INSTITUT DE STATISTIQUE
BIOSTATISTIQUE ET
SCIENCES ACTUARIELLES
(ISBA)

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

DISCUSSION
PAPER

2012/27

Worst-case actuarial calculations consistent with single- and multiple-decrement life tables

CHRISTIANSEN, M.C. and M. DENUIT

WORST-CASE ACTUARIAL CALCULATIONS CONSISTENT WITH SINGLE- AND MULTIPLE-DECREMENT LIFE TABLES

MARCUS C. CHRISTIANSEN
Institute of Insurance Science
University of Ulm
D-89069 Ulm, Germany

MICHEL M. DENUIT
Institut de Statistique, Biostatistique et Sciences Actuarielles
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium

June 15, 2012

Abstract

The present work complements the recent paper by Barz and Müller (2012). Specifically, upper and lower bounds are derived for the force of mortality when one-year death probabilities are given, assuming a monotonic or convex shape. Based on these bounds, worst-case scenarios are derived depending on the mix of benefits in case of survival and of death comprised in a specific insurance policy.

Key Words: life insurance, multistate models, first-order life tables.

1 Introduction

In actuarial studies, life tables are based upon an analytical framework in which death is viewed as an event whose occurrence is probabilistic in nature. In the single-decrement case, the different causes of death are not distinguished. Life tables create a hypothetical cohort of, say, 100,000 persons at age 0 and subject it to all-cause age-specific one-year death probabilities q_x (the number of deaths per 1,000 or 10,000 or 100,000 persons of a given age) observed in a given population. In doing this, researchers can trace how the 100,000 hypothetical persons (called a synthetic cohort) would shrink in numbers due to deaths as the group ages. Denoting as ℓ_x the expected number of survivors at age x, we have the recurrence relation $\ell_{x+1} = \ell_x(1 - q_x)$, starting from a fixed number ℓ_0 of newborns. Here, $p_x = 1 - q_x$ is the one-year survival probability at age x.

Multiple decrement life tables distinguish between different causes of death (traffic accident, suicide, cancer, etc.). They are useful when the benefit in case of death may depend on the cause of death (typically, higher benefit in case of accidental death, or no benefit in case death is due to some particular cause). Such life tables display the expected number ℓ_x of individuals alive at age x together with the expected number of death due to cause 1 to m, say. The cause of death is defined as the disease or injury that initiated the sequence of events leading directly to death. It is selected from the International Classification of Diseases (ICD), for instance. All-cause death rates are then replaced with cause-specific death rates. The 4 leading causes of death are generally heart disease, stroke, accidents, and cancer. See, e.g., Jemal et al. (2005) for more details.

As explained above, a life table can be seen as a sequence of one-year survival probability p_x or death probability q_x indexed by integer age x ranging from birth (or some other initial age α) to an ultimate age ω. The knowledge of the life table itself does not characterize the probability distribution of a random lifetime T conforming to it. For integer x, we have $\Pr[T > x] = \frac{\ell_x}{\ell_0}$ but these probabilities are not known for fractional ages. This is why actuaries and demographers developed so-called fractional age assumptions allowing to interpolate survival functions between consecutive values $\frac{\ell_x}{\ell_0}$ and $\frac{\ell_{x+1}}{\ell_0}$ for integer $x = \alpha, \alpha + 1, \ldots, \omega - 1$. Some recent developments in this regard are Jones and Mereu (2000, 2002), Frostig (2002, 2003), and Hossain (2011). Barz and Müller (2012) ordered the different fractional age assumptions and built random variables that are stochastically larger than other lifetimes subject to increasing force of mortality and given one-year survival probabilities.

Assuming that the present value of benefits is a monotonic function of policyholder's remaining lifetime (as in the whole life insurance cover or in an annuity contract, for instance), Barz and Müller (2012) derived bounds on actuarial quantities under the assumption that the force of mortality is increasing and coherent with a given sequence of one-year survival probabilities. The increasingness of the forces of mortality seems to be in line with observations (at least at adult ages). Typically, one-year death probabilities are relatively high in the first year after birth, decline rapidly to a low point around age 10, and thereafter rise, in a roughly exponential fashion, before decelerating (or slowing their rate of increase) at the end of the life span. Except around the accident hump due to violent deaths (suicide and traffic accidents, mainly) at young adult ages and some controversy about mortality of supercentenarians, it is reasonable to assume that all-cause forces of mortality increase monotonically with age.

The present paper aims to relax the two main assumptions made in Barz and Müller (2012): increasing death rates and insurer's liabilities monotonic with respect to the policyholder's remaining lifetime. Indeed, the majority of insurance products mix benefits in case of death and in case of survival so that the present value of benefits may not be monotonic in the policyholder's remaining lifetime. Moreover, when multiple causes of death are considered, the monotonicity condition imposed to death rates does not apply to certain causes (like suicide or traffic accidents, for instance) while it remains plausible for others (like cancer, for example). The present paper relaxes these rather strong assumptions and derives worst-case scenarios in line with a specific life table and given the mix comprised in a specific insurance product.

Precisely, we show how the worst-case techniques of Christiansen and Denuit (2010) and Christiansen (2010) can be used to obtain bounds where the methods of Barz and Müller (2012) are not applicable. The next table summarizes the extensions to Barz and Müller (2012) derived in the present paper, stressing the technical differences arising between the two studies:

	Barz and Müller, 2012	present paper
pattern of states	single-decrement alive-dead model	single- or multiple-decrement model
contract design	liability monotone with respect to lifetime	all mix of survival and death benefits
bounds	(partially) sharp	not sharp but accurate enough for practical purposes

The paper is organized as follows. Section 2 presents the multistate model used in this paper and makes the link with the approach of Barz and Müller (2012). Section 3 considers the 2-state alive-dead model, or single-decrement case whereas Section 4 extends the results to the multiple-decrement case. Based on the monotonicity or convexity of forces of mortality, bounds are derived on every age interval. Considering the sign of the sum-at-risk, worst-case scenario and best-case scenario for the reserves correspond to these lower or upper bounds. Some numerical illustrations demonstrate the accuracy of these values.

Note that lapses can also be accounted for by adding a state $m+1$ corresponding to the exit from the portfolio, with appropriate lapse rates. This is another major improvement to Barz and Müller (2012).

2 Survival model

2.1 Multiple-decrement model

Assume that the history of each policyholder is described by a right-continuous Markovian process $\{X_t,\ t \geq 0\}$ with state space $\{0, 1, ..., m\}$, where 0 is the initial state "alive" and 1 to m correspond to m different causes of death. As mentioned in the introduction, one of the states 1 to m may correspond to lapse or surrender. Here, time t measures the seniority of the policy (i.e., the time elapsed since policy issue). Policyholder's age at policy issue is denoted as x, so that the age at time t is $x+t$. Henceforth, all quantities are indexed by t (the corresponding age being $x+t$).

The probability distribution of the Markov process $\{X_t,\ t \geq 0\}$ is uniquely described by the forces of mortality μ_{0j}, $j = 1, \ldots, m$, defined as

$$\mu_{0j}(t) = \lim_{\Delta t \to 0} \frac{\Pr[X_{t+\Delta t} = j | X_t = 0]}{\Delta t}.$$

We consider a general life insurance contract including benefits in case of survival (sojourn in state 0) and in case of death (transition from state 0 to some state 1 to m). We write $B_0(t)$ for the aggregated survival benefits minus premiums on $[0, t]$ and $b_{0j}(t)$ for the benefit payments that are due in case of a death due to cause j occurring at time t (b_{0j} may also represent the surrender value). For mathematical technical reasons we assume that the functions $t \mapsto B_0(t)$ and $t \mapsto b_{0j}(t)$, $j = 1, \ldots, m$, have finite variation on compacts and that $B_0(t)$ is right-continuous.

The contract terminates at time $\omega_x = \omega - x < \infty$, at the latest. The policyholder's remaining lifetime T can be defined by means of the process $\{X_t,\ t \geq 0\}$ by

$$T = \sup\{t \geq 0 | X_t = 0\}.$$

In words, T is the instant when the process $\{X_t,\ t \geq 0\}$ jumps from state 0 to one of the absorbing states 1 to m.

Let $\varphi(t)$ be the interest intensity (or spot rate). The present value of future benefits minus premiums for a policyholder alive at time t is then given by

$$A(t) = \int_{(t,T)} \exp\left(-\int_0^s \varphi(u)\mathrm{d}u\right) \mathrm{d}B_0(s) + \sum_{j=1}^{m} \exp\left(-\int_0^T \varphi(u)\mathrm{d}u\right) b_{0j}(T)\,.$$

The so-called prospective reserve at time t in state 0 is defined as

$$V(t) = E[A(t)|X_t = 0] = E[A(t)|T > t].$$

2.2 Single-decrement model

In the two-state model with state space $\{0, 1\}$, 0="alive" and 1="dead", Barz and Müller (2012) assumed that the present value of future benefits minus premiums at time 0 is of the form $f(T)$, where f is a monotone function. By choosing $B_0(t) = 0$ and

$$b_{01}(t) = \exp\left(\int_0^t \varphi(s)\mathrm{d}s\right) f(t)$$

we obtain $A(0) = f(T)$, which means that we can integrate the modeling framework of Barz and Müller (2012) into our framework. Note that f has finite variation on compacts since it is monotone.

3 Worst-case scenario in the single-decrement model

3.1 Increasing force of mortality

In this section, we deal with a two-state space $\{0, 1\}$ corresponding to the single-decrement model. We assume that

$$\text{the values of } p_n = \exp\left(-\int_n^{n+1} \mu_{01}(t)\,\mathrm{d}t\right) \text{ are known for every integer } n, \tag{3.1}$$

$$\text{the transition intensity } \mu_{01}(t) \text{ is monotonic increasing.} \tag{3.2}$$

Note that the quantities are indexed by contract seniority, not by age (this means that p_n is the one-year survival probability at age $x + n$).

Let M_{01} be the set of all transition intensities μ_{01} that satisfy (3.1) and (3.2). The information contained in (3.1) and (3.2) does not suffice to uniquely define the transition intensity μ_{01}, and so M_{01} has infinitely many elements. We want to find upper and lower bounds for the prospective reserve $V(t)$, given that $\mu_{01} \in M_{01}$.

From the two conditions (3.1)-(3.2) above we can conclude that we must have

$$\mu_{01}(n) \leq -\ln p_n \leq \mu_{01}(n+1)$$

for integer times n, which implies that

$$-\ln p_{n-1} \leq \mu_{01}(n) \leq -\ln p_n$$

for all positive integers n. Since $\mu_{01}(t)$ is monotonic increasing, we get

$$-\ln p_{n-1} \leq \mu_{01}(t) \leq -\ln p_{n+1}\,, \quad n \leq t \leq n+1. \tag{3.3}$$

Remark 3.1. The same approach applies to decreasing forces of mortality, even if this situation should not happen except in some very particular cases (life settlements for individuals who just undergo serious surgery, whose chances of survival increase as time passes, for instance). If μ_{01} is decreasing, we get $\mu_{01}(n) \geq -\ln p_n \geq \mu_{01}(n+1)$. Thus, we can show that, provided the force of mortality is monotonic (either decreasing or increasing), the inequalities

$$\min\{-\ln p_{n-1}, -\ln p_{n+1}\} \leq \mu_{01}(t) \leq \max\{-\ln p_{n-1}, -\ln p_{n+1}\}$$

hold true for $n \leq t \leq n+1$.

3.2 Convex or concave forces of mortality

Even if increasingness is generally a reasonable assumption for all-cause death rates, let us now derive upper and lower bounds for other shapes of mortality intensity. This assumption may apply to some cause-specific death rates, like suicide or traffic accidents, for instance. The next results allow us to derive bounds on convex forces of mortality.

Proposition 3.2. *Assume that μ_{01} is convex. If $\mu_{01}(t) \leq -\ln p_n + c$ for $t \in [n, n+1]$, then $\mu_{01}(t) \geq -\ln p_n - c$ for all $t \in [n, n+1]$.*

Proof. In order to get $\inf_{t\in[n,n+1]}\mu_{01}(t)$ under the condition $\int_n^{n+1}\mu_{01}(u)\,\mathrm{d}u = -\ln p_n$, the optimal shape for the convex function μ_{01} is a triangle with the apex pointing downwards. As all triangles have an upper bound of $-\ln p_n + c$ and as the areas where μ_{01} is above and below $-\ln p_n$ must be equal, the apex of the triangles cannot be below $-\ln p_n - c$. □

Proposition 3.3. *If μ_{01} is convex then the inequality*

$$\mu_{01}(t) \le \max\{-\ln p_{n-1}, -\ln p_{n+1}\} \text{ holds for all } n \le t \le n+1.$$

Proof. If the interval where the convex function $\mu_{01}(t)$ is minimal is to the left of n, then $\mu_{01}(t)$ is increasing for all $t \ge n$ and $\mu_{01}(t) \le -\ln p_{n+1}$ for all $t \in [n, n+1]$. Analogously, if the interval where $\mu_{01}(t)$ is minimal is to the right of $n+1$, then $\mu_{01}(t) \le -\ln p_{n-1}$ for all $t \in [n, n+1]$. If $\xi \in [n, n+1]$ is a minimum of μ_{01}, then μ_{01} is decreasing till ξ and increasing from ξ on, which implies that on $[n, \xi]$ and $[\xi, n+1]$ the function μ_{01} is bounded by $-\ln p_{n-1}$ and $-\ln p_{n+1}$, respectively. □

Combining the two propositions above, we get the following inequalities on a convex force of mortality:

$$2(-\ln p_n) - \max\{-\ln p_{n-1}, -\ln p_{n+1}\} \le \mu_{01}(t) \le \max\{-\ln p_{n-1}, -\ln p_{n+1}\} \tag{3.4}$$

for all $t \in [n, n+1]$.

Assuming that μ_{01} is concave, the same reasoning yields

$$\min\{-\ln p_{n-1}, -\ln p_{n+1}\} \le \mu_{01}(t) \le 2(-\ln p_n) + \min\{-\ln p_{n-1}, -\ln p_{n+1}\} \tag{3.5}$$

for all $t \in [n, n+1]$.

3.3 Worst-case and best-case scenarios for the reserve

Let us start with the case where the all-cause force of mortality μ_{01} is increasing. Let us denote as N_{01} the set of all increasing μ_{01} that satisfy (3.3). Let $l_{01}(t)$ and $u_{01}(t)$ be the lower and upper bound (according to (3.3)) on $\mu_{01}(t)$ at time t. Clearly, $M_{01} \subset N_{01}$, and, hence, upper and lower bounds with respect to N_{01} are also upper and lower bounds with respect to M_{01}.

With the help of the worst-case technique of Christiansen and Denuit (2010) for mortality intensities, we obtain the following result.

Proposition 3.4. *The best-case reserve $\underline{V}(t) = \inf_{\mu_{01}\in N_{01}} V(t;\mu_{01})$ and the worst-case reserve $\overline{V}(t) = \sup_{\mu_{01}\in N_{01}} V(t;\mu_{01})$ uniquely solve the integral equations*

$$\begin{aligned}
\underline{V}(t) =& B_0(\omega_x) - B_0(t) - \int_{(t,\omega_x]} \underline{V}(s)\,\varphi(s)\,\mathrm{d}s \\
&+ \frac{1}{2}\int_{(t,\omega_x]} \Big((b_{01}(s) - \underline{V}(s))\,(u_{01}(s) + l_{01}(s)) - |b_{01}(s) - \underline{V}(s)|\,(u_{01}(s) - l_{01}(s)) \Big)\mathrm{d}t, \\
\overline{V}(t) =& B_0(\omega_x) - B_0(t) - \int_{(t,\omega_x]} \overline{V}(s)\,\varphi(s)\,\mathrm{d}s \\
&+ \frac{1}{2}\int_{(t,\omega_x]} \Big((b_{01}(s) - \overline{V}(s))\,(u_{01}(s) + l_{01}(s)) + |b_{01}(s) - \overline{V}(s)|\,(u_{01}(s) - l_{01}(s)) \Big)\mathrm{d}t.
\end{aligned}$$

Furthermore, applying the results in Christiansen (2010) allows us to associate $\underline{V}(t)$ and $\overline{V}(t)$ to specific mortality intensities, as shown next.

Proposition 3.5. *Define*

$$\underline{\mu}_{01}(t) = \mathbf{1}_{b_{01}(t)\geq \underline{V}(t)} l_{01}(t) + \mathbf{1}_{b_{01}(t)< \underline{V}(t)} u_{01}(t),$$
$$\overline{\mu}_{01}(t) = \mathbf{1}_{b_{01}(t)\geq \overline{V}(t)} u_{01}(t) + \mathbf{1}_{b_{01}(t)< \overline{V}(t)} l_{01}(t).$$

Then, $\underline{V}(t) = V(t;\underline{\mu}_{01})$ *and* $\overline{V}(t) = V(t;\overline{\mu}_{01})$ *for all* t.

Depending on the sign of the sum-at-risk at time t (difference of the reserve and $b_{01}(t)$, or net cost of a transition at time t for the insurance company) in case of a death occurring at time t, the worst-case and best-case reserves then correspond to the lower or to the upper bound on the all-cause force of mortality μ_{01}.

While the best-case and worst-case intensities of Barz and Müller (2012) are elements of M_{01}, the intensities $\underline{\mu}_{01}(t)$ and $\overline{\mu}_{01}(t)$ are either in M_{01} or in $N_{01} \setminus M_{01}$. Thus, our bounds for the prospective reserve are less tight than the bounds given by Barz and Müller (2012). However, Barz and Müller (2012) assume that $A(0)$ is monotone with respect to the remaining lifetime T, whereas our modeling framework does not need that monotonicity, allowing to study also mixed contracts with any combination of sojourn benefits, premium payments and transition benefits.

The same approach applies to decreasing, convex or concave μ_{01}. It suffices to replace the bounds in (3.3) with those of Remark 3.1, (3.4) or (3.5).

The following numerical example compares the bounds derived from Propositions 3.4-3.5 to those in Barz and Müller (2012).

Example 3.6. As in Barz and Müller (2012), we assume that

$$\mu_{01}(t) = 0.0007 + 0.00005 \cdot 1.096478196^t\,,$$

but that only the p_n at integer times n (see (3.1)) are actually known. Consider a whole-life insurance cover with a unit death benefit and 6% annual interest rate. Table 1 shows upper and lower bounds for the prospective reserves $\overline{A}_{30}$ and $\overline{A}_{50}$ for monotonic or convex force of mortality. Compared to the values reported in Barz and Müller (2012), our bounds are less accurate but remain nevertheless sharp enough for practical purposes.

In the last two columns the table gives bounds for the prospective reserves of endowment insurances that pay a survival benefit of 2 at age 65 or a death benefit of 1, whichever occurs first. For the endowment insurances the present values of future payments are not monotone with respect to the policyholder's remaining lifetime, and so the method of Barz and Müller (2012) is not applicable here while the method developed in the present paper still is. The conclusions stated before still apply in this case: the bounds are accurate enough to be informative.

monotonicity assumption	$\overline{A}_{30}$	$\overline{A}_{50}$	${}_{\lvert 35}\overline{A}_{30} + 2\ {}_{35}E_{30}$	${}_{\lvert 15}\overline{A}_{50} + 2\ {}_{15}E_{50}$
upper bound	0.1104711	0.2673840	0.2670577	0.8028143
true value	0.1055055	0.2564015	0.2651185	0.7997839
lower bound	0.1008367	0.2459664	0.2632304	0.7966317
convexity assumption	$\overline{A}_{30}$	$\overline{A}_{50}$	${}_{\lvert 35}\overline{A}_{30} + 2\ {}_{35}E_{30}$	${}_{\lvert 15}\overline{A}_{50} + 2\ {}_{15}E_{50}$
upper bound	0.1104697	0.2673812	0.2671363	0.8030740
true value	0.1055055	0.2564015	0.2651185	0.7997839
lower bound	0.1003729	0.2449193	0.2631413	0.7966154

Table 1: Worst-case and best-case scenarios for whole-life insurance cover and endowment insurance described in Example 3.6.

4 Worst-case scenario for multiple-decrement models

Let us now extend the ideas of the previous section to multiple-decrement models. Define the one-year death probability due to cause j

$$q_n^{(j)} = \Pr[X_{n+1} = j | X_n = 0] = \int_n^{n+1} \exp\left(-\int_n^{\xi} \sum_{k=1}^m \mu_{0k}(\eta) d\eta\right) \mu_{0j}(\xi) d\xi$$

and the one-year survival probability

$$p_n = 1 - \sum_{j=1}^m q_n^{(j)} = \exp\left(-\int_n^{n+1} \sum_{j=1}^m \mu_{0j}(\eta) d\eta\right).$$

Clearly, $q_n = \sum_{j=1}^m q_n^{(j)}$.

Instead of (3.1) and (3.2), let us now assume that

$$\text{the values of } q_n^{(j)} \text{ are known at integer times } n. \tag{4.1}$$

As explained above, when multiple causes of death are considered, increasingness may become unrealistic for some specific decrements. This is why we consider death rates with various shapes in the next result, all complying with assumption (4.1).

Proposition 4.1. *(i) If μ_{0j} is either increasing or decreasing then*

$$\min\{q_{n-1}^{(j)}, q_{n+1}^{(j)}\} \le \mu_{0j}(t) \le \max\left\{\frac{q_{n-1}^{(j)}}{1 - q_{n-1}}, \frac{q_{n+1}^{(j)}}{1 - q_{n+1}}\right\}, \quad n \le t \le n+1.$$

(ii) If μ_{0j} is convex then

$$q_n^{(j)} - \left(\max\left\{\frac{q_{n-1}^{(j)}}{1 - q_{n-1}}, \frac{q_{n+1}^{(j)}}{1 - q_{n+1}}\right\} - \frac{q_n^{(j)}}{1 - q_n}\right) \le \mu_{0j}(t) \tag{4.2}$$

$$\le \max\left\{\frac{q_{n-1}^{(j)}}{1 - q_{n-1}}, \frac{q_{n+1}^{(j)}}{1 - q_{n+1}}\right\}, \quad t \in [n, n+1].$$

(iii) If μ_{0j} is concave then

$$\min\{q_{n-1}^{(j)}, q_{n+1}^{(j)}\} \le \mu_{0j}(t) \le \frac{q_n^{(j)}}{1-q_n} + \left(q_n^{(j)} - \min\{q_{n-1}^{(j)}, q_{n+1}^{(j)}\}\right), \quad t \in [n, n+1]. \tag{4.3}$$

Proof. Let us establish (i). Since the intensities $\mu_{0j}(u)$ are non-negative, we have

$$\int_n^{n+1} \mu_{0j}(u)\,(1-q_n)\mathrm{d}u \le q_n^{(j)} \le \int_n^{n+1} \mu_{0j}(u)\,\mathrm{d}u\,,$$

and hence

$$q_n^{(j)} \le \int_n^{n+1} \mu_{0j}(u)\,\mathrm{d}u \le \frac{q_n^{(j)}}{1-q_n}, \tag{4.4}$$

whence the announced inequality follows.

Let us now turn to (ii). As in the alive-dead model, convexity implies that we first have decreasingness and then increasingness. Hence,

$$\mu_{0j}(t) \le \max\left\{\frac{q_{n-1}^{(j)}}{1-q_{n-1}}, \frac{q_{n+1}^{(j)}}{1-q_{n+1}}\right\}, \quad n \le t \le n+1.$$

Also similar to the alive-dead model, we have that if $\mu_{0j}(t) \le \frac{q_n^{(j)}}{1-q_n} + c$ on $t \in [n, n+1]$, then $\mu_{0j}(t) \ge q_n^{(j)} - c$ for all $t \in [n, n+1]$, which leads to the lower bound in (4.2).

The proof for (iii) is similar. □

Let N be the set of all transition intensities μ_{0j}, $j = 1, \ldots, m$, that satisfy (4.1). Furthermore, we write $l_{0j}(t)$ and $u_{0j}(t)$ for the lower and upper bound on $\mu_{0j}(t)$ at time t (according to Proposition 4.1(i), (ii) or (iii)). By applying the worst-case method of Christiansen (2010), we obtain the following generalization of Proposition 3.4.

Proposition 4.2. *The best-case reserve $\underline{V}(t) = \inf_{\mu \in N} V(t;\mu)$ and worst-case reserves $\overline{V}(t) = \sup_{\mu\in N} V(t;\mu)$ uniquely solve the integral equations*

$$\begin{aligned}
\underline{V}(t) =& B_0(\omega_x) - B_0(t) - \int_{(t,\omega_x]} \underline{V}(s)\,\varphi(s)\,\mathrm{d}s \\
&+ \frac{1}{2}\sum_{j=1}^{n} \int_{(t,\omega_x]} \Big((b_{0j}(s) - \underline{V}(s))\,(u_{0j}(s) + l_{0j}(s)) - |b_{0j}(s) - \underline{V}(s)|\,(u_{0j}(s) - l_{0j}(s)) \Big)\mathrm{d}t, \\
\overline{V}(t) =& B_0(\omega_x) - B_0(t) - \int_{(t,\omega_x]} \overline{V}(s)\,\varphi(s)\,\mathrm{d}s \\
&+ \frac{1}{2}\sum_{j=1}^{n} \int_{(t,\omega_x]} \Big((b_{0j}(s) - \overline{V}(s))\,(u_{0j}(s) + l_{0j}(s)) + |b_{0j}(s) - \overline{V}(s)|\,(u_{0j}(s) - l_{0j}(s)) \Big)\mathrm{d}t.
\end{aligned}$$

monotony assumption	reserve at age 30	reserve at age 50
upper bound	0.3417897	0.0.8397701
true value	0.3387780	0.8354620
lower bound	0.3358757	0.8310205
convexity assumption	reserve at age 30	reserve at age 50
upper bound	0.3418961	0.8401417
true value	0.3387780	0.8354620
lower bound	0.3356995	0.8309075

Table 2: Worst-case and best-case scenarios for the combination of critical illness cover with endowment insurance described in Example 4.4.

For the worst-case reserve the proof is given in Christiansen (2010). The proof for the best-case reserve is analogous. From the best-case and worst-case reserves we can derive the corresponding transition intensities, as stated next.

Proposition 4.3. *Define*

$$\underline{\mu}_{0j}(t) = \mathbf{1}_{b_{0j}(t)\geq \underline{V}(t)} l_{0j}(t) + \mathbf{1}_{b_{0j}(t)<\underline{V}(t)} u_{0j}(t),$$
$$\overline{\mu}_{0j}(t) = \mathbf{1}_{b_{0j}(t)\geq \overline{V}(t)} u_{0j}(t) + \mathbf{1}_{b_{0j}(t)<\overline{V}(t)} l_{0j}(t).$$

Then, $\underline{V}(t) = V(t;\underline{\mu})$ *and* $\overline{V}(t) = V(t;\overline{\mu})$ *for all* t.

The following numerical example illustrates the accuracy of the bounds derived in Proposition 4.2.

Example 4.4. Consider a mixture of an endowment insurance and a critical illness insurance. If the policyholder gets a critical illness before age 65, a payment of 2 is made and the contract terminates. If the policyholder does not incur a critical illness but dies before age 65, a death benefit of 1 is paid. If none of these two events occurs, a survival benefit of 2 is made at age 65. Only the transitions active to dead and active to ill are relevant here. We assume that the mortality intensity has the same form as in Example 3.6 and that the morbidity intensity has the form

$$\mu_{02}(t) = 0.003 + 0.00005 \exp(0.065t),$$

but that the insurer has only information on $q_n^{(1)}$ and $q_n^{(2)}$ at integer times (see (4.1)). Table 2 shows lower and upper bounds for the prospective reserve in state active. We see that the bounds derived from Proposition 4.2 are accurate enough for practical purposes.

Acknowledgements

Michel Denuit acknowledges the financial support of AG Insurance under the K.U.Leuven Health Insurance Chair.

References

Barz, C., Müller, A., 2012. Comparison and bounds for functionals of future lifetimes consistent with life tables. Insurance: Mathematics and Economics 50, 229-235.

Christiansen, M.C., 2010. Biometric worst-case scenarios for multi-state life insurance policies. Insurance: Mathematics and Economics 47, 190-197.

Christiansen, M.C., Denuit, M.M., 2010. First-order mortality rates and safe-side actuarial calculations in life insurance. ASTIN Bulletin 40, 587-614.

Frostig, E., 2002. Comparison between future lifetime distribution and its approximation. North American Actuarial Journal 6(2), 11-17.

Frostig, E., 2003. Properties of the power family of fractional age approximations. Insurance:Mathematics and Economics 33, 163-171.

Hossain, S.A., 2011. Quadratic fractional age assumption revisited. Lifetime Data Anal 17, 321-332.

Jemal, A., Ward, E., Hao, Y., Thun, M., 2005. Trends in the leading causes of death in the United States, 1970-2002. Journal of the American Medical Association 294, 1255-1259.

Jones B.L., Mereu J.A., 2000. A family of fractional age assumptions. Insurance:Mathematics and Economics 27, 261-276.

Jones B.L., Mereu J.A., 2002. A critique of fractional age assumptions. Insurance:Mathematics and Economics 30, 363-370.