $\boldsymbol{i} = (i_1, \ldots, i_d) \in \{1, \ldots, n\}^d$ the random vector $\boldsymbol{X_i} = (X_{i_1 1}, \ldots, X_{i_d d})$. Now, it is easily seen that

$$\left( \sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) = \frac{1}{n^{d-1}} \sum_{\boldsymbol{i} \in \{1,\ldots,n\}^d} \boldsymbol{X_i}.$$

Hence, Proposition 2.1 establishes that

$$\begin{aligned} \boldsymbol{X_i} \preceq_{\text{uo-cx}} \boldsymbol{Y_i} \text{ for all } \boldsymbol{i} \in \{1, \ldots, n\}^d \quad &\Rightarrow \sum_{\boldsymbol{i} \in \{1,\ldots,n\}^d} \boldsymbol{X_i} \preceq_{\text{uo-cx}} \sum_{\boldsymbol{i} \in \{1,\ldots,n\}^d} \boldsymbol{Y_i} \\ &\Leftrightarrow \left( \sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \preceq_{\text{uo-cx}} \left( \sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id} \right). \end{aligned}$$

Notice that condition A2 greatly simplifies when the random vectors $(X_{11}, \ldots, X_{n1})$, $(X_{12}, \ldots, X_{n2})$, …, $(X_{1d}, \ldots, X_{nd})$ are independent, as well as $(Y_{11}, \ldots, Y_{n1})$, $(Y_{12}, \ldots, Y_{n2})$, …, $(Y_{1d}, \ldots, Y_{nd})$ as it is generally assumed in the literature. The stability of $\preceq_{\text{uo-cx}}$ with respect to marginalization and concatenation (see, e.g. Properties 5.3 and 5.4 with $s = 2$ in Denuit and Mesfioui, 2010) then ensures that

$$\begin{aligned} & (X_{1j}, \ldots, X_{nj}) \preceq_{\text{uo-cx}} (Y_{1j}, \ldots, Y_{nj}) \text{ for } j = 1, \ldots, d \\ \Leftrightarrow\ & \boldsymbol{X_i} \preceq_{\text{uo-cx}} \boldsymbol{Y_i} \text{ for all } \boldsymbol{i} \in \{1, \ldots, n\}^d \\ \Rightarrow\ & \left( \sum_{i=1}^{n} X_{i1}, \ldots, \sum_{i=1}^{n} X_{id} \right) \preceq_{\text{uo-cx}} \left( \sum_{i=1}^{n} Y_{i1}, \ldots, \sum_{i=1}^{n} Y_{id} \right) \end{aligned}$$

as $\preceq_{\text{uo-cx}}$ is closed under convolution for independent random vectors (see, e.g. Property 5.2(iii) with $s = 2$ in Denuit and Mesfioui, 2010).

Let us now discuss a particular case where the $\preceq_{\text{uo-cx}}$-inequality assumed under A2 in Proposition 2.1 can be replaced with easier stochastic inequalities provided the random variables exhibit some strong positive dependence structure. Recall that the random vector $(X_1, \ldots, X_d)$ is said to be conditionally increasing in sequence (CIS) if $E[g(X_i)|X_1 = x_1, \ldots, X_{i-1} = x_{i-1}]$ is non-decreasing in $x_1, \ldots, x_{i-1}$ for every non-decreasing function $g$ for which the expectation is defined, $i = 2, 3, \ldots, d$. Furthermore, the random vector $(X_1, \ldots, X_d)$ is said to be conditionally increasing (CI) if $(X_{\pi(1)}, \ldots, X_{\pi(d)})$ is CIS for every permutation $\pi$ of $\{1, \ldots, d\}$, that is, if $E[g(X_i)|X_j = x_j,\ \ j \in J]$ is non-decreasing in $x_j,\ j \in J$ for every $J \subset \{1, \ldots, d\}$, $i \notin J$, and non-decreasing function $g$ for which the expectation is defined, $i = 2, 3, \ldots, d$. The assumption A2 can be replaced with the simpler $X_{ij} \preceq_{\text{icx}} Y_{ij}$ for all $i, j$ when $(X_{i_1 1}, \ldots, X_{i_d d})$ and $(Y_{i_1 1}, \ldots, Y_{i_d d})$ have a common CI copula. This is because $(X_{i_1 1}, \ldots, X_{i_d d})$ and $(Y_{i_1 1}, \ldots, Y_{i_d d})$ are then ordered in the increasing directionally convex sense by Theorem 2.4 in Balakrishnan et al. (2012), and thus also in $\preceq_{\text{uo-cx}}$.

In the next result, we derive useful stochastic inequalities when the copula $C_{i_1,\ldots,i_d}$ of the random vector $(X_{i_1 1}, \ldots, X_{i_d d})$ can be bounded above by some copula $C_U$. Notice that this condition is stronger than $C_{i_1,\ldots,i_d} \preceq_{\text{uo-cx}} C_U$ but cannot be weakened, as it will be seen from the proof of Corollary 2.2.

**Corollary 2.2.** *Let $X_{ij}$ be continuous random variables with distribution functions $F_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_{i_1,\ldots,i_d}$ be the copula associated to the random vectors $(X_{i_1 1}, \ldots, X_{i_d d})$, $i_j = 1, \ldots, n$, $j = 1, \ldots, d$. Assume that there exists a copula $C_U$ such that the inequality*

$$C_{i_1,\ldots,i_d}(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$$

*holds for all $u_1, \ldots, u_d \in [0, 1]$. Let $(U_1, \ldots, U_d)$ be a random vector distributed as $C_U$ and set $S_j = \sum_{i=1}^n X_{ij}$. Then*

$$(S_1, \ldots, S_d) \preceq_{uo\text{-}cx} \left( \sum_{i=1}^n F_{i1}^{-1}(U_1), \ldots, \sum_{i=1}^n F_{id}^{-1}(U_d) \right).$$

*Proof.* Since $C_{i_1,\ldots,i_d}(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$ for all $u_1, \ldots, u_d \in [0, 1]$, then

$$(X_{i_1 1}, \ldots, X_{i_d d}) \preceq_{\text{uo-cx}} (F_{i_1 1}^{-1}(U_1), \ldots, F_{i_d d}^{-1}(U_d)) \tag{2.5}$$

which leads to the announced result by Proposition 2.1, because the random variables $F_{ij}^{-1}(U_j)$, $i = 1, \ldots, n$, are comonotonic for every $j = 1, \ldots, d$. □

As mentioned previously, requiring $C_{i_1,\ldots,i_d} \preceq_{\text{uo-cx}} C_U$ is not enough as it does not guarantee that (2.5) holds, unless the quantile functions are convex. This is because an ordering in the $\preceq_{\text{uo-cx}}$-sense is maintained only by non-decreasing and convex transformations of the components of the random vectors (see, e.g., Theorem 7.A.41 in Shaked and Shanthikumar, 2007).

Many parametric families of copulas are ordered when their parameters vary so that there are natural candidates for $C_U$ in many situations, as shown in the next example.

**Example 2.3.** If the random vector $(X_{11}, \ldots, X_{nd})$ has an Archimedean copula with generator $\phi$ then every sub-vector possesses the same copula. For instance, if this random vector has a dependence structure governed by the Clayton copula with parameter $\alpha > 0$, i.e. by

$$C(u_1, \ldots, u_{nd}) = \left( \sum_{i=1}^{nd} u_i^{-\alpha} - nd + 1 \right)^{-1/\alpha},$$

then we can take for $C_U$ a Clayton copula with parameter $\theta \geq \alpha$ in Corollary 2.2.

**Remark 2.4.** Of course, we can always take the Fréchet-Höffding upper bound copula

$$C_U(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\}$$

in Corollary 2.2. In this case, $U_1 = \cdots = U_d$ and

$$(S_1, \ldots, S_d) \preceq_{\text{uo-cx}} \left( \sum_{i=1}^n F_{i1}^{-1}(U), \ldots, \sum_{i=1}^n F_{id}^{-1}(U) \right) \tag{2.6}$$

where $U$ is a random variable uniformly distributed over $[0, 1]$. However, the stochastic inequality (2.6) appears to be weaker than the well-known supermodular comparison of

$(S_1, \ldots, S_d)$ with its comonotonic version $(F_{S_1}^{-1}(U), \ldots, F_{S_d}^{-1}(U))$. Defining $S_j^c = \sum_{i=1}^n F_{ij}^{-1}(U)$, we know that $S_j \preceq_{\text{icx}} S_j^c$ holds for $j = 1, \ldots, d$. Therefore, as the random vectors $(S_1^c, \ldots, S_d^c)$ and $(F_{S_1}^{-1}(U), \ldots, F_{S_d}^{-1}(U))$ are both comonotonic, $(F_{S_1}^{-1}(U), \ldots, F_{S_d}^{-1}(U))$ is smaller than $(S_1^c, \ldots, S_d^c)$ in the directional convex order sense and (2.6) directly follows as a consequence. Therefore, taking for $C_U$ the Fréchet-Höffding upper bound copula does not provide an accurate upper bound.

Let us now reinforce Corollary 2.2 when the dominating copula $C_U$ is CI. In this case, we are allowed to replace the marginal distribution of the summands with $\preceq_{\text{icx}}$-larger ones.

**Corollary 2.5.** *Let $X_{ij}$ and $Y_{ij}$ be continuous random variables with distribution functions $F_{ij}$ and $G_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n$. Let $C_{i_1,\ldots,i_d}$ be the copula associated to the random vectors $(X_{i_1 1}, \ldots, X_{i_d d})$, $i_j = 1, \ldots, n$, $j = 1, \ldots, d$. Assume that there exists a copula $C_U$ such that $C_{i_1,\ldots,i_d}(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$ for all $u_1, \ldots, u_d \in [0, 1]$. Assume that the random vector $(U_1, \ldots, U_d)$ distributed as $C_U$ is CI and set $S_j = \sum_{i=1}^n X_{ij}$. Then*

$$X_{ij} \preceq_{icx} Y_{ij} \text{ for all } i, j \;\Rightarrow\; (S_1, \ldots, S_d) \preceq_{uo\text{-}cx} \left( \sum_{i=1}^n G_{i1}^{-1}(U_1), \ldots, \sum_{i=1}^n G_{id}^{-1}(U_d) \right).$$

*Proof.* Since the random vectors $(F_{i_1 1}^{-1}(U_1), \ldots, F_{i_d d}^{-1}(U_d))$ and $(G_{i_1 1}^{-1}(U_1), \ldots, G_{i_d d}^{-1}(U_d))$ have a common CI copula and $X_{ij} \preceq_{\text{icx}} Y_{ij}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, d$, then by Theorem 2.4 in Balakrishnan et al. (2012) the random vector $(G_{i_1 1}^{-1}(U_1), \ldots, G_{i_d d}^{-1}(U_d))$ dominates $(F_{i_1 1}^{-1}(U_1), \ldots, F_{i_d d}^{-1}(U_d))$ in the increasing directional convex order so that the stochastic inequality

$$(F_{i_1 1}^{-1}(U_1), \ldots, F_{i_d d}^{-1}(U_d)) \preceq_{\text{uo-cx}} (G_{i_1 1}^{-1}(U_1), \ldots, G_{i_d d}^{-1}(U_d))$$

is valid. The result then follows from Proposition 2.1. □

## 2.3 Compound sums

In this section, we now consider vectors of compound sums obtained by letting the numbers of terms appearing in the sums become themselves random variables. Following Pellerey (1999), let us give an example of application where this construction is meaningful. Consider $d$ devices subjected to shocks and let $N_1, N_2, \ldots, N_d$ be the random numbers of shocks until failure of these devices. The inter-times between consecutive shocks for device $j$ are denoted as $X_{ij}$, $i = 1, 2, \ldots$ and are assumed to be independent of the shock numbers $N_1, N_2, \ldots, N_d$. Then, $S_j = \sum_{i=1}^{N_j} X_{ij}$ represents the random lifetime for this device. Now, assume that we replace the inter-times $X_{ij}$ with $Y_{ij}$. We aim to derive conditions under which the resulting random vector of lifetimes $(S_1, \ldots, S_d)$ increases or decreases in the $\preceq_{\text{uo-cx}}$-sense. Pellerey (1999) considered multicomponent systems in which the $d$ components of each system have non-independent tolerances to shocks. In other words, this author allowed for possible dependence in the sequence $N_1, \ldots, N_d$ of the random numbers of shocks until failure of the components of the $d$ systems. Notice that no particular assumption is made about the counting processes producing the shocks. In the special case where all inter-times are independent and obey the Negative Exponential distribution with the same mean, shocks occur according to an homogeneous Poisson process. Here, we also allow for correlated inter-times between consecutive shocks.

Of course, random vectors of compound sums also naturally arise in other fields of applications. In actuarial science, for example, $S_j$ typically represents the total amount paid by an insurer for the $N_j$ claims filed by policyholder $j$, or generated by the $j$th class of business in a given time period. See, e.g., Denuit et al. (2002). In queueing theory, the same random sums may stand for the total service time for random numbers $N_1, \ldots, N_d$ clients lining up in $d$ parallel queues. See, e.g., Jean-Marie and Liu (1992).

In the next result, we keep the same random numbers of shocks until failure but allow for changes in the inter-times between consecutive shocks. To this end, we have to provide an appropriate condition on the two sets of inter-times, as shown next.

**Proposition 2.6.** *Let $X_{ij}$ and $Y_{ij}$, $j = 1, \ldots, d$, $i = 1, 2, \ldots$, be two arrays of non-negative random variables satisfying assumptions A1-A2 in Proposition 2.1 for every $n$. Let $\boldsymbol{N} = (N_1, \ldots, N_d)$ be a random vector of counting random variables, independent of the $X_{ij}$'s and of the $Y_{ij}$'s. Defining $S_{j,n_j} = \sum_{i=1}^{n_j} X_{ij}$ and $T_{j,n_j} = \sum_{i=1}^{n_j} Y_{ij}$, with the convention that the empty sums are zero, we have*

$$(S_{1,N_1}, \ldots, S_{d,N_d}) \preceq_{uo-cx} (T_{1,N_1}, \ldots, T_{d,N_d}).$$

*Proof.* Allowing for different number of terms $n_1, n_2, \ldots, n_d$ in the proof of Proposition 2.1 shows that the stochastic inequality $(S_{1,n_1}, \ldots, S_{d,n_d}) \preceq_{\text{uo-cx}} (T_{1,n_1}, \ldots, T_{d,n_d})$ holds for all $n_1, \ldots, n_d$, which in turn implies that, for all non-negative increasing convex $g_1, \ldots, g_d$,

$$\begin{aligned}
E\left[g_1(S_{1,N_1}) \cdots g_d(S_{d,N_d})\right] &= \sum_{n_1,\ldots,n_d} E\left[g_1(S_{1,n_1}) \cdots g_d(S_{d,n_d})\right] P[N_1 = n_1, \ldots, N_d = n_d] \\
&\leq \sum_{n_1,\ldots,n_d} E\left[g_1(T_{1,n_1}) \cdots g_d(T_{d,n_d})\right] P[N_1 = n_1, \ldots, N_d = n_d] \\
&= E\left[g_1(T_{1,N_1}) \cdots g_d(T_{d,N_d})\right].
\end{aligned}$$

This completes the proof. □

The next result provides a situation where the conditions of Proposition 2.6 are fulfilled. It generalizes Corollary 2.2 to context of compound sums.

**Corollary 2.7.** *Let $X_{ij}$ and $Y_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n_j$ be two arrays of non-negative random variables with distribution functions $F_{ij}$ and $G_{ij}$, $j = 1, \ldots, d$, $i = 1, \ldots, n_j$, respectively. Let $C_{i_1,\ldots,i_d}$ be the copula associated to the random vectors $(X_{i_1 1}, \ldots, X_{i_d d})$, $i_j = 1, \ldots, n_j$, $j = 1, \ldots, d$. Assume that there exists a copula $C_U$ such that $C_{i_1,\ldots,i_d}(u_1, \ldots, u_d) \leq C_U(u_1, \ldots, u_d)$ for all $u_1, \ldots, u_d \in [0, 1]$. Let $(U_1, \ldots, U_d)$ be a random vector distributed as $C_U$ and set $S_{n_j} = \sum_{i=1}^{n_j} X_{ij}$. Assume that the copula $C_U$ is CI and Let $\boldsymbol{N} = (N_1, \ldots, N_d)$ be a random vector of counting random variables, independent of $(U_1, \ldots, U_d)$ and of the $X_{ij}$'s and $Y_{ij}$'s. Then*

$$X_{ij} \preceq_{icx} Y_{ij} \text{ for all } i, j \ \Rightarrow\ (S_{N_1}, \ldots, S_{N_d}) \preceq_{uo\text{-}cx} \left( \sum_{i=1}^{N_1} G_{i1}^{-1}(U_1), \ldots, \sum_{i=1}^{N_d} G_{id}^{-1}(U_d) \right).$$

# 3 Improved $\preceq_{\text{uo-cx}}$-bounds

In this section, we aim to improve the upper bound on the random vector $(S_1,\ldots,S_d)$ derived in the preceding section and to provide a lower bound as well, in the $\preceq_{\text{uo-cx}}$-sense. This is done by means of the bounds derived in Kaas et al. (2000) for each component $S_j$ that we extend to the multivariate case.

## 3.1 Improved comonotonic upper bound

For any random variable $Z$, introduce after Kaas et al. (2000) the notation $F^{-1}_{X_{ij}|Z}(U_j)$ for the random variable $f_{ij}(U,Z)$ where the function $f_{ij}$ is defined as $f_{ij}(u,z)=F^{-1}_{X_{ij}|Z=z}(u)$. Define

$$S_j^u=\sum_{i=1}^n F^{-1}_{X_{ij}|Z}(U_j) \text{ and } S_j^u(z)=\sum_{i=1}^n F^{-1}_{X_{ij}|Z=z}(U_j) \text{ for } j=1,\ldots,d.$$

The next result generalizes Proposition 2 in Kaas et al. (2000) who established that $S_j\preceq_{\text{icx}}S_j^u$ holds for $j=1,\ldots,d$. Here, we extend this comparison to the random vectors $(S_1,\ldots,S_d)$ and $(S_1^u,\ldots,S_d^u)$ by means of the $\preceq_{\text{uo-cx}}$ order.

**Proposition 3.1.** *In the notation of Corollary 2.2, let $(U_1,\ldots,U_d)$ be a random vector distributed as $C_U$ such that $C_{i_1,\ldots,i_d}(u_1,\ldots,u_d)\leq C_U(u_1,\ldots,u_d)$ for all $u_1,\ldots,u_d\in[0,1]$. Let $Z$) be a random variable with distribution function $F_Z$. Assume that $Z$ and $(U_1,\ldots,U_d)$ are independent. Then, we have*

$$(S_1,\ldots,S_d)\preceq_{uo-cx}(S_1^u,\ldots,S_d^u).$$

*Proof.* For any $j=1,\ldots,d$, the random variables $F^{-1}_{X_{ij}|Z=z}(U_j)$, $i=1,\ldots,n$ are comonotonic. Corollary 2.2 then ensures that for any non-negative increasing convex functions $g_1,\ldots,g_d$

$$E\left[g_1(S_1)\ldots g_d(S_d)|Z=z\right]\leq E\left[g_1(S_1^u(z))\ldots g_d(S_d^u(z))\right]$$

Thus,

$$\begin{aligned} E\left[g_1(S_1)\ldots g_d(S_d)\right] &= \int E\left[g_1(S_1)\ldots g_d(S_d)|Z=z\right]dF_Z(z)\\ &\leq \int E\left[g_1(S_1^u(z))\ldots g_d(S_d^u(z))|Z=z\right]dF_Z(z)\\ &= E\left[g_1(S_1^u)\ldots g_d(S_d^u)\right]\end{aligned}$$

which ends the proof. □

Now, we are in position to show that this bound is better than that given in Corollary 2.2.

**Corollary 3.2.** *The random vector $(S_1^u,\ldots,S_d^u)$ in Proposition 3.1 is such that*

$$(S_1^u,\ldots,S_d^u)\preceq_{uo-cx}\left(\sum_{i=1}^n F_{i1}^{-1}(U_1),\ldots,\sum_{i=1}^n F_{id}^{-1}(U_d)\right).$$

*Proof.* It suffices to show that for every $\{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$, the random vectors

$$(F^{-1}_{X_{i_1 1}|Z}(U_1), \ldots, F^{-1}_{X_{i_d d}|Z}(U_d)) \text{ and } (F^{-1}_{X_{i_1 1}}(U_1), \ldots, F^{-1}_{X_{i_d d}}(U_d))$$

are identically distributed. This is the case, because

$$\begin{aligned}
& P\left[F^{-1}_{X_{i_1 1}|Z}(U_1) \leq x_1, \ldots, F^{-1}_{X_{i_d d}|Z}(U_d) \leq x_d\right] \\
= & \int P\left[F^{-1}_{X_{i_1 1}|Z=z}(U_1) \leq x_1, \ldots, F^{-1}_{X_{i_d d}|Z=z}(U_d) \leq x_d | Z = z\right] dF_Z(z) \\
= & \int P\left[U_1 \leq F_{X_{i_1 1}|Z=z}(x_1), \ldots, U_d \leq F_{X_{i_d d}|Z=z}(x_d)\right] dF_Z(z) \\
= & \int C_U\left(F_{X_{i_1 1}|Z=z}(x_1), \ldots, F_{X_{i_d d}|Z=z}(x_d)\right) dF_Z(z) \\
= & \; C_U\left(F_{X_{i_1 1}}(x_1), \ldots, F_{X_{i_d d}}(x_d)\right) \\
= & \; P\left[F^{-1}_{X_{i_1 1}}(U_1) \leq x_1, \ldots, F^{-1}_{X_{i_d d}}(U_d) \leq x_d\right].
\end{aligned}$$

The result then follows from Corollary 2.2. □

## 3.2 Lower bounds

Denote $\boldsymbol{X}_j = (X_{1j}, \ldots, X_{nj})$, $j = 1, \ldots, d$ and let $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ be a random vector with distribution function $F_{\boldsymbol{Z}}$. In order to derive a lower bound $(S_1^l, \ldots, S_d^l)$ on $(S_1, \ldots, S_d)$ with respect $\preceq_{\text{uo-cx}}$, let us consider the following random variables

$$S_j^l = \sum_{i=1}^n \mathbb{E}\left[X_{ij}|\boldsymbol{Z}\right] = \mathbb{E}\left[S_j|\boldsymbol{Z}\right], \quad j = 1, \ldots, d.$$

Kaas et al. (2000, Proposition 3) have shown that $S_j^l \preceq_{\text{icx}} S_j$ holds. The next result extends these marginal stochastic inequalities to the corresponding random vectors.

**Proposition 3.3.** *For any random vectors $\boldsymbol{Z}$ and $\boldsymbol{X}_j$, $j = 1, \ldots, d$, we have*

$$(S_1^l, \ldots, S_d^l) \preceq_{uo-cx} (S_1, \ldots, S_d).$$

*Proof.* For any non-negative increasing convex functions $g_1, \ldots, g_d$, the function $g(x_1, \ldots, x_d) = \prod_{i=1}^d g_i(x_i)$ is convex. The multivariate Jensen's inequality allows us to write

$$E\left[g_1(S_1)\ldots g_d(S_d)|\boldsymbol{Z} = \boldsymbol{z}\right] \geq g_1\left(E\left[S_1|\boldsymbol{Z} = \boldsymbol{z}\right]\right)\ldots g_d\left(E\left[S_d|\boldsymbol{Z} = \boldsymbol{z}\right]\right)$$

Therefore,

$$\begin{aligned}
E\left[g_1(S_1)\ldots g_d(S_d)\right] &= \int E\left[g_1(S_1)\ldots g_d(S_d)|\boldsymbol{Z} = \boldsymbol{z}\right] dF_{\boldsymbol{Z}}(\boldsymbol{z}) \\
&\geq \int g_1\left(E\left[S_1|\boldsymbol{Z} = \boldsymbol{z}\right]\right)\ldots g_d\left(E\left[S_d|\boldsymbol{Z} = \boldsymbol{z}\right]\right) dF_{\boldsymbol{Z}}(\boldsymbol{z}) \\
&= E\left[g_1(S_1^l)\cdots g_d(S_d^l)\right]
\end{aligned}$$

witch leads to result announced. □

Notice that when the random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ is chosen such that $Z_j$ and $\boldsymbol{X}_k$ are independent for all $j \neq k$, then $S_j^l$ reduces to

$$S_j^l = \sum_{i=1}^{n} \mathbb{E}\left[X_{ij}|Z_j\right] = \mathbb{E}\left[S_j|Z_j\right].$$


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