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Optimal mix between pay as you go and funding for pension liabilities in a stochastic framework

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OPTIMAL MIX BETWEEN PAY AS YOU GO AND FUNDING IN A STOCHASTIC FRAMEWORK

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ABSTRACT

This paper deals with the financing of public pension in a stochastic environment. Traditionally funded and unfunded schemes have been considered as opposite solutions and enemies for a first pillar public pension. But more recently, different countries as Sweden or Poland are exploring mix solutions combining pay as you go and funding mechanisms. The purpose of this paper is to check the rationality of such a combination using portfolio theory arguments and to find the optimal splitting of the contributions between the two systems. We first introduce the classical deterministic model leading to the well-known Samuelson rule where diversification is never optimal. Then we introduce different stochastic models where the main processes become random (wage growth, population growth, financial rate of return). We obtain in particular conditions on the parameters in order to justify the diversification and the explicit optimal sharing between pay as you go and funding.

Keywords: Pay as you go, funding, diversification, portfolio theory.

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1. Introduction

Financing of our pension is one of the most challenging macro economic issues for the next decades. In a lot of countries, populations are rapidly ageing due to the combined effect of decreasing fertility rates and increasing life expectancy. This implies an increasing dependency ratio, the number of retired people relative to the number of active people in a population. For instance, in Belgium, this ratio was equal to 40.5% in 2000. It should grow to 44.7% in 2010 and even to 68.5% in 2030 according to recent demographic projections.

This demographic evolution will clearly induce major financial problems in the long run for pure unfunded pension schemes. In those systems, usually called PAYG (pay-as-you-go), active people finance retired people's pension. The rate of return of such schemes is then linked to the growth rate of the salary mass. But, according to demographic trends, the growth rate of the mean salary is not enough to compensate the increase of dependency ratio. Therefore, either the charges of contributions or taxes must become higher and higher or pension amounts must be less generous. Another solution is to introduce a more fundamental reform in the public pension system by introducing a part of funding inside the social security. Countries like Chile or Sweden have opted for such a mechanism.

In a pure funding, contributions are invested on the financial market for one cohort's own future pension. The rate of return of the system mostly depends on the financial return. This mechanism might seem attractive considering future demographic trends in Europe, but it could lead to major problems in case of financial crash as seen recently.

This debate about the relative attractiveness of funding or PAYG is not new but recently diversification policies have been applied combining both techniques. They are no more considered as opposite solutions but two different assets to diversify the pension investments.

The starting point of this paper is that PAYG and funding schemes are dealing with different risks and have therefore specific advantages and drawbacks. We therefore seek for an equilibrium between those two "extreme" pension scheme inside a country's first pillar, arguing that mixing both systems means diversification between financial and demographic risks. The State would have to determine an optimal mix between those systems, which would be the same for each member of a pension plan. In our model we consider the general case of a pension system in a country, and not from an individual point of view.

This paper proposes some new features compared to the existing literature. Basically there are two important differences. First of all, the idea of blending different pension schemes is not often used in the literature. This method should allow to attain a new and more performant equilibrium. Dutta et al. (1999,2000) present a mixed model funded and unfunded in a mean variance framework. They find that the mixed systems are better because they enable risk diversification. Van Praag and Cardoso (2003) present a model that explains the mix between funded and unfunded pension systems. Matsen and Thogersen (2004) study the optimal size of PAYG system and the optimal split between funded and unfunded pension savings by means of a theoretical portfolio choice framework. Knel (2010) studies the optimal portfolio combination between funded and unfunded pension systems when people care about relative consumption. Guigou et al (2010) develop a model for evaluating the efficiency of a diversified pension system financed partly by capitalization, measuring the efficiency by the long term sustainability of the system.

Second, we do not assume independence between the demographic and financial risks as it is usually done. We will allow those two risks to be correlated and analyse the consequences of this assumption.

The remainder of the paper is organized as follows. Section 2 introduces the subject with a simple case, the classical deterministic model. Section 3 analyzes a two periods overlapping stochastic model with only one risky financial asset. In this section, we examine the optimality of a mixed system. We consider a defined contribution pension system: a fixed contribution is applied on the wages. The pension liability generated by these contributions will depend on the chosen funding mechanism (PAYG or funding or both). Our purpose is to compare the level of pension achieved. Sometimes diversification is not optimal. If a combined system is proved to be more efficient, we determine the optimal fraction to be invested in each pension scheme, in order to maximize the utility of the pension amount received at retirement. Section 4 analyzes a three periods overlapping stochastic model with only one risky financial asset, with two generations of working people and one generation of retired people.

In this paper, we also study the impact of the correlation between the risks on the results. Three risks are considered: the wage growth, the rate of return of the assets (of the funded system) and the population growth rate.

Doing this we will try to answer different questions. Can we obtain a theoretical justification for diversification? What is the optimal level for diversification? How does the correlation between financial and demographic risks impact this level?

2. Classical deterministic models

Let us consider a *defined contributions* pension plan. The contribution rate is fixed, while the pension that people get depends on the return of the scheme. We will compare the pension amount generated with a PAYG or a funding scheme. In the deterministic model every variable (wage growth rate, population growth rate and financial rate of return) is constant and known. This model will be our starting point for the next chapters.

2.1 Notations and assumptions

Demographics Let $L(x;t)$ be the number of persons aged x (with $x=x_0,x_1,...,x_\omega$) at time t. There are three specific ages for every generation. Between x_0 and x_{r-1} the individual is part of the active population and is affiliated to the public pension plan. He gets a salary on which a cotisation is perceived. At x_r, the individual retires and starts receiving a pension annuity that he will get until his death (somewhere before x_ω).

The probability, being alive at age x, to survive during n years (up to age $x+n$) is $p(x,n)$. The annual growth of the entrance function is d. So we have $L(x_0,t)=L(x_0,t-1).(1+d)$.

Economics Π is the contribution rate, constant over time and cohorts. $S(t)$ is the mean salary that active people receive at time t. It increases at the yearly rate g. So we have: $S(t)=S(t-1)(1+g)$. The mean pension at time t is $P(t)$. We define $RR(t)$ the replacement rate at time t as $RR(t)=\frac{P(t)}{S(t)}$. It compares the relative level of wages and pensions at one specific time.

The rate of return of the financial investments is i.

2. 2 Two period's model

We start with a simple "two period overlapping generation model". In this model each generation lives for two periods (working part and pension period) of equal length and $x_0=x_{r-1}$. The pension benefits can be computed:

Funding The contributions paid by the active generation at time $t-1$ are invested for one period and will generate their own pension at time t. The actuarial equilibrium relationship can be written as:

$$L(x_0,t-1)\pi S(t-1)(1+i)=L(x_r,t)P(t) \tag{1}$$

As $L(x_r,t)=L(x_0,t-1)p(x_0,1)$, the generated benefit will simply given by:

$P(t)=\pi S(t-1)\frac{1}{p(x_0,1)}(1+i)$. And the replacement rate is:

$$RR(t)=\pi\frac{1}{p(x_0,1)}\frac{1+i}{1+g} \tag{2}$$

Pay-As-You-Go The contributions paid at time *t* by the active generation are used to pay the pension of the retired generation at time *t*. The equilibrium relation is:

$$L(x_0,t)\pi S(t) = L(x_r,t)P(t) \tag{3}$$

We have $L(x_0,t) = L(x_0,t-1)(1+d)$ and $L(x_r,t) = L(x_0,t-1)\,p(x_0,1)$. The generated benefit is given by $P(t) = \pi S(t)\dfrac{1}{p(x_0,1)}(1+d)$. And the replacement rate is:

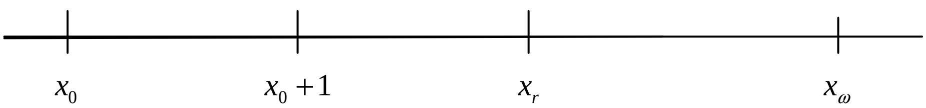
$$RR(t) = \pi \frac{1}{p(x_0,1)}(1+d) \tag{4}$$

Comparison and decision rule We obtain the well known decision rule of pension. If $(1+d) > \dfrac{(1+i)}{(1+g)}$ then the PAYG must be preferred. In this situation the return of the PAYG system is greater than the "funding return". If $(1+d) < \dfrac{(1+i)}{(1+g)}$ then funding must be preferred because it will lead to a greater pension.

In particular we note that diversification between the two techniques is never optimal on a two periods deterministic model. It could only be justified in the very specific situation when both systems have the same return.

2. 3 Three periods model (2,1)

In this model each generation lives for three periods, two periods for working and one for retirement, with $x_r = x_0 + 2$.



We consider the generation who retires at time *t*. They paid the contributions for the 2 previous periods and they receive the pension for one period.

The pension benefits can be computed as follows:

FundingThe contributions paid by active generation at time *(t-1)* and time *(t-2)* are invested for two periods and will generate their own pension at time t. The actuarial equilibrium relationship can be written as:

$$L(x_0,t-2)\pi S(t-2)(1+i)^2+L(x_0+1,t-1)\pi S(t-1)(1+i)=L(x_r,t)P(t) \qquad (5)$$

We have:

$$L(x_0+1,t-1)=L(x_0,t-2)\,p(x_0,1)$$

$$L(x_r,t)=L(x_0,t-2)\,p(x_0,2)=L(x_0,t-2)\,p(x_0,1)\,p(x_0+1,1)$$

$$S(t)=S(t-2)(1+g)^2$$

$$S(t-1)=S(t-2)(1+g)$$

The pension will be:

$$P(t)=\frac{L(x_0,t-2)\pi S(t-2)(1+i)^2+L(x_0,t-2)\,p(x_0,1)\pi S(t-2)(1+g)(1+i)}{L(x_0,t-2)\,p(x_0,1)\,p(x_0+1,1)}$$

$$P(t)=\frac{\pi S(t-2)(1+i)^2+p(x_0,1)\pi S(t-2)(1+g)(1+i)}{p(x_0,1)\,p(x_0+1,1)}$$

The replacement rate will be:

$$RR(t)=\frac{\pi(1+i)^2+p(x_0,1)\pi(1+g)(1+i)}{p(x_0,1)\,p(x_0+1,1)(1+g)^2} \qquad (6)$$

Pay-as-you-go The contributions paid at time t by the active generations are used to pay the pension of the retired generation at time t. The equilibrium relation is:

$$L(x_0,t)\pi S(t)+L(x_0+1,t)\pi S(t)=L(x_r,t)P(t) \qquad (7)$$

The pension will be:

$$P(t)=\frac{L(x_0,t-2)(1+d)^2\,\pi S(t)+L(x_0,t-2)(1+d)\,p(x_0,1)\pi S(t)}{L(x_0,t-2)\,p(x_0,1)\,p(x_0+1,1)}$$

The replacement rate will be:

$$RR(t)=\frac{(1+d)^2\,\pi+(1+d)\,p(x_0,1)\pi}{p(x_0,1)\,p(x_0+1,1)} \qquad (8)$$

Comparison and decision rule:

If $(1+d)^2+(1+d)\,p(x_0,1)>\dfrac{(1+i)^2+p(x_0,1)(1+g)(1+i)}{(1+g)^2}$

$$(1+d)^2+(1+d)\,p(x_0,1)>\frac{(1+i)^2}{(1+g)^2}+p(x_0,1)\frac{(1+i)}{(1+g)}$$

from which we obtain the classical condition:

If $(1+d)>\frac{(1+i)}{(1+g)}$ then the PAYG must be preferred. In this situation the return of the PAYG system is greater than the funding return.

If $(1+d)<\frac{(1+i)}{(1+g)}$ then funding must be preferred because it will lead to a greater pension.

The diversification between PAYG and funding is never optimal on a three period deterministic model. It could only be justified in the very specific situation when both systems have the same return.

2. 4 Four periods model (2,2)

In this model each generation lives for four periods, two periods for working and two for retirement, with $x_r = x_0 + 2$ and $x_\omega = x_r + 2$.

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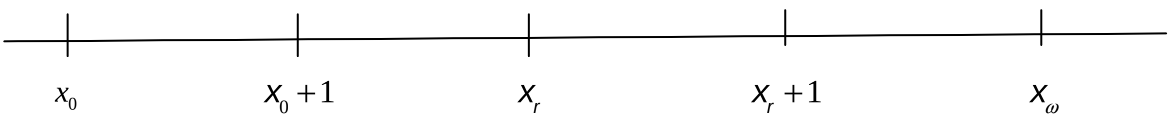

We consider the generation who retires at time *t*. They paid the contributions for two previous periods and they receive the pension for two periods.

The pension benefits can be computed as follows:

Funding The contributions paid by active generation at time *(t-2)* and *(t-1)* are invested and will generate their own pension at time *t* and time *t+1*. The actuarial equilibrium relationship can be written as:

$$L(x_0,t-2)\pi S(t-2)(1+i)^2+L(x_0+1,t)\pi S(t-1)(1+i)=L(x_r,t)P(t)+L(x_{r+1},t+1)\frac{P(t+1)}{(1+i)} \quad (9)$$

$$P(t+1)=P(t)(1+g)$$

$$L(x_0,t-2)\pi S(t-2)(1+i)^2+L(x_0,t-2)\,p(x_0,1)\pi S(t-2)(1+i)(1+g)=$$
$$L(x_0,t-2)\,p(x_0,2)P(t)+L(x_0,t-2)\,p(x_0,3)P(t)\frac{(1+g)}{(1+i)}$$

The pension will be:

$$P(t)=\frac{\pi S(t-2)(1+i)^2+p(x_0,1)\pi S(t-2)(1+i)(1+g)}{p(x_0,2)+p(x_0,3)\frac{(1+g)}{(1+i)}}$$

The replacement rate will be:

$$RR(t)=\frac{P(t)}{S(t)}=\frac{1}{(1+g)^2}\frac{\pi(1+i)^2+p(x_0,1)\pi(1+i)(1+g)}{p(x_0,1)\,p(x_0+1,1)+p(x_0,1)\,p(x_0+1,1)\,p(x_0+2,1)\frac{(1+g)}{(1+i)}}$$

$$RR(t)=\frac{\pi}{p(x_0,1)\,p(x_0+1,1)}\frac{1+i}{(1+g)^2}\frac{(1+i)+p(x_0,1)(1+g)}{1+p(x_0+2,1)\frac{(1+g)}{(1+i)}} \qquad (10)$$

Pay-as-you-go The contributions paid at time t by the active generations are uses to pay the pension of the retired generation at time t. The equilibrium relation is:

$$L(x_0,t)\pi S(t)+L(x_0+1,t)\pi S(t)=L(x_r,t)P(t)+L(x_r+1,t)P(t) \qquad (11)$$

$$L(x_0,t-3)(1+d)^3\pi S(t)+L(x_0,t-3)(1+d)^2\,p(x_0,1)\pi S(t)=$$
$$L(x_0,t-3)(1+d)\,p(x,2)P(t)+L(x_0,t-3)\,p(x,3)P(t)$$

The pension will be:

$$P(t)=\frac{L(x_0,t-3)(1+d)^3\pi S(t)+L(x_0,t-3)(1+d)^2\,p(x_0,1)\pi S(t)}{L(x_0,t-3)(1+d)\,p(x,2)+L(x_0,t-3)\,p(x,3)}$$

$$RR(t)=\pi\frac{(1+d)^3+(1+d)^2\,p(x_0,1)}{(1+d)\,p(x,2)+p(x,3)}$$

$$RR(t) = \pi \frac{(1+d)^2 + \left[(1+d) + p(x_0,1)\right]}{p(x_0,1)\, p(x_0+1,1)\left[(1+d) + p(x_0+2,1)\right]} \tag{12}$$

Comparison and decision rule:

If $\frac{(1+d)^2 + \left[(1+d) + p(x_0,1)\right]}{\left[(1+d) + p(x_0+2,1)\right]} > \frac{1+i}{(1+g)^2} \frac{(1+i) + p(x_0,1)(1+g)}{1 + p(x_0+2,1)\frac{(1+g)}{(1+i)}}$

from which we obtain the classical condition:

If $(1+d) > \frac{(1+i)}{(1+g)}$ then the PAYG must be preferred. In this situation the return of the PAYG system is greater than the funding return.

If $(1+d) < \frac{(1+i)}{(1+g)}$ then funding must be preferred because it will lead to a greater pension.

The diversification between PAYG and funding is never optimal on a three period deterministic model. It could only be justified in the very specific situation when both systems have the same return.

2. 5 Multi periods model

The two, three and four periods results can easily be generalized in a multi-periodic model. In this model people are working between age x_0 and x_{r-1} and retire at age x_r. We consider the generation who retires in t. Let us compare what we get using the one or the other scheme.

Funding In order to determine *P(t)*, we will equalize the present value of the past contributions with the present value of the future benefits. The relationship (1) becomes:

$$\sum_{x=x_0}^{x_{r-1}} L(x, t-x_r+x)\pi S(t-x_r+x)(1+i)^{x_r-x} = P(t) \sum_{x=x_r}^{x_\omega} L(x, t-x_r+x)\left(\frac{1+g}{1+i}\right)^{x_r-x} \tag{13}$$

Doing this we suppose that the pensions are indexed at the rate *g* after the retirement. The relation (2) becomes

$$RR(t)=\frac{P(t)}{S(t)}=\pi\frac{\sum_{x=x_0}^{x_{r-1}} p(x_0,x-x_0)\left(\frac{1+g}{1+i}\right)^{x-x_0}}{\sum_{x=x_r}^{x_\omega} p(x_0,x-x_0)\left(\frac{1+g}{1+i}\right)^{x-x_0}} \tag{14}$$

Pay-as-you-go At time *t*, all the cotization are distributed at the retirees. They all get the same pensions. The equilibrium relationship is given by:

$$\sum_{x=x_0}^{x_{r-1}} L(x,t)\pi S(t)=\sum_{x=x_r}^{x_\omega} L(x,t)P(t) \tag{15}$$

and the relation (4) becomes:

$$RR(t)=\frac{P(t)}{S(t)}=\pi\frac{\sum_{x=x_0}^{x_{r-1}} p(x_0,x-x_0)\left(\frac{1}{1+d}\right)^{x-x_0}}{\sum_{x=x_r}^{x_\omega} p(x_0,x-x_0)\left(\frac{1}{1+d}\right)^{x-x_0}} \tag{16}$$

Comparison and decision rule It is easy to show that the following function is non decreasing:

$$f(\xi)=\frac{\sum_{x=x_0}^{x_{r-1}} p(x_0,x-x_0)\left(\frac{1}{\xi}\right)^{x-x_0}}{\sum_{x=x_r}^{x_\omega} p(x_0,x-x_0)(\xi)^{x-x_0}}$$

So the comparison between the two replacement rates induces the same conclusion as in the two and three period models. If $1+d>\frac{1+i}{1+g}$ PAYG must be preferred. And if $1+d<\frac{1+i}{1+g}$ funding must be preferred.

2. 6 Conclusion

In a deterministic framework the choice between PAYG and funding only depends on the returns of the financial and demographic variables. As those are known or sure the choice is immediate. Diversification is never a good option. It could only be considered if both schemes have identical returns.

But Samuelson’s paradox is only verified in a deterministic model. When the variables become random or, when the processes become stochastic, such conclusions cannot automatically be drawn. This is the object of the next sections.

3. Stochastic model on two periods with one funding asset

In this section we analyze the impact of stochastic assumptions on the previous conclusions.

From now on the stochastic processes i_t, g_t, d_t and the survival probability are defined on a single probability space (Ω, F, P).

Our framework is the overlapping generation set-up with two generations existing at any time, the one working (aged x_0), and the one retiring (aged x_r), with $x_r = x_0 + 1$.

3. 1 Context and maximization problem

We start with a combined system. A fraction of the cotizations of the working generations is invested in a funding system, in order to finance a part of the generation's own future pension. The rest of the cotization is "given" at cohorts attaining retirement age through a PAYG mechanism. We aim at determining the optimal part which should be invested in the funding system in order to maximize the utility of the pension benefit received at retirement time.

Let us consider the cohort aged x_0 at time $t-1$.

At time $(t-1)$ this cohort pays a contribution mass: $L(x_0,t-1)\pi S(t-1)$. Of this, a fraction *a* is invested in the contribution system, the rest, $1-a$, being "invested" in the PAYG system (being paid to the persons retiring). We suppose that this fraction *a* is constant across generations and time. At time t the cohort is aged x_r. As pension amount they receive:

- a "funded part", the money invested in the pension fund and its return after one year:

  $aL(x_0,t-1)\pi S(t-1)(1+i_t)$

- a "non funded" part, fraction $(1-a)$ of the cotization of the cohorts aged x_0 at time *t*:

  $(1-a)L(x_0,t)\pi S(t)$

The mass of pensions for the cohort aged x_r at time *t* is given by:

$$L(x_0,t-1)\pi S(t-1)\left[a(1+i_t)+(1-a)(1+d_t)(1+g_t)\right]$$

At this moment the cohorts counts $L(x_r,t)$ and the mean salary is $S(t)$, which means that the replacement rate of the system for one surviving person at this cohort is:

$$RR(t)=\frac{\pi}{p(x_0,1)}\left[a\frac{1+i_t}{1+g_t}+(1-a)(1+d_t)\right] \tag{17}$$

Let us define: $f_t=\frac{1+i_t}{1+g_t}-1$. It represents the rate of return of the investment deflated by the rate of increase of the mean salary. If the inflation index should only be driven by the evolution of the wages, this process represents the real rate of financial return (the excess/shortage of return over the wage growth). Later we will refer to this as the "financial process" to make the difference with the pure "demographic process", d_t.

As in this section we consider a two generations set up, only the values of the process at time t matter. Therefore, to lighten the formulas, we will not indice the stochastic processes by the time.

And in order to simplify the notations even more we will write: $I=1+i$, $G=1+g$ and $D=1+d$.

We define:

$$Z=a(1+f)+(1-a)(1+d)=a\cdot F+(1-a)\cdot D \tag{18}$$

Z, our "pension portfolio", is the process that we will study for the maximization. Doing so, we consider that the longevity risk $p(x_0,1)$ is independent of f and d. This hypothesis will be relaxed in further research. We aim at determining the optimal value for a.

3. 2 Portfolio maximization.

3.2.1 Framework and efficient border.

We consider a mean variance framework. We therefore define the different efficient solutions as the mean-variance couples which maximize the expectation, given a certain degree of risk, or minimize the risk, for a specific level of expectation.

The maximization problem is: for a certain level of expectation (μ) of our process Z, we seek to minimize its variance.

$$P\equiv\begin{cases}\min\limits_{a}\left(\frac{1}{2}\cdot Var[Z]\right)\\ E[Z]=\mu\end{cases}$$

We can easily determine the mean variance curve's equation, i.e the curve which contains all the optimal risk-return couples.

$$Var[Z]=\left(\frac{1}{E[F-D]}\right)^2\left\{Var[F-D]\cdot\left(E[Z]\right)^2-2H\cdot E[Z]+Var[F\cdot E(D)-D\cdot E(F)]\right\} \tag{19}$$

with $H=E[D]\cdot Var[F]+E[F]-Var[D]-Cov[F,D]\cdot E[Z]-Cov[F,D]\cdot E[D]$

where$E$[.] is the expectation function, Var[.] is the variance function and Cov[.] the covariance function.

The minimum of Var[Z] is attained when $E[Z]^*=\frac{H}{Var[F-D]}$, i.e. when $a^*=\frac{V[D]-Cov[F,D]}{V[F-D]}$. At this point we find: $Var[Z]^*=Var[F]\cdot Var[D]-\left(Cov[F,D]\right)^2$.

The efficient part of the curve is the concave part of this parabola, i.e. the one where, to get a higher return, one need to accept a higher risk. This concave part of the curve represents all the solutions

that the State could choose, at one moment, knowing the financial and the demographical conditions. Every portfolio is not optimal.
Let us only take into consideration the most likely case in western countries: the financial returns are higher than the demographic ones and they are more risky. This first means that no title is better than the other one (the opportunity of arbitrage are not considered). Second, it means that the financial "products" are for the moment more attractive from a return point of view, but this implies a higher risk.
In this case, the values of $a < a^*$ are not optimal (because it is possible, by increasing *a*, to increase the expectation and, at the same time, decrease the variance of *Z*.

3.2.2 Choice of an utility function and optimal funding.

Once the efficient border has been defined, the choice of a specific combination on this curve, a^{SOL}, depends on the individual preferences (i.e. on the State preference for the population). It can be determined by the choice of an *utility function*. The following quadratic utility function will be used: $U\left[Z\right] = E\left[Z\right] - \frac{\varepsilon}{2} Var\left[Z\right]$, where ε > 0 is the risk aversion coefficient chosen by the State. It should correspond to the aggregate risk aversion of the consumers who bear the risk. We find:

$$\frac{\delta U\left[Z\right]}{\delta a} = 0 \quad \Leftrightarrow a^{SOL} = \frac{Var\left[D\right] - Cov\left[F, D\right]}{Var\left[F - D\right]} + \frac{1}{\varepsilon}\frac{E\left[F - D\right]}{Var\left[F - D\right]} - = a^* + \frac{1}{\varepsilon} \cdot \Delta \tag{20}$$

Note that if *F* and *D* have similar expectations, the optimal portfolio is the one that minimize the variance ($a^{SOL} = a^*$).

3.3 Analysis of the results

3.3.1 What is the impact of the hypothesis of a "link" between demography and finance?

Most papers in relation with this topic suppose that the financial and the demographic processes are independent. Let us relax this hypothesis and analyse how the level of correlation between the financial and demographic risk does not impact the fraction *a*.
In case of independence, the solution is given by:

$$a_{SOL}^{IND} = \left(\frac{Var(D)}{VAR[F] + Var[D]}\right) + \frac{1}{\gamma} \cdot \frac{E[F] - E[D]}{VAR[F] + Var[D]}$$

Does the fraction *a* increase or decrease when we introduce a possible link between risks?
Let us again only take into consideration the case where *F* is more risky and therefore offers a higher return. We observe that, compared to the independence situation, the first part a^*, decreases

if the link the risks is positive ($Cov[F,D]>0$). The second part, Δ, increases when the covariance is positive. So we cannot easily predict the variation of a^{SOL} when introducing a covariance.

But we can find a kind of "rule" referring to the level of risk aversion. Let us rewrite a^{SOL} in this way:

$$a^{SOL}=\frac{1}{2}+\frac{\frac{1}{\gamma}\cdot\left(E[F]-E[D]\right)-\frac{1}{2}\left(Var[F]-Var[D]\right)}{Var[F]+Var[D]-2Cov[F,D]}$$

And let us note $\gamma^{*}=2\frac{E[F]-E[D]}{Var[F]-Var[D]}$. We find that a^{SOL} will be higher than a_{IND}^{SOL} in two cases:

- the covariance is positive and the state's risk aversion coefficients higher than γ^{*};
- the covariance is negative and the state's risk aversion coefficient is higher than γ^{*}.

So, there is an impact when we take into account a possible correlation between the risks. But the impact is not always positive or always negative. It depends on two things: the sign of the correlation and the level of the risk aversion.

3.3.2 So, is there diversification between PAYG and funding?

Is a^{SOL} equal to 0 or 1 or does it lay somewhere between? And what if it lays out of these bounds? Let us note first that our framework is not the usual portfolio theory framework. One cannot borrow from the financial market to invest more in PAYG or the other way round; this would make non-sense. State it another way: *a* cannot be less than 0 or more than 1. Therefore the optimal solution will be:

$$a^{OPT}=\left[\min\left(a^{SOL},1\right)\right]^{+}$$

Can we predict when the a^{SOL} will be out of bounds (i.e. the cases where we will have to constraint (*a*)? Remember that $a^{SOL}=a^{*}+\frac{1}{\gamma}\cdot\Delta$. The first term $a^{*}=\frac{Var[D]-Cov[F,D]}{Var[F-D]}$, is linked to the correlation between the two risks. In the case of independence between the financial and the demographic risk, it is guaranteed that a^{*} belongs to [0,1]. If the financial and demographic risks are correlated negatively we can also easily verify that $0\le a^{*}\le 1$. But when the correlation between the processes is positive, we cannot predict anything: a^{*} could get below 0 or over 1.

The second term, $\frac{1}{\gamma}\cdot\Delta=\frac{1}{\gamma}\cdot\frac{E[F-D]}{Var[F-D]}$, will be zero if the two risks have similar expectations. It will be positive in case the financial process has got a better return and better risk (it will be optimal to invest more in the financial asset). This term is first linked to the coefficient of risk aversion (γ): if the risk-aversion is high, one will choose a coefficient *a* which is very closed to the less risky solution (a^*). It is also linked to Δ which measures the excess return per square unit of risk.

The following table summarizes all the possible situations:

Case	E[F]>E[D] ↔ Δ>0	E[F]>E[D] ↔ Δ<0
a^*<0	[1] a^{OPT} ∈[0,1]	[4] a^{OPT}=0
a^*>1	[2] a^{OPT}=1	[5] a^{OPT}=[0,1]
a^*∈ [0,1]	[3] a^{OPT} ∈[a*,1]	[6] a^{OPT}∈[0,a^*]

We observe six different situations. If we only consider the cases where the financial process is a higher return/higher risk process, we only look at the first column.

In cells [2], diversification is not worth it and only funding is the best solution. This case is actually an "arbitrage case" because in this situation the process F has a higher return than D but it is less risky. This is why the optimal solution would be to invest everything in the funding scheme. But we consider that this is not possible.

In cases [1], every strategy can be a possible response. There, a^{OPT}=[min(a^{SOL},1)]$^+$.

The last case [3] lays somewhere between those two extremes, one risk being preferred, but not at the exclusion of the other. In this case a^* does for sure belong to [0,1]. But as Δ is positive, depending on their risk aversion.

3.4 Numerical examples

In this section we present a small application of this theory. As we only consider a two period OLG model, those results are to be considered with prudence. The source of data is the OECD website. Here are the chosen variables: *d* is the rate of increasing of the births, *i* is the short term interest rate and *g* is the rate of increase of the wages.

	a^*	Δ	a^{SOL}	a^{SOL}	a^{SOL}	a^{SOL}	a^{SOL}

			γ=0,1	γ=1	γ=10	γ=20	γ=30
Belgium	0.12	8.36	1	1	0.96	0.54	0.4
France	0.35	4.41	1	1	0.79	0.57	0.49
Germany	0.80	7.19	1	1	1	1	1
United Kingdom	0.58	6.81	1	1	1	0.92	0.81
United States	0.90	5.19	1	1	1	1	1

We observe that a^* is between 0 and 1 for every country. This "lower variance solution" is close to 1 for Germany and United States. Those countries should invest a higher part of the cotisation in funding. The Δ part is positive, because the expectation of the funding process was higher than the demographic one for every country. Its value is high which implies that the part of the cotisation to be invested in funding is very high (almost 100%,except when the risk aversion is high).

Note that we obtained a positive correlation between the two risks (finance and demography) for every country (from 0.11 to 0.28), except for Canada, where the correlation was slightly negative.

Example in a binomial model

	sc1	sc2	P(sc1)	P(sc2)	mean
d	0	0,02	0,5	0,5	0,01
s	0,02	0,03	0,5	0,5	0,025
i	0,04	0,06	0,5	0,5	0,05

a	E[X]	VAR[X]
0	1,035	0,000125
0,1	1,037	0,000102
0,2	1,038	0,000084
0,3	1,040	0,000070
0,4	1,041	0,000061
0,5	1,043	0,000056
0,6	1,044	0,000056
0,7	1,046	0,000060
0,8	1,047	0,000069
0,9	1,049	0,000082
1	1,050	0,000100

3. 5 Modelling in a log-normal environment

So far, we have not specified any stochastic process for the growth of the population, the growth of the wages and the interest rate. In this section, we provide and analyze the results if the processes follow log-normal processes:

$$dX = \mu_x X\,dt + \sigma_x X\,dW_x(t)$$

with $x = d,g,i$ and $X = D,G,I$.

That king of geometric Brownian motion is often used for financial processes. For the demographic process it has the interesting property that it cannot fall down zero.

We find:

- $dF = \mu_f F\, dt + \sigma_f F\, dW_f(t)$
- $D = \frac{D(t)}{D(t-1)} = e^{\left[\mu_d - \frac{1}{2}\sigma_d^2 + \sigma_d\left(W_d(t) - W_d(t-1)\right)\right]}$
- $F = \frac{F(t)}{F(t-1)} = e^{\left[\mu_f - \frac{1}{2}\sigma_f^2 + \sigma_f\left(W_f(t) - W_f(t-1)\right)\right]}$

$E\left[F\right] = e^{\mu_f}$ $\qquad VAR\left[F\right] = e^{2\mu_f} \cdot \left(e^{\sigma_f^2} - 1\right)$

$E\left[D\right] = e^{\mu_d}$ $\qquad VAR\left[D\right] = e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right)$

$Cov\left[F,D\right] = e^{\mu_f + \mu_d} \cdot \left(e^{\sigma_{fd}} - 1\right)$

Let us stay general as possible and consider that the two hazard processes (W_d, W_f) are correlated. Let us use the following notation:

$\Sigma = \left(\sigma_{ij}\right)_{2x2} = \begin{pmatrix} \sigma_f^2 & \sigma_{fd} \\ \sigma_{fd} & \sigma_g^2 \end{pmatrix}$ where $\sigma_x^2 = Var\left[W_x\right]$ and $\sigma_{xy} = Cov\left[W_x, W_y\right]$.

And let us define $A = \left(A_{ij}\right)_{2x2} = \begin{pmatrix} A_{f1} & 0 \\ A_{d1} & A_{d2} \end{pmatrix}$ such as $A \cdot A^t = \Sigma$.

Once we calculate $E[F]$, $Var[F]$, $E[D]$, $Var[D]$ and $Cov[F,D]$, we can inject their value in the above results. For example $a^{SOL} = \frac{V\left[D\right] - Cov\left[F,D\right]}{Var\left[F-D\right]} + \frac{1}{\varepsilon}\frac{E\left[F-D\right]}{Var\left[F-D\right]} - = a^* + \frac{1}{\varepsilon} \cdot \Delta$ with:

$$a^* = \frac{e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right) - e^{\mu_f + \mu_d} \cdot \left(e^{\sigma_{fd}} - 1\right)}{e^{2\mu_f} \cdot \left(e^{\sigma_f^2} - 1\right) + e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right) - 2e^{\mu_f + \mu_d} \cdot \left(e^{\sigma_{fd}} - 1\right)} \qquad (21)$$

$$\Delta = \frac{e^{\mu_f} - e^{\mu_d}}{e^{2\mu_f} \cdot \left(e^{\sigma_f^2} - 1\right) + e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right) - 2e^{\mu_f + \mu_d} \cdot \left(e^{\sigma_{fd}} - 1\right)} \qquad (22)$$

3. 6 Modelling in a mean reverting framework

In this section we analyze the case the growth of the population, the growth of the wages and the interest rate follow a mean reverting process:

$$F(t) = F_0 e^{\int_0^t r(s)ds}$$

with
$$r(t) = \gamma + Y(t)$$

where $Y(t)$ is an Ornstein-Ulembeck process described by the following differential equation:

$$dY(s) = -\beta Y(s)ds + \rho dW(s)$$

with $Y(0) = 0$ and $\beta, \sigma > 0$.
The solution is:

$$Y(t) = \rho \int_0^t e^{-\beta(t-s)} dW(s)$$

$$F(t) = F_0 \cdot e^{\int_0^t \left(\gamma + \rho \int_0^s e^{-\beta(s-u)} dW(u)\right) ds}$$

$$F(t) = F_0 \cdot e^{\gamma t} \cdot e^{\rho \int_0^t \left(\int_0^s e^{-\beta(s-u)} dW(u)\right) ds}$$

$$F(t) = F_0 \cdot e^{\gamma t} \cdot e^{\rho \int_0^t \left(\frac{1-e^{-\beta(t-u)}}{\beta}\right) dW(u)}$$

$$F_t = \frac{F(t)}{F(t-1)} = \frac{F_0 \cdot e^{\gamma_f \cdot t + \rho_f \int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right) dW_f(u)}}{F_0 \cdot e^{\gamma_f \cdot (t-1) + \rho_f \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right) dW_f(u)}} = e^{\gamma_f \cdot t + \rho_f \int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right) dW_f(u) - \gamma_f \cdot (t-1) - \rho_f \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right) dW_f(u)}$$

$$F = e^{\gamma_f + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right) dW_f(u)\right]}$$

$$D = \frac{D(t)}{D(t-1)} = \frac{e^{\gamma_d \cdot t + \rho_d \int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right) dW_d(u)}}{e^{\gamma_d \cdot (t-1) + \rho_d \int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right) dW_d(u)}} = e^{\gamma_d \cdot t + \rho_d \int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right) dW_d(u) - \gamma_d \cdot (t-1) - \rho_d \int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right) dW_d(u)}$$

$$D = e^{\gamma_d + \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right) dW_d(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right) dW_d(u)\right]}$$

$$F = e^{A} \text{with } A = \gamma_f + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f} \right) dW_f(u) \right]$$

$$E[A] = a_f \qquad Var[A] = b_f^2$$

$$a_f = \gamma_f$$

$$Var[A] = E\left\{ \left[\rho_f \left(\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f} \right) dW_f(u) \right) \right]^2 \right\} =$$

$$b_f^2 = \frac{\rho_f^2}{2\beta_f^3} \left[2\beta_f - 2 - e^{-2\beta_f \cdot t} - e^{-2\beta_f(t-1)} + 2e^{-\beta_f} + 2e^{-\beta_f(2t-1)} \right]$$

$$E[F] = e^{a_f + \frac{b_f^2}{2}}$$

$$Var[F] = e^{2a_f + b_f^2}\left(e^{b_f^2} - 1 \right)$$

$$D = e^{A_d}$$

$$E[A_d] = a_d \qquad Var[A_d] = b_d^2$$

$$a_d = \gamma_d$$

$$b_d^2 = \frac{\rho_d^2}{2\beta_d^3} \left[2\beta_d - 2 - e^{-2\beta_d(t-1)} + 2e^{-\beta_d} + 2e^{-\beta_d(2t-1)} \right]$$

$$E[D] = e^{a_d + \frac{b_d^2}{2}}$$

$$Var[D] = e^{2a_d + b_d^2}\left(e^{b_d^2} - 1 \right)$$

$$Cov[F, D] = e^{a_f + a_d + \frac{1}{2}\left(b_f^2 + b_d^2\right)}\left(e^{b_{f,d}} - 1 \right)$$

$$b_{f,d} = E\left[\left(\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f} \right) dW_f(u) \right) \cdot \int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) \right]$$

$$b_{f,d} = \frac{\rho_{f,d}}{\beta_f \beta_d} \left[\begin{array}{l} 1 + \frac{2}{\beta_f + \beta_d} - \frac{e^{-(\beta_f + \beta_d) \cdot t}}{\beta_f + \beta_d} + \frac{e^{-\beta_f}}{\beta_f} - \frac{e^{-\beta_f}}{\beta_f + \beta_d} + \frac{e^{-\beta_f \cdot t - \beta_d(t-1)}}{\beta_f + \beta_d} + \frac{e^{-\beta_d}}{\beta_d} - \frac{e^{-\beta_d}}{\beta_f + \beta_d} + \frac{e^{-\beta_f \cdot (t-1) - \beta_d \cdot t}}{\beta_f + \beta_d} + \\ - \frac{1}{\beta_f} - \frac{1}{\beta_d} - \frac{e^{-(\beta_f + \beta_d) \cdot (t-1)}}{\beta_f + \beta_d} \end{array} \right]$$

where $\rho_{f,d}$ is the correlation between W_f and W_d.[4]

Once we calculate $E[F]$, $Var[F]$, $E[D]$, $Var[D]$ and $Cov[F,D]$, we can inject their value in the above results. For example $a^{SOL} = \frac{V[D]-Cov[F,D]}{Var[F-D]} + \frac{1}{\varepsilon}\frac{E[F-D]}{Var[F-D]} - = a^* + \frac{1}{\varepsilon}\cdot\Delta$ with:

$$a^* = \frac{e^{2a_d+b_d^2}\left(e^{b_d^2}-1\right)-e^{a_f+a_d+\frac{1}{2}\left(b_f^2+b_d^2\right)}\left(e^{b_{f,d}}-1\right)}{e^{2a_f+b_f^2}\left(e^{b_f^2}-1\right)+e^{2a_d+b_d^2}\left(e^{b_d^2}-1\right)-2e^{a_f+a_d+\frac{1}{2}\left(b_f^2+b_d^2\right)}\left(e^{b_{f,d}}-1\right)} \quad (23)$$

$$\Delta = \frac{e^{a_f+\frac{b_f^2}{2}}-e^{a_d+\frac{b_d^2}{2}}}{e^{2a_f+b_f^2}\left(e^{b_f^2}-1\right)+e^{2a_d+b_d^2}\left(e^{b_d^2}-1\right)-2e^{a_f+a_d+\frac{1}{2}\left(b_f^2+b_d^2\right)}\left(e^{b_{f,d}}-1\right)} \quad (24)$$

4. Extension to multi period model

4.1 Stochastic model on three periods and one funding asset (2,1)

In this section we analyze the impact of stochastic assumptions on the previous section.

The stochastic processes i_t, g_t, d_t and the survival probability are defined on a single probability space (Ω, F, P).

Our framework is the overlapping generation set-up with three generations existing at any time, two generations working: young workers (aged x_0) and old workers (aged x_0+1), with $x_r = x_0 + 2$.

4.1.1 Context and maximization problem

We start with a combined system. A fraction of the cotizations of the working generations is invested in a funding system, in order to finance a part of the generation's own future pension. The rest of the cotization is given at cohorts attaining retirement age through a PAYG mechanism. We aim at determining the optimal part which should be invested in the funding system in order to maximize the utility of the pension benefit received at retirement time.

a : fraction invested in funding

$(1-a)$: fraction invested in PAYG

[4]Seethe appendix A for the computations of the expected values, variances and covariances.

Let us consider the cohort aged x_0 at time $t-2$.

At time $(t-2)$ this cohort pays a contribution mass: $L(x_0,t-2)\pi S(t-2)$.

At time $(t-1)$ this cohort pays a contribution mass: $L(x_0+1,t-1)\pi S(t-1)$.

At time t the cohort is aged x_r. As pension amount they receive:

- a funded part, the money invested in the pension fund and its return:

$$a\left[L(x_0,t-2)\pi S(t-2)(1+i_{t-1})(1+i_t)+L(x_0+1,t-1)\pi S(t-1)(1+i_t)\right]=$$
$$=aL(x_0,t-2)\pi S(t-2)\left[(1+i_{t-1})(1+i_t)+p(x_0,1,t-2)(1+g_{t-1})(1+i_t)\right]=$$
$$=aL(x_0,t-2)\pi S(t-2)(1+i_t)\left[(1+i_{t-1})+p(x_0,1,t-2)(1+g_{t-1})\right]$$

- a non funded part, fraction $(1-a)$ of the cotization of the cohorts aged x_0 and x_0+1 at time *t*:

$$(1-a)\left[L(x_0,t)\pi S(t)+L(x_0+1,t)\pi S(t)\right]=$$
$$=(1-a)L(x_0,t-2)\pi S(t-2)(1+g_{t-1})(1+g_t)(1+d_{t-1})\left[(1+d_t)+p(x_0,1,t-1)\right]$$

The mass of pensions for the cohort aged x_r at time *t* is given by:

$$L(x_0,t-2)\pi S(t-2)\begin{Bmatrix}a(1+i_t)\left[(1+i_{t-1})+p(x_0,1,t-2)(1+g_{t-1})\right]+\\(1-a)(1+g_{t-1})(1+g_t)(1+d_{t-1})\left[(1+d_t)\,p(x_0,1,t-1)\right]\end{Bmatrix}$$

At this moment the cohorts counts $L(x_r,t)$ and the mean salary is $S(t)$, which means that the replacement rate of the system for one surviving person at this cohort is:

$$RR(t)=\frac{\pi}{p(x_0,2,t-2)}\left[a\frac{1+i_t}{1+g_t}\left(\frac{1+i_{t-1}}{1+g_{t-1}}+p(x_0,1,t-2)\right)+(1-a)(1+d_{t-1})\big((1+d_t)+p(x_0,1,t-1)\big)\right],$$

where $p(x_0,2,t-2)=p(x_0,1,t-2)\cdot p(x_0+1,1,t-1)$

$p(x_0,1,t-1)=p(x_0,1,t-2)(1+h_{t-1})$, where h represents the increment in the survival probability (longevity).

$$RR(t)=\frac{\pi}{p(x_0+1,1,t-1)}\left[a\frac{1+i_t}{1+g_t}\left(\frac{1+i_{t-1}}{1+g_{t-1}}\frac{1}{p(x_0,1,t-2)}+1\right)+(1-a)(1+d_{t-1})\left(\frac{(1+d_t)}{p(x_0,1,t-2)}+(1+h_{t-1})\right)\right]$$

(25)

Let us define: $f_t=\dfrac{1+i_t}{1+g_t}-1$. It represents the rate of return of the investment deflated by the rate of increase of the mean salary. If the inflation index should only be driven by the evolution of the wages, this process represents the real rate of financial return. Later we will refer to this as the "financial process" to make the difference with the pure "demographic process", d_t.

$$RR(t)=\frac{\pi}{p(x_0+1,1,t-1)}\left[a(1+f_t)\left(\frac{1+f_{t-1}}{p(x_0,1,t-2)}+1\right)+(1-a)(1+d_{t-1})\left(\frac{(1+d_t)}{p(x_0,1,t-2)}+(1+h_{t-1})\right)\right]$$

In order to simplify the notation we write:

$F_t=(1+f_t)$, $D_t=(1+d_t)$, $I_t=(1+i_t)$ and $G_t=(1+g_t)$ the replacement ratio becomes:

$$RR(t)=\frac{\pi}{p(x_0+1,1,t-1)}\left[aF_t\left(\frac{F_{t-1}}{p(x_0,1,t-2)}+1\right)+(1-a)D_{t-1}\left(\frac{D_t}{p(x_0,1,t-2)}+(1+h_{t-1})\right)\right] \quad (26)$$

We define the pension portfolio:

$$Z=aF_t\left(\frac{F_{t-1}}{p(x_0,1,t-2)}+1\right)+(1-a)D_{t-1}\left(\frac{D_t}{p(x_0,1,t-2)}+(1+h_{t-1})\right) \quad (27)$$

If we define: $X=F_t\left(\frac{F_{t-1}}{p(x_0,1,t-2)}+1\right)$ and $Y=D_{t-1}\left(\frac{D_t}{p(x_0,1,t-2)}+(1+h_{t-1})\right)$ our pension portfolio is:

$Z=aX+(1-a)Y$

4.1.2 Portfolio maximization

Framework and efficient border.

We consider a mean variance framework. We therefore define the different efficient solutions as the mean-variance couples which maximize the expectation, given a certain degree of risk, or minimize the risk, for a specific level of expectation.

The maximization problem is: for a certain level of expectation μ of our process *Z*, we seek to minimize its variance.

$$P\equiv\begin{cases}\min_a\left(\frac{1}{2}\cdot Var[Z]\right)\\ E[Z]=\mu\end{cases}$$

We can easily determine the mean variance curve's equation.

Indicating by E[X] the expectation, Var[X] the variance and Cov[X,Y] the covariance between *X* and *Y*, we can write:

$$Var[Z]=\left(\frac{1}{E[X-Y]}\right)^2\left\{Var[X-Y]\cdot(E[Z])^2-2HE[Z]+Var[X\cdot E(Y)-Y\cdot E(X)]\right\} \quad (28)$$

with $H=E[Y]\cdot Var[X]+E[X]-Var[Y]-Cov[X,Y]\cdot E[X]-Cov[X,Y]\cdot E[Y]$

The minimum of V[Z] is attained when $E[Z]^*=\frac{H}{Var[X-Y]}$, when $a^*=\frac{V[Y]-Cov[X,Y]}{V[X-Y]}$.

$Var[Z]^*=Var[X]\cdot Var[Y]-(Cov[X,Y])^2$.

The efficient part of the curve is the concave part of this parabola and it represents all the solutions that the State could choose, at one moment, knowing the financial and the demographical conditions.

Choice of an utility function and optimal funding.

Once the efficient border has been defined, the choice of a specific combination on this curve, a^{SOL}, depends on the individual preferences (i.e. on the State preference for the population). It can be determined by the choice of an utility function. The following quadratic utility function will be used: $U[Z]=E[Z]-\frac{\varepsilon}{2}V[Z]$, where ε > 0 is the risk aversion coefficient chosen by the State. It should correspond to the aggregate risk aversion of the consumers who bear the risk. We find:

$$\frac{\delta U[Z]}{\delta a}=0 \quad \Leftrightarrow a^{SOL}=\frac{Var[Y]-Cov[X,Y]}{Var[X-Y]}+\frac{1}{\varepsilon}\frac{E[X-Y]}{Var[X-Y]}=a^*+\frac{1}{\varepsilon}\cdot\Delta$$

4.1.3 Analysis of the results

In case of independence, the solution is given by:

$$a_{SOL}^{IND}=\left(\frac{Var(Y)}{VAR[X]+Var[Y]}\right)+\frac{1}{\gamma}\cdot\frac{E[X]-E[Y]}{VAR[X]+Var[Y]}$$

Does the fraction *a* increase or decrease when we introduce a possible link between risks?

Let us again only take into consideration the case where *F* is more risky and therefore offers a higher return. We observe that, compared to the independence situation, the first part a^*, decreases if the link the risks is positive ($Cov[X,Y]>0$). The second part, Δ, increases when the covariance is positive. So we cannot easily predict the variation of a^{SOL} when introducing a covariance.

But we can find a kind of "rule" referring to the level of risk aversion. Let us rewrite a^{SOL} in this way:

$$a^{SOL}=\frac{1}{2}+\frac{\frac{1}{\gamma}\cdot(E[X]-E[Y])-\frac{1}{2}(Var[X]-Var[Y])}{Var[X]+Var[Y]-2Cov[X,Y]}$$

And let us note $\gamma^* = 2\dfrac{E[X]-E[Y]}{Var[X]-Var[Y]}$. We find that a^{SOL} will be higher than a_{IND}^{SOL} in two cases:

- the covariance is positive and the state's risk aversion coefficients higher than γ^*;
- the covariance is negative and the state's risk aversion coefficient is higher than γ^*.

So, there is an impact when we take into account a possible correlation between the risks. But the impact is not always positive or always negative. It depends on two things: the sign of the correlation and the level of the risk aversion.

So, is there diversification between PAYG and funding?

Is a^{SOL} equal to 0 or 1 or does it lay somewhere between? And what if it lays out of these bounds? Let us note first that our framework is not the usual portfolio theory framework. One cannot borrow from the financial market to invest more in PAYG or the other way round; this would make non-sense. State it another way: *a* cannot be less than 0 or more than 1. Therefore the optimal solution will be:

$$a^{OPT} = \left[\min\left(a^{SOL},1\right)\right]^+$$

Can we predict when the a^{SOL} will be out of bounds (i.e. the cases where we will have to constraint (*a*)? Remember that $a^{SOL} = a^* + \frac{1}{\gamma}\cdot\Delta$. The first term $a^* = \dfrac{Var[Y]-Cov[X,Y]}{Var[X-Y]}$, is linked to the correlation between the two risks. In the case of independence between the financial and the demographic risk, it is guaranteed that a^* belongs to [0,1]. If the financial and demographic risks are correlated negatively we can also easily verify that $0 \le a^* \le 1$. But when the correlation between the processes is positive, we cannot predict anything: a^* could get below 0 or over 1.

The second term, $\frac{1}{\gamma}\cdot\Delta = \frac{1}{\gamma}\cdot\dfrac{E[X-Y]}{Var[X-Y]}$, will be zero if the two risks have similar expectations. It will be positive in case the financial process has got a better return and better risk (it will be optimal to invest more in the financial asset). This term is first linked to the coefficient of risk aversion (γ): if the risk-aversion is high, one will choose a coefficient *a* which is very closed to the less risky solution (a^*). It is also linked to Δ which measures the excess return per square unit of risk.

The following table summarizes all the possible situations:

Case	E[X]>E[Y] ↔ Δ>0	E[X]>E[Y] ↔ Δ<0

$a^*<0$	[1] $a^{\text{OPT}} \in [0,1]$	[4] $a^{\text{OPT}}=0$
$a^*>1$	[2] $a^{OPT}=1$	[5] $a^{\text{OPT}}=[0,1]$
$a^* \in [0,1]$	[3] $a^{OPT} \in [a*,1]$	[6] $a^{\text{OPT}} \in [0,a^*]$

We observe six different situations. If we only consider the cases where the financial process is a higher return/higher risk process, we only look at the first column.

In cells [2], diversification is not worth it and only funding is the best solution. This case is actually an "arbitrage case" because in this situation the process F has a higher return than D but it is less risky. This is why the optimal solution would be to invest everything in the funding scheme. But we consider that this is not possible.

In cases [1], every strategy can be a possible response. There, $a^{OPT}=[\min(a^{\text{SOL}},1)]^+$.

The last case [3] lays somewhere between those two extremes, one risk being preferred, but not at the exclusion of the other. In this case a^* does for sure belong to [0,1]. But as Δ is positive, depending on their risk aversion.

4.1.4 Numerical examples

4.2 Modelling in a lognormal framework.

So far, we have not specified any stochastic process for the financial and demographic processes.

In this section we assume that the stochastic variables F_t and D_t follow a lognormal distribution, while the survival probability and its increment age are deterministic.

We have:

$$F_{t-1} = \frac{F(t-1)}{F(t-2)} = e^{\left(\mu_f - \frac{1}{2}\sigma_f^2\right) + \sigma_f\left[W_f(t-1) - W_f(t-2)\right]} \qquad D_{t-1} = \frac{D(t-1)}{D(t-2)} = e^{\left(\mu_d - \frac{1}{2}\sigma_d^2\right) + \sigma_d\left[W_d(t-1) - W_d(t-2)\right]}$$

$$F_t = \frac{F(t)}{F(t-1)} = e^{\left(\mu_f - \frac{1}{2}\sigma_f^2\right) + \sigma_f\left[W_f(t) - W_f(t-1)\right]} \qquad D_t = \frac{D(t)}{D(t-1)} = e^{\left(\mu_d - \frac{1}{2}\sigma_d^2\right) + \sigma_d\left[W_d(t) - W_d(t-1)\right]}$$

Let us stay as general as possible and consider that the two hazard processes ($W_f(t), W_d(t)$) are correlated.

Once we calculated E[X], VAR[X], E[Y], VAR[Y] and COV[X,Y] we can inject their value in the results obtained in the previous section.

F_t and F_{t-1} are i.i.d. random variables.

$$E[F_t \cdot F_{t-1}] = E[F_t] \cdot E[F_{t-1}]$$

$$E[F_t] = e^{\mu_f} \qquad VAR[F_t] = e^{2\mu_f} \cdot \left(e^{\sigma_f^2} - 1\right)$$

$$E[F_{t-1}] = e^{\mu_f} \qquad VAR[F_{t-1}] = e^{2\mu_f} \cdot \left(e^{\sigma_f^2} - 1\right)$$

We find[5]:

$$E[X] = e^{\mu_f} \cdot \left(\frac{e^{\mu_f}}{p(x_0,1,t-2)} + 1 \right) = \frac{e^{2\mu_f}}{p(x_0,1,t-2)} + e^{\mu_f} \tag{29}$$

$$VAR[X] = \frac{e^{4\mu_f}}{p^2(x_0,1,t-2)} \left(e^{2\sigma_f^2} - 1\right) + 2e^{\mu_f}\left(e^{\mu_f} - 1\right) + \frac{e^{3\mu_f}}{p(x_0,1,t-2)} \cdot \left(2e^{\sigma_f^2} - 1\right) \tag{30}$$

$$Y = \frac{D_t \cdot D_{t-1}}{p(x_0,1,t-2)} + D_{t-1} \cdot (1 + h_{t-1})$$

$$E[D_t \cdot D_{t-1}] = E[D_t] \cdot E[D_{t-1}]$$

$$E[D_t] = e^{\mu_d} \qquad VAR[D_t] = e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right)$$

$$E[D_{t-1}] = e^{\mu_d} \qquad VAR[D_{t-1}] = e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right)$$

[5]Seethe appendix B for the computations of the expected values, variances and covariances.

$$E\left[Y\right]=e^{\mu_d}\cdot\left(\frac{e^{\mu d}}{p\left(x_0,1,t-2\right)}+\left(1+h_{t-1}\right)\right)=\frac{e^{2\mu_d}}{p\left(x_0,1,t-2\right)}+e^{\mu_d}\cdot\left(1+h_{t-1}\right) \qquad (31)$$

$$Var[Y]=\frac{1}{p^2\left(x_0,1,t-2\right)}\left(e^{4\mu_d+2\sigma_d^2}-e^{4\mu_d}\right)+\left(1+h_{t-1}\right)^2\left(e^{2\mu_d+\sigma_d^2}-e^{2\mu_d}\right)+\frac{2\left(1+h_{t-1}\right)}{p\left(x_0,1,t-2\right)}\left(e^{3\mu_d+\sigma_d^2}-2e^{3\mu_d}\right) \qquad (32)$$

If financial and demographic processes F_t and D_t are independent the covariance between X and Y will be zero.

Otherwise the covariance will be:

$$Cov\left(X,Y\right)=E\left[\left(X-E\left[X\right]\right)\left(Y-E\left[Y\right]\right)\right]$$

$$\begin{aligned}Cov\left[X,Y\right]=&\frac{e^{2\mu_f+2\mu_d}}{p^2\left(x_0,1,t-2\right)}\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_f+2\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(e^{\rho_{f,d}}-1\right)+\\&+\frac{e^{2\mu_f+\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_f+\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)\end{aligned} \qquad (33)$$

$a^{SOL}=\dfrac{V\left[Y\right]-Cov\left[X,Y\right]}{Var\left[X-Y\right]}+\dfrac{1}{\varepsilon}\dfrac{E\left[X-Y\right]}{Var\left[X-Y\right]}=a^*+\dfrac{1}{\varepsilon}\cdot\Delta$ with

$$a^{*}=\left[\begin{array}{l}\frac{1}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{4\mu_{d}+2\sigma_{d}^{2}}-e^{4\mu_{d}}\right)+\left(1+h_{t-1}\right)^{2}\left(e^{2\mu_{d}+\sigma_{d}^{2}}-e^{2\mu_{d}}\right)+\frac{2\left(1+h_{t-1}\right)}{p\left(x_{0},1,t-2\right)}\left(e^{3\mu_{d}+\sigma_{d}^{2}}-2e^{3\mu_{d}}\right)-\\ \left(\frac{e^{2\mu_{f}+2\mu_{d}}}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_{f}+2\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(e^{\rho_{f,d}}-1\right)+\frac{e^{2\mu_{f}+\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_{f}+\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)\right)\end{array}\right]\cdot$$

$$\left[\frac{1}{\begin{array}{l}\frac{e^{4\mu_{f}}}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{2\sigma_{f}^{2}}-1\right)+2e^{\mu_{f}}\left(e^{\mu_{f}}-1\right)+\frac{e^{3\mu_{f}}}{p\left(x_{0},1,t-2\right)}\cdot\left(2e^{\sigma_{f}^{2}}-1\right)+\\ \frac{1}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{4\mu_{d}+2\sigma_{d}^{2}}-e^{4\mu_{d}}\right)+\left(1+h_{t-1}\right)^{2}\left(e^{2\mu_{d}+\sigma_{d}^{2}}-e^{2\mu_{d}}\right)+\frac{2\left(1+h_{t-1}\right)}{p\left(x_{0},1,t-2\right)}\left(e^{3\mu_{d}+\sigma_{d}^{2}}-2e^{3\mu_{d}}\right)+\\ -\left(\begin{array}{l}\frac{e^{2\mu_{f}+2\mu_{d}}}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_{f}+2\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(e^{\rho_{f,d}}-1\right)+\\ \quad+\frac{e^{2\mu_{f}+\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_{f}+\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)\end{array}\right)\end{array}}\right]$$

(34)

$$\Delta=\frac{\frac{e^{2\mu_{f}}}{p\left(x_{0},1,t-2\right)}+e^{\mu_{f}}-\frac{e^{2\mu_{d}}}{p\left(x_{0},1,t-2\right)}-e^{\mu_{d}}\cdot\left(1+h_{t-1}\right)}{\begin{array}{l}\frac{e^{4\mu_{f}}}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{2\sigma_{f}^{2}}-1\right)+2e^{\mu_{f}}\left(e^{\mu_{f}}-1\right)+\frac{e^{3\mu_{f}}}{p\left(x_{0},1,t-2\right)}\cdot\left(2e^{\sigma_{f}^{2}}-1\right)+\\ \frac{1}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{4\mu_{d}+2\sigma_{d}^{2}}-e^{4\mu_{d}}\right)+\left(1+h_{t-1}\right)^{2}\left(e^{2\mu_{d}+\sigma_{d}^{2}}-e^{2\mu_{d}}\right)+\frac{2\left(1+h_{t-1}\right)}{p\left(x_{0},1,t-2\right)}\left(e^{3\mu_{d}+\sigma_{d}^{2}}-2e^{3\mu_{d}}\right)+\\ -\left(\begin{array}{l}\frac{e^{2\mu_{f}+2\mu_{d}}}{p^{2}\left(x_{0},1,t-2\right)}\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_{f}+2\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(e^{\rho_{f,d}}-1\right)+\\ \quad+\frac{e^{2\mu_{f}+\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)+\frac{e^{\mu_{f}+\mu_{d}}}{p\left(x_{0},1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{\rho_{f,d}}-1\right)\end{array}\right)\end{array}} \quad (35)$$

4.3 Modelling in a mean reverting framework.

In this section we assume that the stochastic variables F_t and D_t follow a mean reverting process, while the survival probability and its increment age are deterministic.

Let us stay as general as possible and consider that the two hazard processes $(W_f(t), W_d(t))$ are correlated.

Once we calculated E[X], VAR[X], E[Y], VAR[Y] and COV[X,Y] we can inject their value in the results obtained in the previous paragraph.

$$F_t=\frac{F(t)}{F(t-1)}=\frac{e^{\gamma_f\cdot t+\rho_f\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)}}{e^{\gamma_f\cdot(t-1)+\rho_f\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)}}=e^{\gamma_f\cdot t+\rho_f\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\gamma_f\cdot(t-1)-\rho_f\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)}$$

$$F_t=e^{\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]}$$

$$D_t=\frac{D(t)}{D(t-1)}=\frac{e^{\gamma_d\cdot t+\rho_d\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)}}{e^{\gamma_d\cdot(t-1)+\rho_d\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)}}=e^{\gamma_d\cdot t+\rho_d\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\gamma_d\cdot(t-1)-\rho_d\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)}$$

$$D_t=e^{\gamma_d+\rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right]}$$

$$F_{t-1}=\frac{F(t-1)}{F(t-2)}=\frac{e^{\gamma_f\cdot(t-1)+\rho_f\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)}}{e^{\gamma_f\cdot(t-2)+\rho_f\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)}}=e^{\gamma_f\cdot(t-1)+\rho_f\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\gamma_f\cdot(t-2)-\rho_f\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)}$$

$$F_{t-1}=e^{\gamma_f+\rho_f\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]}$$

$$D_{t-1}=\frac{D(t-1)}{D(t-2)}=\frac{e^{\gamma_d\cdot(t-1)+\rho_d\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)}}{e^{\gamma_f\cdot(t-2)+\rho_f\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)}}=e^{\gamma_d\cdot(t-1)+\rho_d\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\gamma_d\cdot(t-2)-\rho_d\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)}$$

$$D_{t-1}=e^{\gamma_d+\rho_d\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]}$$

$$X=F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}+F_t=\frac{e^{2\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]}}{p(x_0,1,t-2)}+e^{\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]}$$

We find[6]:

(26)

[6]See the appendix C for the computations of the expectedvalues, variances and covariances.

$$E\left[X\right]=\frac{e^{2\gamma_f+\frac{\rho_f^2}{4\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f\cdot t}+4\beta_f-e^{-2\beta_f(t-2)}+4e^{-2\beta_f}-2e^{-\beta_f\cdot t}+2e^{-\beta_f(t-1)}\right]}}{p\left(x_0,1,t-2\right)}+e^{\gamma_f+\frac{\rho_f^2}{4\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f(t-1)}+2e^{-\beta_f}+2e^{-\beta_f(2t-1)}\right]} \tag{36}$$

$$Var\left[X\right]=\frac{e^{4\gamma_f+Var[B]}\cdot\left(e^{Var[B]}-1\right)}{p^2\left(x_0,1,t-2\right)}+e^{2\gamma_f+Var[A]}\cdot\left(e^{Var[A]}-1\right)+e^{3\gamma_f+\frac{1}{2}\left(Var[A]+Var[B]\right)}\cdot\left(e^{b_{A,B}}-1\right) \tag{37}$$

$$\begin{aligned}E[Y]=&\frac{e^{2\gamma_d+\frac{\rho_d^2}{4\beta_d^3}\left[-2-e^{-2\beta_d t}+4\beta_d-e^{-2\beta_d(t-2)}+4e^{-2\beta_d}-2e^{-\beta_d t}+2e^{-2\beta_d(t-1)}\right]}}{p\left(x_0,1,t-2\right)}+\\&+e^{\gamma_d+\frac{\rho_d^2}{4\beta_d^3}\left[2\beta_d-2-e^{-2\beta_d(t-1)}+4e^{-\beta_d t(t-1)}-e^{-2\beta_d(t-2)}+4e^{-\beta_d(t-2)}+2e^{-\beta_d}-4e^{-\beta_d t}-4e^{-\beta_d(t-1)}+2e^{-\beta_d(2t-1)}\right]}\left(1+h_{t-1}\right)\end{aligned} \tag{38}$$

$$Var\left[Y\right]=\frac{e^{4\gamma_d+Var[L]}\cdot\left(e^{Var[L]}-1\right)}{p^2\left(x_0,1,t-2\right)}+e^{2\gamma_d+Var[C]}\cdot\left(e^{Var[C]}-1\right)\left(1+h_{t-1}\right)^2+e^{3\gamma_d+\frac{1}{2}\left(Var[C]+Var[L]\right)}\cdot\left(e^{b_{C,L}}-1\right) \tag{39}$$

$$\begin{aligned}Cov[X,Y]=&\frac{e^{E[M]+\frac{Var[M]}{2}}}{p^2\left(x_0,1,t-2\right)}+\frac{e^{E[N]+\frac{Var[N]}{2}}}{p\left(x_0,1,t-2\right)}\left(1+h_{t-1}\right)+\frac{e^{E[O]+\frac{Var[P]}{2}}}{p\left(x_0,1,t-2\right)}+e^{E[P]+\frac{Var[P]}{2}}\cdot\left(1+h_{t-1}\right)+\\&-\frac{e^{E[B]+\frac{Var[B]}{2}+E[L]+\frac{Var[L]}{2}}}{p^2\left(x_0,1,t-2\right)}-\frac{e^{E[B]+\frac{Var[B]}{2}+E[C]+\frac{Var[C]}{2}}}{p\left(x_0,1,t-2\right)}\left(1+h_{t-1}\right)-\frac{e^{E[A]+\frac{Var[A]}{2}+E[L]+\frac{Var[L]}{2}}}{p\left(x_0,1,t-2\right)}+\\&-e^{E[A]+\frac{Var[A]}{2}+E[C]+\frac{Var[C]}{2}}\cdot\left(1+h_{t-1}\right)\end{aligned} \tag{40}$$

$$a^{SOL}=\frac{V[Y]-Cov[X,Y]}{Var[X-Y]}+\frac{1}{\varepsilon}\frac{E[X-Y]}{Var[X-Y]}=a^*+\frac{1}{\varepsilon}\cdot\Delta \text{ with}$$

$$\begin{aligned}a^*=&\left[\frac{e^{4\gamma_d+Var[L]}\cdot\left(e^{Var[L]}-1\right)}{p^2\left(x_0,1,t-2\right)}+e^{2\gamma_d+Var[C]}\cdot\left(e^{Var[C]}-1\right)\left(1+h_{t-1}\right)^2+e^{3\gamma_d+\frac{1}{2}\left(Var[C]+Var[L]\right)}\cdot\left(e^{b_{C,L}}-1\right)-\frac{e^{E[M]+\frac{Var[M]}{2}}}{p^2\left(x_0,1,t-2\right)}+\frac{e^{E[N]+\frac{Var[N]}{2}}}{p\left(x_0,1,t-2\right)}\left(1+h_{t-1}\right)+\frac{e^{E[O]+\frac{Var[P]}{2}}}{p\left(x_0,1,t-2\right)}-e^{E[P]+\frac{Var[P]}{2}}\cdot\left(1+h_{t-1}\right)+\right.\\&\left.+\frac{e^{E[B]+\frac{Var[B]}{2}+E[L]+\frac{Var[L]}{2}}}{p^2\left(x_0,1,t-2\right)}+\frac{e^{E[B]+\frac{Var[B]}{2}+E[C]+\frac{Var[C]}{2}}}{p\left(x_0,1,t-2\right)}\left(1+h_{t-1}\right)+\frac{e^{E[A]+\frac{Var[A]}{2}+E[L]+\frac{Var[L]}{2}}}{p\left(x_0,1,t-2\right)}+e^{E[A]+\frac{Var[A]}{2}+E[C]+\frac{Var[C]}{2}}\cdot\left(1+h_{t-1}\right)\cdot\right]\\&\left[\frac{1}{\frac{e^{4\gamma_f+Var[B]}\cdot\left(e^{Var[B]}-1\right)}{p^2\left(x_0,1,t-2\right)}+e^{2\gamma_f+Var[A]}\cdot\left(e^{Var[A]}-1\right)+e^{3\gamma_f+\frac{1}{2}\left(Var[A]+Var[B]\right)}\cdot\left(e^{b_{A,B}}-1\right)+\frac{e^{4\gamma_d+Var[L]}\cdot\left(e^{Var[L]}-1\right)}{p^2\left(x_0,1,t-2\right)}+e^{2\gamma_d+Var[C]}\cdot\left(e^{Var[C]}-1\right)\left(1+h_{t-1}\right)^2+e^{3\gamma_d+\frac{1}{2}\left(Var[C]+Var[L]\right)}\cdot\left(e^{b_{C,L}}-1\right)}-\right.\\&\left.\left(\frac{e^{E[M]+\frac{Var[M]}{2}}}{p^2\left(x_0,1,t-2\right)}+\frac{e^{E[N]+\frac{Var[N]}{2}}}{p\left(x_0,1,t-2\right)}\left(1+h_{t-1}\right)+\frac{e^{E[O]+\frac{Var[P]}{2}}}{p\left(x_0,1,t-2\right)}+e^{E[P]+\frac{Var[P]}{2}}\cdot\left(1+h_{t-1}\right)-\frac{e^{E[B]+\frac{Var[B]}{2}+E[L]+\frac{Var[L]}{2}}}{p^2\left(x_0,1,t-2\right)}-\frac{e^{E[B]+\frac{Var[B]}{2}+E[C]+\frac{Var[C]}{2}}}{p\left(x_0,1,t-2\right)}\left(1+h_{t-1}\right)-\frac{e^{E[A]+\frac{Var[A]}{2}+E[L]+\frac{Var[L]}{2}}}{p\left(x_0,1,t-2\right)}+\right.\right.\\&\left.\left.-e^{E[A]+\frac{Var[A]}{2}+E[C]+\frac{Var[C]}{2}}\cdot\left(1+h_{t-1}\right)\right)\right]\end{aligned}$$

(41)

$$\Delta = \frac{E[X-Y]}{Var[X-Y]}$$

$$\Delta = \left[\frac{e^{2\gamma_f + \frac{\rho_f^2}{4\beta_f^3}\left[2\beta_f - 2 - e^{-2\beta_f \cdot t} + 4\beta_f - e^{-2\beta_f(t-2)} + 4e^{-2\beta_f} - 2e^{-\beta_f \cdot t} + 2e^{-\beta_f(t-1)}\right]}}{p(x_0,1,t-2)} + e^{\gamma_f + \frac{\rho_f^2}{4\beta_f^3\left[2\beta_f - 2 - e^{-2\beta_f(t-1)} + 2e^{-\beta_f} + 2e^{-\beta_f(2t-1)}\right]}} - \frac{e^{2\gamma_d + \frac{\rho_d^2}{4\beta_d^3}\left[-2 - e^{-2\beta_d t} + 4\beta_d - e^{-2\beta_d(t-2)} + 4e^{-2\beta_d} - 2e^{-\beta_d t} + 2e^{-2\beta_d(t-1)}\right]}}{p(x_0,1,t-2)} + \right.$$
$$\left. - e^{\gamma_d + \frac{\rho_d^2}{4\beta_d^3}\left[2\beta_d - 2 - e^{-2\beta_d(t-1)} + 4e^{-\beta_d t(t-1)} - e^{-2\beta_d(t-2)} + 4e^{-\beta_d(t-2)} + 2e^{-\beta_d} - 4e^{-\beta_d t} - 4e^{-\beta_d(t-1)} + 2e^{-\beta_d(2t-1)}\right]}\left(1+h_{t-1}\right) \right]$$
$$\cdot \left[\frac{1}{\frac{e^{4\gamma_f + Var[B]} \cdot \left(e^{Var[B]} - 1\right)}{p^2(x_0,1,t-2)} + e^{2\gamma_f + Var[A]} \cdot \left(e^{Var[A]} - 1\right) + e^{3\gamma_f + \frac{1}{2}(Var[A] + Var[B])} \cdot \left(e^{b_{A,B}} - 1\right) + \frac{e^{4\gamma_d + Var[L]} \cdot \left(e^{Var[L]} - 1\right)}{p^2(x_0,1,t-2)} + e^{2\gamma_d + Var[C]} \cdot \left(e^{Var[C]} - 1\right)\left(1+h_{t-1}\right)^2 + e^{3\gamma_d + \frac{1}{2}(Var[C] + Var[L])} \cdot \left(e^{b_{C,L}} - 1\right)} \right]$$

(42)

Conclusioni. Differenza con modello a due periodi.

4.4 Stochastic model on four periods and one funding asset (2,2)

The stochastic processes i_t, g_t, d_t and the survival probability are defined on a single probability space (Ω, F, P).

Our framework is the overlapping generation set-up with four generations existing at any time, two generations working : young workers (aged x_0) and old workers (aged x_0+1), and two generations of pensioners: young pensioners (aged x_r), with $x_r = x_0 + 1$, and old pensioners (aged $x_r + 1$).

Context and maximization problem

We start with a combined system. A fraction of the cotizations of the working generation is invested in a funding system, in order to finance a part of the generation's own future pension. The rest of the cotization is given at cohorts attaining retirement age through a PAYG mechanism. We aim at determining the optimal part which should be invested in the funding system in order to maximize the utility of the pension benefit received at retirement time.

a : fraction invested in funding

$(1-a)$: fraction invested in PAYG

Funding The actuarial equilibrium relationship can be written as:

$$L(x_0,t-2)\pi S(t-2)(1+i_{t-1})(1+i_t)+L(x+1,t-1)\pi S(t-1)(1+i_t)=L(x_r,t)P(t)+L(x_r+1,t+1)\frac{P(t+1)}{1+i_{t+1}}$$

$$L(x_0,t-2)\pi S(t-2)(1+i_{t-1})(1+i_t)+L(x_0,t-2)\,p(x_0,1,t-2)\pi S(t-2)(1+g_{t-1})(1+i_t)=$$
$$L(x_0,t-2)\,p(x_0,2,t-2)P(t)+L(x_0,t-2)\,p(x_0,3,t-2)P(t)\frac{(1+g_{t+1})}{(1+i_{t+1})}$$

$$p(x_0,2,t-2)=p(x_0,1,t-2)\,p(x_0+1,t-1)$$

The pension will be:

$$P(t)=\frac{\pi S(t-2)(1+i_{t-1})(1+i_t)+p(x_0,1,t-2)\pi S(t-2)(1+g_{t-1})(1+i_t)}{p(x_0,2,t-2)+p(x_0,3,t-2)\frac{(1+g_{t+1})}{(1+i_{t+1})}}$$

The replacement rate will be:

$$RR(t)=\pi\frac{1}{(1+g_{t-1})\cdot(1+g_t)}\frac{(1+i_{t-1})\cdot(1+i_t)+p(x_0,1,t-2)(1+g_{t-1})(1+i_t)}{p(x_0,1,t-2)+p(x_0,3,t-2)\frac{(1+g_{t+1})}{(1+i_{t+1})}}$$

$$RR(t)=\pi\frac{(1+i_t)}{(1+g_{t-1})\cdot(1+g_t)}\frac{(1+i_{t-1})+p(x_0,1,t-2)(1+g_{t-1})}{p(x_0,1,t-2)+p(x_0,3,t-2)\frac{(1+g_{t+1})}{(1+i_{t+1})}}$$

Pay-as-you-go The contributions paid at time t by the active generations are uses to pay the pension of the retired generation at time t. The equilibrium relation is:

$$L(x_0,t)\pi S(t)+L(x_0+1,t)\pi S(t)=L(x_r,t)P(t)+L(x_r+1,t)P(t)$$

$$L(x_0,t-3)(1+d_{t-2})(1+d_{t-1})(1+d_t)\pi S(t)+L(x_0,t-3)(1+d_{t-1})(1+d_t)\,p(x_0,1,t-2)\pi S(t)=$$
$$L(x_0,t-3)(1+d_{t-2})\,p(x_0,2,t-2)P(t)+L(x_0,t-3)\,p(x_0,3,t-2)P(t)$$

The pension will be:

$$P(t)=\frac{L(x_0,t-3)(1+d_{t-2})(1+d_{t-1})(1+d_t)\pi S(t)+L(x_0,t-3)(1+d_{t-2})(1+d_{t-1})\,p(x_0,1,t-1)\pi S(t)}{L(x_0,t-3)(1+d_{t-2})\,p(x_0,2,t-2)P(t)+L(x_0,t-3)\,p(x_0,3,t-2)P(t)}$$

The replacement rate will be:

$$RR(t) = \pi \frac{(1+d_{t-2})(1+d_{t-1})\left[(1+d_t) + p(x_0,1,t-1)\right]}{(1+d_{t-2})\, p(x_0,2,t-2) + p(x_0,3,t-2)}$$

$$RR(t) = \pi \frac{(1+d_{t-2})(1+d_{t-1})\left[(1+d_t) + p(x_0,1,t-2)(1+h_{t-1})\right]}{(1+d_{t-2})\, p(x_0,1,t-2)\cdot p(x_0+1,1,t-1) + p(x_0,3,t-2)}$$

$$RR(t) = \pi \left[\begin{array}{l} a \cdot \dfrac{(1+i_t)}{(1+g_{t-1})\cdot(1+g_t)} \dfrac{(1+i_{t-1}) + p(x_0,1,t-2)(1+g_{t-1})}{p(x_0,1,t-2) + p(x_0,3,t-2)\dfrac{(1+g_{t+1})}{(1+i_{t+1})}} + \\ +(1-a)\dfrac{(1+d_{t-2})(1+d_{t-1})\left[(1+d_t) + p(x_0,1,t-2)(1+h_{t-1})\right]}{(1+d_{t-2})\, p(x_0,1,t-2)\, p(x_0+1,1,t-1) + p(x_0,3,t-2)} \end{array} \right] \quad (43)$$

5. Asset allocation problem

In this section we assume that the contributions are invested in two different assets, a bond index F_1 and a stock index F_2, with $E[F_1]<E[F_2]$ and $Var[F_1]<Var[F_2]$:

$$F = \beta \cdot F_1 + (1-\beta)\cdot F_2$$

where β is the fraction of the financial assets invested in bonds, and (1-β) is the fraction of the financial assets invested in stocks.

Then the portfolio becomes:

$$Z = a\cdot\beta\cdot F_1 + a(1-\beta)\cdot F_2 + (1-a)\cdot D$$

$$E[Z] = a\cdot\beta\cdot E[F_1] + a(1-\beta)\cdot E[F_2] + (1-a)\cdot E[D]$$

$$\begin{aligned} Var[Z] &= a^2\cdot\beta^2\cdot Var[F_1] + a^2(1-\beta)^2\cdot Var[F_2] + (1-a)^2\cdot Var[D] + \\ &+ 2a^2\beta(1-\beta)Cov[F_1,F_2] + 2a(1-a)\beta Cov[F_1,D] 2a(1-a)(1-\beta)Cov[F_2,D] \end{aligned}$$

$$U(Z) = E[Z] - \frac{\gamma}{2} Var[Z]$$

We assume that F_1 and F_2 are dependent, while are independent from D and the survival probability is deterministic.

The problem is to find α and β to maximize the utility function:

$$\frac{\partial U(Z)}{\partial a} = \beta E[F_1] + (1-\beta)E[F_2] - E[D] +$$
$$-\gamma\left\{a\beta^2 Var[F_1] + a(1-\beta)^2 Var[F_2] - (1-a)Var[D] + 2a\beta(1-\beta)Cov[F_1,F_2]\right\} = 0$$

$$\frac{\partial U(Z)}{\partial \beta} = aE[F_1] - aE[F_2] +$$
$$-\gamma\left\{a^2\beta Var[F_1] - a^2(1-\beta)Var[F_2] + a^2(1-2\beta)Cov[F_1,F_2]\right\} = 0$$

The system has two solutions:

$$\begin{cases} a = 0 \\ \beta = -\dfrac{E[F_2] + \gamma Var[D]}{E[F_1] - E[F_2]} \end{cases}$$

Setting:

$$A = 2Var[F_2]E[F_1 - F_2] - 3E[F_1 - F_2]Cov[F_1,F_2],$$
$$B = -3E[F_1 - F_2]Var[F_2] + 3E[F_1 - F_2]Cov[F_1,F_2] + E[D]Var[F_1] - E[D]Var[F_2] - E[D]Cov[F_1,F_2] +$$
$$-\gamma Var[F_1]Var[D] + \gamma Var[D] - \gamma Var[D]Cov[F_1,F_2]$$
$$C = Var[F_2]E[F_1 - F_2] + Var[D]E[F_1 - F_2] + E[D]Var[F_2] - E[D]Cov[F_1,F_2] - \gamma Var[F_2] +$$
$$+\gamma Var[D]Cov[F_1,F_2]$$

$$\begin{cases} \beta^{SOL} = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ a^{SOL} = \dfrac{E[F_1] - E[F_2]}{\gamma\left[\beta^{SOL} Var[F_1] + (1-\beta^{SOL})Var[F_2] + (1-\beta^{SOL})Cov[F_1,F_2]\right]} \end{cases}$$

6. Longevity risk in two periods model
Stochastic model on two periods and one funding asset and stochastic longevity risk

$$RR(t) = \frac{\pi}{p(x_0,1)}\left[a\frac{1+i_t}{1+g_t} + (1+a)(1+d_t)\right] \quad (21)$$

If we relax the hypothesis of a deterministic survival probability and assume that it is stochastic we have:

$$RR(t) = \pi\left[a\frac{1+i_t}{1+g_t}(1+l_{t-1}) + (1+a)(1+d_t)(1+l_{t-1})\right]$$

If $I = 1+i$, $G = 1+g$, $D = 1+d$ and $L = 1+l$
We define:

$$Z = a \cdot L \cdot F + (1-a) \cdot L \cdot D$$

Z, our “pension portfolio”, is the process that we will study for the maximization.

6.1. All the variables are independent

If L,F and D are independent each other, then we have:

$$Z = a \cdot L \cdot F + (1-a) \cdot L \cdot D$$

$$E[Z] = a \cdot E[L] \cdot E[F] + (1-a) \cdot E[L] \cdot E[D]$$

$$Var[Z] = a^2 Var[L \cdot F] + (1-a)^2 Var[L \cdot D] + 2a(1-a) Cov[LF, LD]$$

$$\frac{\partial U(Z)}{\partial a} = E[L] \cdot E[F] - E[L]E[D] - \gamma\{aVar[LF] - (1-a)Var[LD] + (1-2a)Cov[LF, LD]\} = 0$$

$$a^{SOL} = \frac{Var[LD] - Cov[LF, LD]}{Var[LF - LD]} + \frac{1}{\gamma}\frac{E[L]\{E[F] - E[D]\}}{Var[LF - LD]}$$

$$a^{SOL} = \frac{Var[L]\left\{Var[D] + \left[E[D]\right]^2 - E[F]E[D]\right\} + \left[E[L]\right]^2 Var[D]}{Var[L]\left\{Var[F] + Var[D] + \left[E[F]\right]^2 - 2E[F]E[D]\right\} + \left[E[L]\right]^2 \{Var[F] + Var[D]\}} +$$

$$+\frac{1}{\gamma}\frac{E[L]\left[E[F] - E[D]\right]}{Var[L]\left\{Var[F] + Var[D] + \left[E[F]\right]^2 - 2E[F]E[D]\right\} + \left[E[L]\right]^2 \{Var[F] + Var[D]\}}$$

6.2 D and L are dependent, F is independent of L and D

$$Z = a \cdot L \cdot F + (1-a) \cdot L \cdot D$$

$$E[Z] = a \cdot E[L] \cdot E[F] + (1-a) \cdot \left[E[L] \cdot E[D] + Cov[L, D]\right]$$

$$Var[Z] = a^2 Var[L \cdot F] + (1-a)^2 Var[L \cdot D] + 2a(1-a) Cov[LF, LD]$$

$$U(Z) = E[Z] - \frac{\gamma}{2} Var[Z]$$

$$\frac{\partial U(Z)}{\partial a} = E[L]E[F] - E[L]E[D] - Cov[L, D] - \gamma\left[aVar[LF] - (1-a)Var[LD] + (1-2a)Cov[LF, LD]\right]$$

$$a^{SOL} = \frac{Var[LD] - Cov[LD]}{Var[LF] + Var[LD] - 2Cov[LF, LD]} + \frac{1}{\gamma}\frac{E[L]E[F-D] - Cov[L, D]}{Var[LF] + Var[LD] - 2Cov[LF, LD]}$$

$$Var[L \cdot F] = Var[L]Var[F] + \left[E[L]\right]^2 Var[F] + Var[L]\left[E[L]\right]^2$$

$$Var[L \cdot D] = Var[L]Var[D] + \left[E[L]\right]^2 Var[D] + Var[L]\left[E[D]\right]^2 +$$
$$- \left[Cov[L,D]\right]^2 - 2E[L]E[D]Cov[L,D] + Cov\left[L^2, D^2\right]$$

$$Cov[LF, LD] = E[F]E[D]Var[L] - E[L]E[F]Cov[L,D] + E[F]Cov\left[L^2, D\right]$$

$$a^{SOL} = \frac{Var[L]Var[D] + \left[E[L]\right]^2 Var[D] + Var[L]\left[E[D]\right] - \left[Cov[L,D]\right]^2 - 2E[L]E[D]Cov[L,D] + Cov\left[L^2, D^2\right] - Cov[L,D]}{Var[L]\left\{Var[F] + Var[D] + \left[E[F] - E[D]\right]^2\right\} + \left[E[L]\right]^2\left\{Var[F] + Var[D]\right\} + 2E[L]Cov[L,D]\left\{E[F-D]\right\} - \left[Cov[L,D]\right]^2 + Cov\left[L^2, D^2\right] - 2E[F]Cov\left[L^2, D\right]}$$
$$+ \frac{1}{\gamma}\frac{E[L]E[F-D] - Cov[L,D]}{Var[L]\left\{Var[F] + Var[D] + \left[E[F] - E[D]\right]^2\right\} + \left[E[L]\right]^2\left\{Var[F] + Var[D]\right\} + 2E[L]Cov[L,D]\left\{E[F-D]\right\} - \left[Cov[L,D]\right]^2 + Cov\left[L^2, D^2\right] - 2E[F]Cov\left[L^2, D\right]}$$

6.3 All the variables are dependent

$$Z = a \cdot L \cdot F + (1-a) \cdot L \cdot D$$

$$E[Z] = a \cdot E[L] \cdot E[F] + aCov[L,F] + (1-a) \cdot E[L] \cdot E[D] + (1-a)Cov[L,D]$$

$$Var[Z] = a^2 Var[L \cdot F] + (1-a)^2 Var[L \cdot D] + 2a(1-a)Cov[LF, LD]$$

$$U(Z) = E[Z] - \frac{\gamma}{2}Var[Z]$$

$$\frac{\partial U(Z)}{\partial a} = E[L]E[F] - E[L]E[D] - Cov[L,D] - \gamma\left[aVar[LF] - (1-a)Var[LD] + (1-2a)Cov[LF, LD]\right]$$

$$a^{SOL} = \frac{Var[LD] - Cov[LD]}{Var[LF] + Var[LD] - 2Cov[LF, LD]} + \frac{1}{\gamma}\frac{E[L]E[F-D] - Cov[L,D]}{Var[LF] + Var[LD] - 2Cov[LF, LD]}$$

$$Var[L \cdot F] = Var[L]Var[F] + \left[E[L]\right]^2 Var[F] + Var[L]\left[E[L]\right]^2 +$$
$$- \left[Cov[L,F]\right]^2 - 2E[L]E[F]Cov[L,F] + Cov\left[L^2, F^2\right]$$

$$Var[L\cdot D]=Var[L]Var[D]+\left[E[L]\right]^2Var[D]+Var[L]\left[E[D]\right]^2+$$
$$-\left[Cov[L,D]\right]^2-2E[L]E[D]Cov[L,D]+Cov\left[L^2,D^2\right]$$

$$Cov[LF,LD]=E[F]E[D]Var[L]+\left[E[L]\right]^2Cov[F,D]+Var[L]Cov[F,D]+Cov\left[L^2,FD\right]$$
$$-E[L]E[F]Cov[L,D]-E[L]E[D]Cov[L,F]-Cov[L,F]Cov[L,D]$$

$$a^{SOL}=\frac{Var[L]Var[D]+\left[E[L]\right]^2Var[F]+Var[L]\left[E[D]\right]^2-\left[Cov[L,D]\right]^2-2E[L]E[D]Cov[L,D]+Cov\left[L^2,D^2\right]-Cov[L,D]}{Var[L]\left\{Var[F-D]+E[F-D]^2\right\}+\left[E[L]\right]^2\cdot Var[F-D]-\left[Cov[L,F]\right]^2+2E[L]E[F-D]\left\{Cov[L,F]-Cov[L,D]\right\}+Cov\left[L^2,D^2\right]-2Cov\left[L^2,FD\right]+2Cov[L,F]Cov[L,D]}+$$
$$+\frac{1}{\gamma}\frac{E[L]\left[E[F-D]\right]}{Var[L]\left\{Var[F-D]+E[F-D]^2\right\}+\left[E[L]\right]^2\cdot Var[F-D]-\left[Cov[L,F]\right]^2+2E[L]E[F-D]\left\{Cov[L,F]-Cov[L,D]\right\}+Cov\left[L^2,D^2\right]-2Cov\left[L^2,FD\right]+2Cov[L,F]Cov[L,D]}$$

6.4 Modelling in a lognormal framework

In this section we develop the model presented in section 6.1, assuming that the stochastic variables F , L and D follow a lognormal process

- $D=\dfrac{D(t)}{D(t-1)}=e^{\left[\mu_d-\frac{1}{2}\sigma_d^2+\sigma_d\left(W_d(t)-W_d(t-1)\right)\right]}$
- $F=\dfrac{F(t)}{F(t-1)}=e^{\left[\mu_f-\frac{1}{2}\sigma_f^2+\sigma_f\left(W_f(t)-W_f(t-1)\right)\right]}$

$E\left[F\right]=e^{\mu_f}$ $\qquad \left[E[F]\right]^2=e^{2\mu_f}$ $\qquad VAR\left[F\right]=e^{2\mu_f}\cdot\left(e^{\sigma_f^2}-1\right)$

$E\left[D\right]=e^{\mu_d}$ $\qquad \left[E[D]\right]^2=e^{2\mu_d}$ $\qquad VAR\left[D\right]=e^{2\mu_d}\cdot\left(e^{\sigma_d^2}-1\right)$

$E[L]=e^{\mu_l}$ $\qquad \left[E[L]\right]^2=e^{2\mu_l}$ $\qquad VAR[F]=e^{2\mu_l}\cdot\left(e^{\sigma_l^2}-1\right)$

$$a^{SOL}=\frac{e^{2\mu_l}\cdot\left(e^{\sigma_l^2}-1\right)\left\{e^{2\mu_d}\cdot\left(e^{\sigma_d^2}-1\right)+e^{2\mu_d}-e^{\mu_f+\mu_d}\right\}+e^{2\mu_l+2\mu_d}\cdot\left(e^{\sigma_d^2}-1\right)}{e^{2\mu_l}\cdot\left(e^{\sigma_l^2}-1\right)\left\{e^{2\mu_f}\cdot\left(e^{\sigma_f^2}-1\right)+e^{2\mu_d}\cdot\left(e^{\sigma_d^2}-1\right)+e^{2\mu_f}-2e^{\mu_f+\mu_d}\right\}+e^{2\mu_l}\left\{e^{2\mu_f}\cdot\left(e^{\sigma_f^2}-1\right)+e^{2\mu_d}\cdot\left(e^{\sigma_d^2}-1\right)\right\}}+$$
$$+\frac{1}{\gamma}\frac{e^{\mu_l}\left[e^{\mu_f}+e^{\mu_d}\right]}{e^{2\mu_l}\cdot\left(e^{\sigma_l^2}-1\right)\left\{e^{2\mu_f}\cdot\left(e^{\sigma_f^2}-1\right)+e^{2\mu_d}\cdot\left(e^{\sigma_d^2}-1\right)+e^{2\mu_f}-2e^{\mu_f+\mu_d}\right\}+e^{2\mu_l}\left\{e^{2\mu_f}\cdot\left(e^{\sigma_f^2}-1\right)+e^{2\mu_d}\cdot\left(e^{\sigma_d^2}-1\right)\right\}}$$

If F and D follow a lognormal process while L is modelled with a Lee Carter model:

L=

7. Numerical applications

8. Conclusion

Funded and PAYG pension schemes may seem completely different but are quite complementary because they deal with different risks. Time has come to re-think the financing of public pension, especially in countries that have opted for PAYG schemes.

In a deterministic framework, Samuelson rule applies: the relative returns of both schemes indicate which one to choose. One scheme is always better than the other one and there is no room for a blending of them.

But, in a stochastic framework it is different. We have two main conclusions.

First, to achieve a better public pension, mixing both schemes might be an interesting alternative: we observe that the optimal fraction might be somewhere between *0* and *1* in some cases.

Second, we study the impact of introducing a correlation between the financial and the demographic processes, which is usually not done in the literature. When the covariance is positive and if the state's risk aversion coefficient is lower than a certain level, the fraction to invest in the funded scheme will increase.

Appendix A Modelling in a mean reverting framework (2 periods model)

$$F = e^{\gamma_f + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f (t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f (t-1-u)}}{\beta_f} \right) dW_f(u) \right]}$$

$F = e^A$ $\quad$ F_t is lognormal

$$A = \gamma_f + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f (t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f (t-1-u)}}{\beta_f} \right) dW_f(u) \right]$$

$$E[A] = \gamma_f$$

$$Var\left[A\right]=E\left\{\left[\rho_f\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\right]^2\right\}=$$

$$=E\left[\rho_f^2\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2\right]=$$

$$=E\left[\rho_f^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2+\\-2\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\end{array}\right]\right]=$$

$$=E\left[\rho_f^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2+\\-2\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)+\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\end{array}\right]\right]=$$

$$=E\left[\begin{array}{l}\rho_f^2\left[\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2\right]+\\-2\left[\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)+\left(\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\right]\end{array}\right]=$$

$$=\rho_f^2\left[\begin{array}{l}\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)^2du+\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)^2du-2\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)du+\\-2E\left[\left(\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\right]\end{array}\right]=$$

$$E\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\cdot\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right]=0$$

$$= \rho_f^2 \left[\int_0^t \left(\frac{1-e^{-\beta_f (t-u)}}{\beta_f} \right)^2 du + \int_0^{t-1} \left(\frac{1-e^{-\beta_f (t-1-u)}}{\beta_f} \right)^2 du - 2\int_0^{t-1} \left(\frac{1-e^{-\beta_f (t-u)}}{\beta_f} \right)\left(\frac{1-e^{-\beta_f (t-1-u)}}{\beta_f} \right) du \right] =$$

$$= \rho_f^2 \left[\int_0^t \frac{1+e^{-2\beta_f (t-u)} - 2e^{-\beta_f (t-u)}}{\beta_f^2} du + \int_0^{t-1} \frac{1+e^{-2\beta_f (t-1-u)} - 2e^{-\beta_f (t-1-u)}}{\beta_f^2} du - 2\int_0^{t-1} \frac{1-e^{-\beta_f (t-u)} - e^{-\beta_f (t-1-u)} + e^{-\beta_f (2t-2u-1)}}{\beta_f^2} du \right] =$$

$$= \frac{\rho_f^2}{\beta_f^2} \left[\left[u + \frac{e^{-2\beta_f (t-u)}}{2\beta_f} - \frac{2e^{-\beta_f (t-u)}}{\beta_f} \right]_0^t + \left[u + \frac{e^{-2\beta_f (t-1-u)}}{2\beta_f} - \frac{2e^{-\beta_f (t-1-u)}}{\beta_f} \right]_0^{t-1} - 2\left[u - \frac{e^{-\beta_f (t-u)}}{\beta_f} - \frac{e^{-\beta_f (t-1-u)}}{\beta_f} + \frac{e^{-\beta_f (2t-2u-1)}}{2\beta_f} \right]_0^{t-1} \right] =$$

$$= \frac{\rho_f^2}{\beta_f^2} \left[\begin{array}{l} t + \frac{1}{2\beta_f} - \frac{2}{\beta_f} - \frac{e^{-2\beta_f t}}{2\beta_f} + \frac{2e^{-\beta_f t}}{\beta_f} + t - 1 + \frac{1}{2\beta_f} - \frac{2}{\beta_f} - \frac{e^{-2\beta_f (t-1)}}{2\beta_f} + \frac{2e^{-\beta_f (t-1)}}{\beta_f} + \\ -2t + 2 + \frac{2e^{-\beta_f}}{\beta_f} + \frac{2}{\beta_f} - \frac{e^{-\beta_f}}{\beta_f} - \frac{2e^{-\beta_f t}}{\beta_f} - \frac{2e^{-\beta_f (t-1)}}{\beta_f} + \frac{e^{-\beta_f (2t-1)}}{\beta_f} \end{array} \right] =$$

$$Var\left[A\right] = \frac{\rho_f^2}{2\beta_f^3}\left[2\beta_f - 2 - e^{-2\beta_f \cdot t} - e^{-2\beta_f (t-1)} + 2e^{-\beta_f} + 2e^{-\beta_f (2t-1)} \right]$$

$$F = e^A$$

$$E\left[A\right] = a_f \qquad Var\left[A\right] = b_f^2$$

$$a_f = \gamma_f$$

$$b_f^2 = \frac{\rho_f^2}{2\beta_f^3}\left[2\beta_f - 2 - e^{-2\beta_f \cdot t} - e^{-2\beta_f (t-1)} + 2e^{-\beta_f} + 2e^{-\beta_f (2t-1)} \right]$$

$$E\left[F\right] = e^{a_f + \frac{b_f^2}{2}}$$

$$Var\left[F\right] = e^{2a_f + b_f^2}\left(e^{b_f^2} - 1 \right)$$

$$D = e^{A_d}$$

$$E\left[A_d\right] = a_d \qquad Var\left[A_d\right] = b_d^2$$

$$a_d = \gamma_d$$

$$b_d^2 = \frac{\rho_d^2}{2\beta_d^3}\left[2\beta_d - 2 - e^{-2\beta_d \cdot t} - e^{-2\beta_d (t-1)} + 2e^{-\beta_d} + 2e^{-\beta_d (2t-1)} \right]$$

$$E\left[D\right]=e^{a_d+\frac{b_d^2}{2}}$$

$$Var\left[D\right]=e^{2a_d+b_d^2}\left(e^{b_d^2}-1\right)$$

$$Cov\left[F,D\right]=e^{a_f+a_d+\frac{1}{2}\left(b_f^2+b_d^2\right)}\left(e^{b_{f,d}}-1\right)$$

$$b_{f,d}=E\left[\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f\left(u\right)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d\left(u\right)-\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d\left(u\right)\right)\right]$$

$$b_{f,d}=E\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d\left(u\right)\right)-\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d\left(u\right)\right)+\\-\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d\left(u\right)\right)+\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d\left(u\right)\right)\end{array}\right]$$

$$b_{f,d}=E\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d\left(u\right)\right)+\\-\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d\left(u\right)\right)+\\-\left(\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d\left(u\right)\right)+\\-\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d\left(u\right)\right)+\\-\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_{t-1}^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d\left(u\right)\right)+\\+\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f\left(u\right)\right)\cdot\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d\left(u\right)\right)\end{array}\right]$$

$$\begin{aligned}b_{f,d}=&\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)\rho_{f,d}d(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)\rho_{f,d}d(u)+\\&-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)\rho_{f,d}d(u)+\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)\rho_{f,d}d(u)\end{aligned}$$

$$b_{f,d}=\int_0^t \frac{1-e^{-\beta_f(t-u)}-e^{-\beta_d(t-u)}+e^{-(\beta_f+\beta_d)(t-u)}}{\beta_f\beta_d}\rho_{f,d}d(u)-\int_0^{t-1}\frac{1-e^{-\beta_f(t-u)}-e^{-\beta_d(t-1-u)}+e^{-\beta_f(t-u)-\beta_d(t-1-u)}}{\beta_f\beta_d}\rho_{f,d}d(u)+$$
$$-\int_0^{t-1}\frac{1-e^{-\beta_f(t-1-u)}-e^{-\beta_d(t-u)}+e^{-\beta_f(t-1-u)-\beta_d(t-u)}}{\beta_f\beta_d}\rho_{f,d}d(u)+\int_0^{t-1}\frac{1-e^{-\beta_f(t-1-u)}-e^{-\beta_d(t-1-u)}+e^{-(\beta_f+\beta_d)(t-1-u)}}{\beta_f\beta_d}\rho_{f,d}d(u)$$

$$b_{f,d}=\frac{\rho_{f,d}}{\beta_f\beta_d}\left[\begin{array}{l}\left[u-\frac{e^{-\beta_f(t-u)}}{\beta_f}-\frac{e^{-\beta_d(t-u)}}{\beta_d}+\frac{e^{-(\beta_f+\beta_d)(t-u)}}{\beta_f+\beta_d}\right]_0^t-\left[u-\frac{e^{-\beta_f(t-u)}}{\beta_f}-\frac{e^{-\beta_d(t-1-u)}}{\beta_d}+\frac{e^{-\beta_f(t-u)-\beta_d(t-1-u)}}{\beta_f+\beta_d}\right]_0^{t-1}+\\-\left[u-\frac{e^{-\beta_f(t-1-u)}}{\beta_f}-\frac{e^{-\beta_d(t-u)}}{\beta_d}+\frac{e^{-\beta_f(t-1-u)-\beta_d(t-u)}}{\beta_f+\beta_d}\right]_0^{t-1}+\left[u-\frac{e^{-\beta_f(t-1-u)}}{\beta_f}-\frac{e^{-\beta_d(t-1-u)}}{\beta_d}+\frac{e^{-(\beta_f+\beta_d)(t-1-u)}}{\beta_f+\beta_d}\right]_0^{t-1}\end{array}\right]$$

$$b_{f,d}=\frac{\rho_{f,d}}{\beta_f\beta_d}\left[\begin{array}{l}1+\frac{2}{\beta_f+\beta_d}-\frac{e^{-(\beta_f+\beta_d)\cdot t}}{\beta_f+\beta_d}+\frac{e^{-\beta_f}}{\beta_f}-\frac{e^{-\beta_f}}{\beta_f+\beta_d}+\frac{e^{-\beta_f\cdot t-\beta_d(t-1)}}{\beta_f+\beta_d}+\frac{e^{-\beta_d}}{\beta_d}-\frac{e^{-\beta_d}}{\beta_f+\beta_d}+\frac{e^{-\beta_f\cdot(t-1)-\beta_d\cdot t}}{\beta_f+\beta_d}+\\-\frac{1}{\beta_f}-\frac{1}{\beta_d}-\frac{e^{-(\beta_f+\beta_d)\cdot(t-1)}}{\beta_f+\beta_d}\end{array}\right]$$

Appendix B Modelling in a lognormal framework (3 periods model)

$$E[F_t\cdot F_{t-1}]=E[F_t]\cdot E[F_{t-1}]$$

$$E[F_t]=e^{\mu_f} \qquad VAR[F_t]=e^{2\mu_f}\cdot\left(e^{\sigma_f^2}-1\right)$$

$$E[F_{t-1}]=e^{\mu_f} \qquad VAR[F_{t-1}]=e^{2\mu_f}\cdot\left(e^{\sigma_f^2}-1\right)$$

$$E[X]=E\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}+F_t\right]=$$

$$=\frac{E[F_t]\cdot E[F_{t-1}]}{p(x_0,1,t-2)}+E[F_t]=$$

$$=E[F_t]\cdot\left[\frac{E[F_{t-1}]}{p(x_0,1,t-2)}+1\right]=$$

$$E[X]=e^{\mu_f}\cdot\left(\frac{e^{\mu_f}}{p(x_0,1,t-2)}+1\right)=\frac{e^{2\mu_f}}{p(x_0,1,t-2)}+e^{\mu_f}$$

$$Var[X]=E\left[X-E[X]\right]^2$$

$$Var[X]=E\left[\frac{F_t\cdot F_{t-1}}{p(x_0,1,t-2)}+F_t-E\left[\frac{F_t\cdot F_{t-1}}{p(x_0,1,t-2)}+F_t\right]\right]^2=$$

$$=E\left[\begin{array}{l}\frac{e^{\left(\mu_f-\frac{1}{2}\sigma_f^2\right)+\sigma_f\left[W_f(t)-W_f(t-1)\right]}\cdot e^{\left(\mu_f-\frac{1}{2}\sigma_f^2\right)+\sigma_f\left[W_f(t-1)-W_f(t-2)\right]}}{p(x_0,1,t-2)}+e^{\left(\mu_f-\frac{1}{2}\sigma_f^2\right)+\sigma_f\left[W_f(t)-W_f(t-1)\right]}+\\-\frac{e^{2\mu_f}}{p(x_0,1,t-2)}-e^{\mu_f}\end{array}\right]^2$$

$$=E\left[\frac{e^{\left(2\mu_f-\sigma_f^2\right)+\sigma_f\left[W_f(t)-W_f(t-2)\right]}}{p(x_0,1,t-2)}+e^{\left(\mu_f-\frac{1}{2}\sigma_f^2\right)+\sigma_f\left[W_f(t)-W_f(t-1)\right]}-\frac{e^{2\mu_f}}{p(x_0,1,t-2)}-e^{\mu_f}\right]^2=$$

$$=E\left[\frac{e^{(2\mu_f)}}{p(x_0,1,t-2)}\cdot\left(e^{-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}-1\right)+e^{(\mu_f)}\cdot\left(e^{-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}-1\right)\right]^2=$$

$$=E\left[\begin{array}{l}\frac{e^{4\mu_f}}{p^2(x_0,1,t-2)}\cdot\left(e^{-2\sigma_f^2+2\sigma_f\left[W_f(t)-W_f(t-2)\right]}+1-2e^{-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}\right)+\\e^{2\mu_f}\cdot\left(e^{-\sigma_f^2+2\sigma_f\left[W_f(t)-W_f(t-1)\right]}+1-2e^{-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}\right)+\\+2\frac{e^{3\mu_f}}{p(x_0,1,t-2)}\cdot\left(e^{-\frac{3}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)+W_f(t)-W_f(t-1)\right]}-e^{-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}-e^{-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}+1\right)\end{array}\right]$$

$$W_f(t)-W_f(t-2)+W_f(t)-W_f(t-1)=W_f(t)-W_f(t-1)+W_f(t-1)-W_f(t-2)+W_f(t)-W_f(t-1)=$$
$$=W_f(t-1)-W_f(t-2)+2\left[W_f(t)-W_f(t-1)\right]$$

$$=E\left[\begin{array}{l}\frac{1}{p^2(x_0,1,t-2)}\left(e^{4\mu_f-2\sigma_f^2+2\sigma_f\left[W_f(t)-W_f(t-2)\right]}+e^{4\mu_f}-2e^{4\mu_f-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}\right)+\\+e^{2\mu_f-\sigma_f^2+2\sigma_f\left[W_f(t)-W_f(t-1)\right]}+e^{2\mu_f}-2e^{2\mu_f-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}+\\+\frac{2}{p(x_0,1,t-2)}\left(e^{3\mu_f-\frac{3}{2}\sigma_f^2+\sigma_f\left[W_f(t-1)-W_f(t-2)+2\left[W_f(t)-W_f(t-1)\right]\right]}-e^{3\mu_f-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}-e^{3\mu_f-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}+e^{3\mu_f}\right)\end{array}\right]$$

$$VAR[X]=\frac{1}{p^2(x_0,1,t-2)}\left(e^{4\mu_f-2\sigma_f^2+4\sigma_f^2}+e^{4\mu_f}-2e^{4\mu_f-\sigma_f^2+\sigma_f^2}\right)+e^{2\mu_f-\sigma_f^2+\sigma_f^2}+e^{2\mu_f}-2e^{\mu_f-\frac{1}{2}\sigma_f^2+\frac{1}{2}\sigma_f^2}+$$
$$+\frac{1}{p(x_0,1,t-2)}\cdot\left(2e^{3\mu_f-\frac{3}{2}\sigma_f^2+\frac{1}{2}\sigma_f^2+2\sigma_f^2}-e^{3\mu_f-\sigma_f^2+\sigma_f^2}-e^{3\mu_f-\frac{1}{2}\sigma_f^2+\frac{1}{2}\sigma_f^2}+e^{3\mu_f}\right)$$

$$VAR[X] = \frac{1}{p^2(x_0,1,t-2)}\left(e^{4\mu_f+2\sigma_f^2} + e^{4\mu_f} - 2e^{4\mu_f}\right) + e^{2\mu_f} + e^{2\mu_f} - 2e^{\mu_f} +$$

$$+ \frac{1}{p(x_0,1,t-2)} \cdot \left(2e^{3\mu_f+\sigma_f^2} - e^{3\mu_f} - e^{3\mu_f} + e^{3\mu_f}\right)$$

$$VAR[X] = \frac{1}{p^2(x_0,1,t-2)}\left(e^{4\mu_f+2\sigma_f^2} - e^{4\mu_f}\right) + 2e^{2\mu_f} - 2e^{\mu_f} + \frac{1}{p(x_0,1,t-2)} \cdot \left(2e^{3\mu_f+\sigma_f^2} - e^{3\mu_f}\right)$$

$$VAR[X] = \frac{e^{4\mu_f}}{p^2(x_0,1,t-2)}\left(e^{2\sigma_f^2} - 1\right) + 2e^{\mu_f}\left(e^{\mu_f} - 1\right) + \frac{e^{3\mu_f}}{p(x_0,1,t-2)} \cdot \left(2e^{\sigma_f^2} - 1\right)$$

$$Y = \frac{D_t \cdot D_{t-1}}{p(x_0,1,t-2)} + D_{t-1} \cdot (1 + h_{t-1})$$

$$E[D_t \cdot D_{t-1}] = E[D_t] \cdot E[D_{t-1}]$$

$$E[D_t] = e^{\mu_d} \qquad VAR[D_t] = e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right)$$

$$E[D_{t-1}] = e^{\mu_d} \qquad VAR[D_{t-1}] = e^{2\mu_d} \cdot \left(e^{\sigma_d^2} - 1\right)$$

$$E[Y] = E\left[\frac{D_t \cdot D_{t-1}}{p(x_0,1,t-2)} + D_{t-1} \cdot (1 + h_{t-1})\right] =$$

$$= \frac{E[D_t] \cdot E[D_{t-1}]}{p(x_0,1,t-2)} + E[D_{t-1}] \cdot (1 + h_{t-1}) =$$

$$E[Y] = e^{\mu_d} \cdot \left(\frac{e^{\mu d}}{p(x_0,1,t-2)} + (1 + h_{t-1})\right) = \frac{e^{2\mu_d}}{p(x_0,1,t-2)} + e^{\mu_d} \cdot (1 + h_{t-1})$$

$$Var[Y] = E\left[Y - E[Y]\right]^2$$

$$Var\left[Y\right]=E\left[\frac{D_t\cdot D_{t-1}}{p\left(x_0,1,t-2\right)}+D_{t-1}\cdot\left(1+h_{t-1}\right)-E\left[\frac{D_t\cdot D_{t-1}}{p\left(x_0,1,t-2\right)}+D_{t-1}\cdot\left(1+h_{t-1}\right)\right]\right]^2=$$

$$=E\left[\begin{array}{l}\dfrac{e^{\mu_d-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-1)\right]}\cdot e^{\mu_d-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}}{p\left(x_0,1,t-2\right)}+e^{\mu_d-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\cdot\left(1+h_{t-1}\right)+\\ -\dfrac{e^{2\mu_d}}{p\left(x_0,1,t-2\right)}-e^{\mu_d}\cdot\left(1+h_{t-1}\right)\end{array}\right]^2$$

$$=E\left[\begin{array}{l}\dfrac{e^{2\mu_d-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}}{p\left(x_0,1,t-2\right)}+e^{\mu_d-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\cdot\left(1+h_{t-1}\right)+\\ -\dfrac{e^{2\mu_d}}{p\left(x_0,1,t-2\right)}-e^{\mu_d}\cdot\left(1+h_{t-1}\right)\end{array}\right]^2=$$

$$=E\left[\frac{e^{2\mu_d}}{p\left(x_0,1,t-2\right)}\left(e^{-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}-1\right)+e^{\mu_d}\cdot\left(1+h_{t-1}\right)\left(e^{-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}-1\right)\right]^2=$$

$$=E\left[\begin{array}{l}\dfrac{e^{4\mu_d}}{p^2\left(x_0,1,t-2\right)}\left(e^{-2\sigma_d^2+2\sigma_d\left[W_d(t)-W_d(t-2)\right]}+1-2e^{-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}\right)+\\ +e^{2\mu_d}\cdot\left(1+h_{t-1}\right)^2\left(e^{-\sigma_d^2+2\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}+1-2e^{-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\right)\\ +\dfrac{2e^{3\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{-\frac{3}{2}\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)+W_d(t-1)-W(t-2)\right]}-e^{-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}-e^{-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\right)\end{array}\right]$$

$$=E\left[\begin{array}{l}\dfrac{1}{p^2\left(x_0,1,t-2\right)}\left(e^{4\mu_d-2\sigma_d^2+2\sigma_d\left[W_d(t)-W_d(t-2)\right]}+e^{4\mu_d}-2e^{4\mu_d-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}\right)+\\ \left(1+h_{t-1}\right)^2\left(e^{2\mu_d-\sigma_d^2+2\sigma_d\left[W_d(t)-W_d(t-1)\right]}+e^{2\mu_d}-2e^{2\mu_d-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\right)+\\ +\dfrac{2\left(1+h_{t-1}\right)}{p\left(x_0,1,t-2\right)}\left(e^{3\mu_d-\frac{3}{2}\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-1)+2\left[W_d(t-1)-W_d(t-2)\right]\right]}-e^{3\mu_d-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}-e^{3\mu_d-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\right)\end{array}\right]$$

$$=\left[\begin{array}{l}\dfrac{1}{p^2\left(x_0,1,t-2\right)}\left(e^{4\mu_d-2\sigma_d^2+4\sigma_d^2}+e^{4\mu_d}-2e^{4\mu_d-\sigma_d^2+\sigma_d^2}\right)+\\ \left(1+h_{t-1}\right)^2\left(e^{2\mu_d-\sigma_d^2+2\sigma_d^2}+e^{2\mu_d}-2e^{2\mu_d-\frac{1}{2}\sigma_d^2+\frac{1}{2}\sigma_d^2}\right)+\\ +\dfrac{2\left(1+h_{t-1}\right)}{p\left(x_0,1,t-2\right)}\left(e^{3\mu_d-\frac{3}{2}\sigma_d^2+\frac{1}{2}\sigma_d^2+2\sigma_d^2}-e^{3\mu_d-\sigma_d^2+\sigma_d^2}-e^{3\mu_d-\frac{1}{2}\sigma_d^2+\frac{1}{2}\sigma_d^2}\right)\end{array}\right]$$

$$=\frac{1}{p^2\left(x_0,1,t-2\right)}\left(e^{4\mu_d+2\sigma_d^2}-e^{4\mu_d}\right)+\left(1+h_{t-1}\right)^2\left(e^{2\mu_d+\sigma_d^2}-e^{2\mu_d}\right)+\frac{2\left(1+h_{t-1}\right)}{p\left(x_0,1,t-2\right)}\left(e^{3\mu_d+\sigma_d^2}-2e^{3\mu_d}\right)$$

If financial and demographic processes F_t and D_t are independent the covariance between X and Y will be zero.

Otherwise the covariance will be:

$Cov\left(X,Y\right)=E\left[\left(X-E\left[X\right]\right)\left(Y-E\left[Y\right]\right)\right]=$

$$=E\left[\begin{array}{l}\left[\frac{e^{2\mu_f}}{p\left(x_0,1,t-2\right)}\cdot\left(e^{-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}-1\right)+e^{\mu_f}\cdot\left(e^{-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}-1\right)\right]\\ \left[\frac{e^{2\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(e^{-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}-1\right)+e^{\mu_d}\cdot\left(1+h_{t-1}\right)\cdot\left(e^{-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}-1\right)\right]\end{array}\right]=$$

$$E\left[\begin{array}{l}\frac{e^{2\mu_f+2\mu_d}}{p^2\left(x_0,1,t-2\right)}\left(e^{-\sigma_f^2-\sigma_d^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]+\sigma_d\left[W_d(t)-W_d(t-2)\right]}+1-e^{-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}-e^{-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}\right)+\\ +\frac{e^{\mu_f+2\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(e^{-\frac{1}{2}\sigma_f^2-\sigma_d^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]+\sigma_d\left[W_d(t)-W_d(t-2)\right]}+1-e^{-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}-e^{-\sigma_d^2+\sigma_d\left[W_d(t)-W_d(t-2)\right]}\right)+\\ +\left(1+h_{t-1}\right)\cdot\frac{e^{2\mu_f+\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(e^{-\sigma_f^2-\frac{1}{2}\sigma_d^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}+1-e^{-\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-2)\right]}-e^{-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\right)+\\ +\left(1+h_{t-1}\right)\cdot\frac{e^{\mu_f+\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(e^{-\frac{1}{2}\sigma_f^2-\frac{1}{2}\sigma_d^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}+1-e^{-\frac{1}{2}\sigma_f^2+\sigma_f\left[W_f(t)-W_f(t-1)\right]}-e^{-\frac{1}{2}\sigma_d^2+\sigma_d\left[W_d(t-1)-W_d(t-2)\right]}\right)\end{array}\right]$$

$$\begin{array}{l}=\frac{e^{2\mu_f+2\mu_d}}{p^2\left(x_0,1,t-2\right)}\left[e^{-\sigma_f^2-\sigma_d^2}\left(e^{\sigma_f^2+\sigma_d^2}+e^{\sigma_f^2+\sigma_d^2}\left(e^{\rho_{f,d}}-1\right)\right)+1-e^{-\sigma_f^2+\sigma_f^2}-e^{-\sigma_d^2+\sigma_d^2}\right]+\\ +\frac{e^{\mu_f+2\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left[e^{-\frac{1}{2}\sigma_f^2-\sigma_d^2}\left(e^{\frac{1}{2}\sigma_f^2+\sigma_d^2}+e^{\frac{1}{2}\sigma_f^2+\sigma_d^2}\left(e^{\rho_{f,d}}-1\right)\right)+1-e^{-\frac{1}{2}\sigma_f^2+\frac{1}{2}\sigma_f^2}-e^{-\sigma_d^2+\sigma_d^2}\right]+\\ +\frac{e^{2\mu_f+\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left[e^{-\sigma_f^2-\frac{1}{2}\sigma_d^2}\left(e^{\sigma_f^2+\frac{1}{2}\sigma_d^2}+e^{\sigma_f^2+\frac{1}{2}\sigma_d^2}\left(e^{\rho_{f,d}}-1\right)\right)+1-e^{-\sigma_f^2+\sigma_f^2}-e^{-\frac{1}{2}\sigma_d^2+\frac{1}{2}\sigma_d^2}\right]+\\ +\frac{e^{\mu_f+\mu_d}}{p\left(x_0,1,t-2\right)}\cdot\left(1+h_{t-1}\right)\cdot\left[e^{-\frac{1}{2}\sigma_f^2-\frac{1}{2}\sigma_d^2}\left(e^{\frac{1}{2}\sigma_f^2+\frac{1}{2}\sigma_d^2}+e^{\frac{1}{2}\sigma_f^2+\frac{1}{2}\sigma_d^2}\left(e^{\rho_{f,d}}-1\right)\right)+1-e^{-\frac{1}{2}\sigma_f^2+\frac{1}{2}\sigma_f^2}-e^{-\frac{1}{2}\sigma_d^2+\frac{1}{2}\sigma_d^2}\right]\end{array}$$

$$Cov[X,Y] = \frac{e^{2\mu_f+2\mu_d}}{p^2(x_0,1,t-2)}\left(e^{\rho_{f,d}}-1\right) + \frac{e^{\mu_f+2\mu_d}}{p(x_0,1,t-2)}\cdot\left(e^{\rho_{f,d}}-1\right) +$$
$$+\frac{e^{2\mu_f+\mu_d}}{p(x_0,1,t-2)}\cdot(1+h_{t-1})\cdot\left(e^{\rho_{f,d}}-1\right) + \frac{e^{\mu_f+\mu_d}}{p(x_0,1,t-2)}\cdot(1+h_{t-1})\cdot\left(e^{\rho_{f,d}}-1\right)$$

Appendix C Modelling in a mean reverting framework (3 periods model)

$$F_t = e^{\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]}$$

$F_t = e^A$ $\quad$ F_t is lognormal

$$A = \gamma_f + \rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]$$

$$E[A] = \gamma_f$$

$$Var[A] = E\left\{\left[\rho_f\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\right]^2\right\} =$$

$$= E\left[\rho_f^2\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2\right] =$$

$$= E\left[\rho_f^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2 + \left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2 + \\ -2\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\end{array}\right]\right] =$$

$$= E\left[\rho_f^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2+\\ -2\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)+\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\end{array}\right]\right]=$$

$$= E\left[\begin{array}{l}\rho_f^2\left[\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)^2\right]+\\ -2\left[\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)+\left(\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\right]\end{array}\right]=$$

$$= \rho_f^2\left[\begin{array}{l}\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)^2du+\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)^2du-2\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)du+\\ -2E\left[\left(\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right)\right]\end{array}\right]=$$

$$E\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\cdot\int_{t-1}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right]=0$$

$$= \rho_f^2\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)^2du+\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)^2du-2\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)du\right]=$$

$$= \rho_f^2\left[\int_0^t\frac{1+e^{-2\beta_f(t-u)}-2e^{-\beta_f(t-u)}}{\beta_f^2}du+\int_0^{t-1}\frac{1+e^{-2\beta_f(t-1-u)}-2e^{-\beta_f(t-1-u)}}{\beta_f^2}du-2\int_0^{t-1}\frac{1-e^{-\beta_f(t-u)}-e^{-\beta_f(t-1-u)}+e^{-\beta_f(2t-2u-1)}}{\beta_f^2}du\right]=$$

$$= \frac{\rho_f^2}{\beta_f^2}\left[\left[u+\frac{e^{-2\beta_f(t-u)}}{2\beta_f}-\frac{2e^{-\beta_f(t-u)}}{\beta_f}\right]_0^t+\left[u+\frac{e^{-2\beta_f(t-1-u)}}{2\beta_f}-\frac{2e^{-\beta_f(t-1-u)}}{\beta_f}\right]_0^{t-1}-2\left[u-\frac{e^{-\beta_f(t-u)}}{\beta_f}-\frac{e^{-\beta_f(t-1-u)}}{\beta_f}+\frac{e^{-\beta_f(2t-2u-1)}}{2\beta_f}\right]_0^{t-1}\right]=$$

$$=\frac{\rho_f^2}{\beta_f^2}\left[\begin{array}{l} t+\frac{1}{2\beta_f}-\frac{2}{\beta_f}-\frac{e^{-2\beta_f t}}{2\beta_f}+\frac{2e^{-\beta_f t}}{\beta_f}+t-1+\frac{1}{2\beta_f}-\frac{2}{\beta_f}-\frac{e^{-2\beta_f(t-1)}}{2\beta_f}+\frac{2e^{-\beta_f(t-1)}}{\beta_f}+ \\ -2t+2+\frac{2e^{-\beta_f}}{\beta_f}+\frac{2}{\beta_f}-\frac{e^{-\beta_f}}{\beta_f}-\frac{2e^{-\beta_f t}}{\beta_f}-\frac{2e^{-\beta_f(t-1)}}{\beta_f}+\frac{e^{-\beta_f(2t-1)}}{\beta_f} \end{array}\right]=$$

$$Var\left[A\right]=\frac{\rho_f^2}{2\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f(t-1)}+2e^{-\beta_f}+2e^{-\beta_f(2t-1)}\right]$$

$$F_t=e^A$$

$$E\left[A\right]=a_f \qquad Var\left[A\right]=b_f^2$$

$$a_f=\gamma_f$$

$$b_f^2=\frac{\rho_f^2}{2\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f(t-1)}+2e^{-\beta_f}+2e^{-\beta_f(2t-1)}\right]$$

$$E\left[F_t\right]=e^{a_f+\frac{b_f^2}{2}}$$

$$Var\left[F_t\right]=e^{2a_f+b_f^2}\left(e^{b_f^2}-1\right)$$

$$X=F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}+F_t=\frac{e^{2\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]}}{p(x_0,1,t-2)}+e^{\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]}$$

$$E\left[X\right]=E\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}+F_t\right]=E\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}\right]+E\left[F_t\right]$$

$$E\left[F_t\cdot F_{t-1}\right]=E\left[e^{2\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]}\right]=E\left[e^B\right]$$

with

$$B=2\gamma_f+\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]$$

$$E[B]=2\gamma_f$$

$$Var[B]=E\left\{\left[\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]\right]^2\right\}=$$

$$=E\left\{\rho_f^2\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]^2\right\}=$$

$$=E\left\{\rho_f^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)^2+\\-2\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)\end{array}\right]\right\}=$$

$$=E\left[\rho_f^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)^2+\\-2\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)+\int_{t-2}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)\end{array}\right]\right]=$$

$$=E\left[\begin{array}{l}\rho_f^2\left[\left(\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)^2\right]+\\-2\left[\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)+\left(\int_{t-2}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)\right]\end{array}\right]=$$

$$=\rho_f^2\left[\begin{array}{l}\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)^2du+\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)^2du-2\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)du+\\-2E\left[\left(\int_{t-2}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right)\right]\end{array}\right]=$$

$$E\left[\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\cdot\int_{t-2}^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\right]=0$$

$$=\rho_f^2\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)^2du+\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)^2du-2\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)du\right]=$$

$$=\rho_f^2\left[\int_0^t\frac{1+e^{-2\beta_f(t-u)}-2e^{-\beta_f(t-u)}}{\beta_f^2}du+\int_0^{t-2}\frac{1+e^{-2\beta_f(t-2-u)}-2e^{-\beta_f(t-2-u)}}{\beta_f^2}du-2\int_0^{t-2}\frac{1-e^{-\beta_f(t-u)}-e^{-\beta_f(t-2-u)}+e^{-\beta_f(2t-2u-2)}}{\beta_f^2}du\right]=$$

$$=\frac{\rho_f^2}{\beta_f^2}\left[\left[u+\frac{e^{-2\beta_f(t-u)}}{2\beta_f}-\frac{2e^{-\beta_f(t-u)}}{\beta_f}\right]_0^t+\left[u+\frac{e^{-2\beta_f(t-2-u)}}{2\beta_f}-\frac{2e^{-\beta_f(t-2-u)}}{\beta_f}\right]_0^{t-2}-2\left[u-\frac{e^{-\beta_f(t-u)}}{\beta_f}-\frac{e^{-\beta_f(t-2-u)}}{\beta_f}+\frac{e^{-2\beta_f(t-1-u)}}{2\beta_f}\right]_0^{t-2}\right]=$$

$$=\frac{\rho_f^2}{\beta_f^2}\left[\begin{array}{l}t+\frac{1}{2\beta_f}-\frac{2}{\beta_f}-\frac{e^{-2\beta_f t}}{2\beta_f}+\frac{2e^{-\beta_f t}}{\beta_f}+t-2+\frac{1}{2\beta_f}-\frac{2}{\beta_f}-\frac{e^{-2\beta_f(t-2)}}{2\beta_f}+\frac{2e^{-\beta_f(t-2)}}{\beta_f}+\\-2t+4+\frac{2e^{-2\beta_f}}{\beta_f}+\frac{2}{\beta_f}-\frac{e^{-\beta_f}}{\beta_f}-\frac{2e^{-\beta_f t}}{\beta_f}-\frac{2e^{-\beta_f(t-2)}}{\beta_f}+\frac{e^{-2\beta_f(t-1)}}{\beta_f}\end{array}\right]=$$

$$=\frac{\rho_f^2}{\beta_f^2}\left[-\frac{1}{\beta_f}-\frac{e^{-2\beta_f t}}{2\beta_f}+2-\frac{e^{-2\beta_f(t-2)}}{2\beta_f}+\frac{2e^{-2\beta_f}}{\beta_f}-\frac{e^{-\beta_f}}{\beta_f}+\frac{e^{-2\beta_f(t-1)}}{\beta_f}\right]=$$

$$Var[B]=\frac{\rho_f^2}{2\beta_f^3}\left[-2-e^{-2\beta_f t}+4\beta_f-e^{-2\beta_f(t-2)}+4e^{-2\beta_f}-2e^{-\beta_f t}+2e^{-2\beta_f(t-1)}\right]$$

$$E[X]=\frac{e^{2\gamma_f+\frac{\rho_f^2}{4\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f t}+4\beta_f-e^{-2\beta_f(t-2)}+4e^{-2\beta_f}-2e^{-\beta_f t}+2e^{-\beta_f(t-1)}\right]}}{p(x_0,1,t-2)}+e^{\gamma_f+\frac{\rho_f^2}{4\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f(t-1)}+2e^{-\beta_f}+2e^{-\beta_f(2t-1)}\right]}$$

$$Var[X]=Var\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}+F_t\right]=Var\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}\right]+Var[F_t]+2Cov\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)},F_t\right]$$

$$Var\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)}\right]=\frac{e^{4\gamma_f+Var[B]}\cdot\left(e^{Var[B]}-1\right)}{p^2(x_0,1,t-2)}$$

$$Var[F_t]=e^{2\gamma_f+Var[A]}\cdot\left(e^{Var[A]}-1\right)$$

$Cov\left[F_t\cdot\frac{F_{t-1}}{p(x_0,1,t-2)},F\right]$ is the covariance of 2 lognormal variables

$$Cov\left[F_t \cdot F_{t-1}, F_t\right] = e^{\left[E[A]+E[B]+\frac{1}{2}(Var[A]+Var[B])\right]} \cdot \left(e^{b_{A,B}} - 1\right)$$

$$b_{A,B} = E\left[\begin{array}{l}\left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right) dW_f(u)\right] \cdot \\ \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right) dW_f(u)\right]\end{array}\right]$$

$$b_{A,B} = \frac{1}{2\beta_f^3}\left[-1 - e^{-2\beta_f \cdot t} + 6e^{-\beta_f \cdot t} + 2\beta_f - e^{\beta_f} + 2e^{-\beta_f} - 2e^{-\beta_f(2t-1)} + 2e^{-\beta_f(t-1)}\right]$$

$$Var\left[X\right] = Var\left[F_t \cdot \frac{F_{t-1}}{p(x_0,1,t-2)} + F_t\right] = Var\left[F_t \cdot \frac{F_{t-1}}{p(x_0,1,t-2)}\right] + Var\left[F_t\right] + 2Cov\left[F_t \cdot \frac{F_{t-1}}{p(x_0,1,t-2)}, F_t\right]$$

$$Var\left[X\right] = \frac{e^{4\gamma_f + Var[B]} \cdot \left(e^{Var[B]} - 1\right)}{p^2\left(x_0,1,t-2\right)} + e^{2\gamma_f + Var[A]} \cdot \left(e^{Var[A]} - 1\right) + e^{3\gamma_f + \frac{1}{2}(Var[A]+Var[B])} \cdot \left(e^{b_{A,B}} - 1\right)$$

$$D_t = e^{\gamma_d + \rho_d\left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right) dW_d(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right) dW_d(u)\right]}$$

$$D_{t-1} = e^{\gamma_d + \rho_d\left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right) dW_d(u)\right]}$$

$$Y = \frac{D_t \cdot D_{t-1}}{p(x_0,1,t-2)} + D_{t-1} \cdot \left(1 + h_{t-1}\right)$$

$$Y = \frac{e^{2\gamma_d + \rho_d\left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right) dW_d(u)\right]}}{p(x_0,1,t-2)} + e^{\gamma_d + \rho_d\left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right) dW_d(u)\right]}$$

$$E\left[D_{t-1}\right]=E\left[e^{C}\right]=e^{E\left[C\right]+\frac{Var\left[C\right]}{2}}$$

$$C=\gamma_d+\rho_d\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]$$

$$E[C]=\gamma_d$$

$$Var[C]=E\left\{\left[\rho_d\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_2}\right)dW_d(u)\right)\right]^2\right\}=$$

$$=E\left[\rho_d^2\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)^2\right]=$$

$$=E\left[\rho_d^2\left[\begin{array}{l}\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)^2+\\-2\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)\end{array}\right]\right]=$$

$$=E\left[\rho_d^2\left[\begin{array}{l}\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)^2+\\-2\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_d}\right)dW_d(u)+\int_{t-2}^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)\end{array}\right]\right]=$$

$$
= E\left[\begin{array}{l}\rho_d^2\left[\left(\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)^2\right]+\\ -2\left[\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)+\left(\int_{t-2}^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)\right]\end{array}\right]=
$$

$$
=\rho_d^2\left[\begin{array}{l}\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)^2du+\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)^2du-2\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)du+\\ -2E\left[\left(\int_{t-2}^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)\right]\end{array}\right]=
$$

$$
E\left[\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\cdot\int_{t-2}^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)\right]=0
$$

$$
=\rho_d^2\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)^2du+\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)^2du-2\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)du\right]=
$$

$$
=\rho_d^2\left[\int_0^{t-1}\frac{1+e^{-2\beta_d(t-1-u)}-2e^{-\beta_d(t-1-u)}}{\beta_d^2}du+\int_0^{t-2}\frac{1+e^{-2\beta_d(t-2-u)}-2e^{-\beta_d(t-2-u)}}{\beta_d^2}du-2\int_0^{t-2}\frac{1-e^{-\beta_d(t-1-u)}-e^{-\beta_d(t-2-u)}+e^{-\beta_d(2t-2u-3)}}{\beta_d^2}du\right]=
$$

$$
=\frac{\rho_d^2}{\beta_d^2}\left[\left[u+\frac{e^{-2\beta_d(t-1-u)}}{2\beta_d}-\frac{2e^{-\beta_d(t-1-u)}}{\beta_d}\right]_0^{t-1}+\left[u+\frac{e^{-2\beta_d(t-2-u)}}{2\beta_d}-\frac{2e^{-\beta_d(t-2-u)}}{\beta_d}\right]_0^{t-2}-2\left[u-\frac{e^{-\beta_d(t-1-u)}}{\beta_d}-\frac{e^{-\beta_d(t-2-u)}}{\beta_d}+\frac{e^{-\beta_d(2t-2u-3)}}{2\beta_d}\right]_0^{t-2}\right]=
$$

$$
=\frac{\rho_d^2}{\beta_d^2}\left[\begin{array}{l}t-1+\frac{1}{2\beta_d}-\frac{2}{\beta_d}-\frac{e^{-2\beta_f(t-1)}}{2\beta_d}+\frac{2e^{-\beta_f t(t-1)}}{\beta_d}+t-2+\frac{1}{2\beta_d}-\frac{2}{\beta_d}-\frac{e^{-2\beta_f(t-2)}}{2\beta_d}+\frac{2e^{-\beta_f(t-2)}}{\beta_d}+\\ -2t+4+\frac{2e^{-\beta_f}}{\beta_d}+\frac{2}{\beta_d}-\frac{e^{-\beta_f}}{\beta_d}-\frac{2e^{-\beta_f t}}{\beta_d}-\frac{2e^{-\beta_f(t-1)}}{\beta_d}+\frac{e^{-\beta_f(2t-1)}}{\beta_d}\end{array}\right]=
$$

$$
=\frac{\rho_d^2}{\beta_d^2}\left[\begin{array}{l}1-\frac{1}{\beta_d}-\frac{e^{-2\beta_d(t-1)}}{2\beta_d}+\frac{2e^{-\beta_d t(t-1)}}{\beta_d}-\frac{e^{-2\beta_d(t-2)}}{2\beta_d}+\frac{2e^{-\beta_d(t-2)}}{\beta_d}+\\ +\frac{e^{-\beta_d}}{\beta_d}-\frac{2e^{-\beta_d t}}{\beta_d}-\frac{2e^{-\beta_d(t-1)}}{\beta_d}+\frac{e^{-\beta_d(2t-1)}}{\beta_d}\end{array}\right]=
$$

$$
Var[C]=\frac{\rho_d^2}{2\beta_d^3}\left[2\beta_d-2-e^{-2\beta_d(t-1)}+4e^{-\beta_d t(t-1)}-e^{-2\beta_d(t-2)}+4e^{-\beta_d(t-2)}+2e^{-\beta_d}-4e^{-\beta_d t}-4e^{-\beta_d(t-1)}+2e^{-\beta_d(2t-1)}\right]
$$

$$E\left[D_t \cdot D_{t-1}\right] = E\left[e^{2\gamma_d+\rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]}\right] = E\left[e^L\right]$$

with

$$L = 2\gamma_d + \rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]$$

$$E[L] = 2\gamma_d$$

$$Var[L] = E\left\{\left[\rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\right]^2\right\} =$$

$$= E\left\{\rho_d^2\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]^2\right\} =$$

$$= E\left\{\rho_d^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)^2+\\-2\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)\end{array}\right]\right\} =$$

$$= E\left[\rho_d^2\left[\begin{array}{l}\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)^2+\\-2\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)+\int_{t-2}^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)\end{array}\right]\right] =$$

$$= E\left[\begin{array}{l}\rho_d^2\left[\left(\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)\right)^2+\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)^2\right]+\\-2\left[\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)+\left(\int_{t-2}^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)\right)\left(\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right)\right]\end{array}\right] =$$

$$
= \rho_d^2 \left[\begin{array}{l} \int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right)^2 du + \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right)^2 du - 2\int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right)\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) du + \\ -2E\left[\left(\int_{t-2}^{t} \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) \right)\left(\int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right) \right] \end{array} \right] =
$$

$$
E\left[\int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \cdot \int_{t-2}^{t} \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) \right] = 0
$$

$$
= \rho_d^2 \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right)^2 du + \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right)^2 du - 2\int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right)\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) du \right] =
$$

$$
= \rho_d^2 \left[\int_0^t \frac{1+e^{-2\beta_d(t-u)} - 2e^{-\beta_d(t-u)}}{\beta_d^2} du + \int_0^{t-2} \frac{1+e^{-2\beta_d(t-2-u)} - 2e^{-\beta_d(t-2-u)}}{\beta_d^2} du - 2\int_0^{t-2} \frac{1-e^{-\beta_d(t-u)} - e^{-\beta_d(t-2-u)} + e^{-\beta_d(2t-2u-2)}}{\beta_d^2} du \right] =
$$

$$
= \frac{\rho_d^2}{\beta_d^2} \left[\left[u + \frac{e^{-2\beta_d(t-u)}}{2\beta_d} - \frac{2e^{-\beta_d(t-u)}}{\beta_d} \right]_0^t + \left[u + \frac{e^{-2\beta_d(t-2-u)}}{2\beta_d} - \frac{2e^{-\beta_d(t-2-u)}}{\beta_d} \right]_0^{t-2} - 2\left[u - \frac{e^{-\beta_d(t-u)}}{\beta_d} - \frac{e^{-\beta_d(t-2-u)}}{\beta_d} + \frac{e^{-2\beta_d(t-1-u)}}{2\beta_d} \right]_0^{t-2} \right] =
$$

$$
= \frac{\rho_d^2}{\beta_d^2} \left[\begin{array}{l} t + \frac{1}{2\beta_d} - \frac{2}{\beta_d} - \frac{e^{-2\beta_d t}}{2\beta_d} + \frac{2e^{-\beta_d t}}{\beta_d} + t - 2 + \frac{1}{2\beta_d} - \frac{2}{\beta_d} - \frac{e^{-2\beta_d(t-2)}}{2\beta_d} + \frac{2e^{-\beta_d(t-2)}}{\beta_d} + \\ -2t + 4 + \frac{2e^{-2\beta_d}}{\beta_d} + \frac{2}{\beta_d} - \frac{e^{-\beta_d t}}{\beta_d} - \frac{2e^{-\beta_d t}}{\beta_d} - \frac{2e^{-\beta_d(t-2)}}{\beta_d} + \frac{e^{-2\beta_d(t-1)}}{\beta_d} \end{array} \right] =
$$

$$
= \frac{\rho_d^2}{\beta_d^2} \left[-\frac{1}{\beta_d} - \frac{e^{-2\beta_d t}}{2\beta_d} + 2 - \frac{e^{-2\beta_d(t-2)}}{2\beta_d} + \frac{2e^{-2\beta_d}}{\beta_d} - \frac{e^{-\beta_d t}}{\beta_d} + \frac{e^{-2\beta_d(t-1)}}{\beta_d} \right] =
$$

$$
Var[L] = \frac{\rho_d^2}{2\beta_d^3} \left[-2 - e^{-2\beta_d t} + 4\beta_d - e^{-2\beta_d(t-2)} + 4e^{-2\beta_d} - 2e^{-\beta_d t} + 2e^{-2\beta_d(t-1)} \right]
$$

$$
E[Y] = \frac{e^{2\gamma_d + \frac{\rho_d^2}{4\beta_d^3}\left[-2 - e^{-2\beta_d t} + 4\beta_d - e^{-2\beta_d(t-2)} + 4e^{-2\beta_d} - 2e^{-\beta_d t} + 2e^{-2\beta_d(t-1)} \right]}}{p(x_0, 1, t-2)} +
$$

$$
+ e^{\gamma_d + \frac{\rho_d^2}{4\beta_d^3}\left[2\beta_d - 2 - e^{-2\beta_d(t-1)} + 4e^{-\beta_d t(t-1)} - e^{-2\beta_d(t-2)} + 4e^{-\beta_d(t-2)} + 2e^{-\beta_d} - 4e^{-\beta_d t} - 4e^{-\beta_d(t-1)} + 2e^{-\beta_d(2t-1)} \right]} \left(1 + h_{t-1}\right)
$$

$$Var\left[Y\right]=Var\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)}+D_{t-1}\left(1+h_{t-1}\right)\right]=$$

$$=Var\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)}\right]+Var\left[D_{t-1}\left(1+h_{t-1}\right)\right]+2Cov\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)},D_{t-1}\left(1+h_{t-1}\right)\right]$$

$$Var\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)}\right]=\frac{e^{4\gamma_d+Var\left[L\right]}\left(e^{Var\left[L\right]}-1\right)}{p^2\left(x_0,1,t-2\right)}$$

$$Var\left[D_{t-1}\left(1+h_{t-1}\right)\right]=e^{2\gamma_d+Var\left[C\right]}\cdot\left(e^{Var\left[C\right]}-1\right)\cdot\left(1+h_{t-1}\right)^2$$

$Cov\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)},D_{t-1}\left(1+h_{t-1}\right)\right]$ is the covariance of 2 lognormal variables

$$Cov\left[D_t\cdot D_{t-1},D_{t-1}\right]=e^{\left[E[C]+E[L]+\frac{1}{2}\left(Var[C]+Var[L]\right)\right]}\cdot\left(e^{b_{C,L}}-1\right)$$

$$b_{C,L}=E\left[\begin{array}{l}\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\cdot\\\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\end{array}\right]$$

$$b_{C,L}=\frac{1}{2\beta_d^3}\left[2\beta_d-2e^{-\beta_d\cdot t}+e^{-\beta_d(2t-1)}+3e^{-2\beta_d}-4e^{-\beta_d(t-1)}-4e^{-\beta_d(t-2)}+3e^{-\beta_d(2t-3)}\right]$$

$$Var\left[Y\right]=Var\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)}+D_{t-1}\left(1+h_{t-1}\right)\right]=$$
$$=Var\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)}\right]+Var\left[D_{t-1}\left(1+h_{t-1}\right)\right]+2Cov\left[D_t\cdot\frac{D_{t-1}}{p\left(x_0,1,t-2\right)},D_{t-1}\left(1+h_{t-1}\right)\right]$$

$$Var\left[Y\right]=\frac{e^{4\gamma_d+Var\left[L\right]}\cdot\left(e^{Var\left[L\right]}-1\right)}{p^2\left(x_0,1,t-2\right)}+e^{2\gamma_d+Var\left[C\right]}\cdot\left(e^{Var\left[C\right]}-1\right)\left(1+h_{t-1}\right)^2+e^{3\gamma_d+\frac{1}{2}\left(Var\left[C\right]+Var\left[L\right]\right)}\cdot\left(e^{b_{C,L}}-1\right)$$

$$X = F_t \cdot \frac{F_{t-1}}{p(x_0,1,t-2)} + F_t = \frac{e^{2\gamma_f + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right]}}{p(x_0,1,t-2)} + e^{\gamma_f + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f} \right) dW_f(u) \right]}$$

$$Y = D_t \frac{D_{t-1}}{p(x_0,1,t-2)} + D_{t-1} \cdot (1+h_{t-1})$$

$$Y = \frac{e^{2\gamma_d + \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right]}}{p(x_0,1,t-2)} + e^{\gamma_d + \rho_d \left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right]} \cdot (1+h_{t-1})$$

$$X = \frac{F_t \cdot F_{t-1}}{p(x_0,1,t-2)} + F_t \qquad\qquad Y = \frac{D_t \cdot D_{t-1}}{p(x_0,1,t-2)} + D_{t-1} \cdot (1+h_{t-1})$$

$$Cov[X,Y] = E[X \cdot Y] - E[X] \cdot E[Y]$$

$$E[X \cdot Y] = \frac{F_t \cdot F_{t-1} \cdot D_t \cdot D_{t-1}}{p^2(x_0,1,t-2)} + \frac{F_t \cdot F_{t-1}}{p(x_0,1,t-2)} \cdot D_{t-1} \cdot (1+h_{t-1}) + \frac{F_t \cdot D_t \cdot D_{t-1}}{p(x_0,1,t-2)} + F_t \cdot D_{t-1} \cdot (1+h_{t-1})$$

$$F_t \cdot F_{t-1} \cdot D_t \cdot D_{t-1} = e^{2\gamma_f + 2\gamma_d + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] + \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right]}$$

$$F_t \cdot F_{t-1} \cdot D_t \cdot D_{t-1} = e^M$$

$$M = 2\gamma_f + 2\gamma_d + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] + \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right]$$

$$E[M] = 2\gamma_f + 2\gamma_d$$

$$Var[M] = E\left\{ \left[\rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] + \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] \right]^2 \right\}$$

$$
= E\left\{ \begin{array}{l} \rho_f^2 \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right]^2 + \rho_d^2 \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] + \\ +2\rho_f \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] \end{array} \right\}
$$

$$
E\left[\left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] \right] =
$$

$$
= E\left[\begin{array}{l} \int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) + \\ -\int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) + \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \end{array} \right]
$$

$$
= \int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) \rho_{f,d} du - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) \rho_{f,d} du +
$$

$$
-\int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) \rho_{f,d} du + \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) \rho_{f,d} du
$$

$$
= \int_0^t \frac{1-e^{-\beta_f(t-u)} - e^{-\beta_d(t-u)} + e^{-(\beta_f+\beta_d)(t-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du - \int_0^{t-2} \frac{1-e^{-\beta_f(t-u)} - e^{-\beta_d(t-2-u)} + e^{-\beta_f(t-u)-\beta_d(t-2-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du +
$$

$$
-\int_0^{t-2} \frac{1-e^{-\beta_f(t-2-u)} - e^{-\beta_d(t-u)} + e^{-\beta_f(t-2-u)-\beta_d(t-2-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du + \int_0^{t-2} \frac{1-e^{-\beta_f(t-2-u)} - e^{-\beta_d(t-2-u)} + e^{-(\beta_f+\beta_d)(t-2-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du
$$

$$
= \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-u)}}{\beta_f} - \frac{e^{-\beta_d(t-u)}}{\beta_d} + \frac{e^{-(\beta_f+\beta_d)(t-u)}}{\beta_f+\beta_d} \right]_0^t - \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-u)}}{\beta_f} - \frac{e^{-\beta_d(t-2-u)}}{\beta_d} + \frac{e^{-\beta_f(t-u)-\beta_d(t-2-u)}}{\beta_f+\beta_d} \right]_0^{t-2} +
$$

$$
-\frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-2-u)}}{\beta_f} - \frac{e^{-\beta_d(t-u)}}{\beta_d} + \frac{e^{-\beta_f(t-2-u)-\beta_d(t-u)}}{\beta_f+\beta_d} \right]_0^{t-2} - \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-2-u)}}{\beta_f} - \frac{e^{-\beta_d(t-2-u)}}{\beta_d} + \frac{e^{-(\beta_f+\beta_d)(t-2-u)}}{\beta_f+\beta_d} \right]_0^{t-2}
$$

$$
= \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[\begin{array}{l} \frac{2}{\beta_f+\beta_d} - \frac{e^{-(\beta_f+\beta_d)\cdot t}}{\beta_f+\beta_d} + \frac{e^{-2\beta_f}}{\beta_f} + \frac{e^{-2\beta_f}}{\beta_f+\beta_d} + \frac{e^{-\beta_f \cdot t-\beta_d(t-2)}}{\beta_f+\beta_d} + 2 + \frac{e^{-2\beta_d}}{\beta_d} \frac{e^{-2\beta_d}}{\beta_f+\beta_d} + \\ + \frac{e^{-\beta_f(t-2)-\beta_d \cdot t}}{\beta_f+\beta_d} - \frac{1}{\beta_f} - \frac{1}{\beta_d} - \frac{e^{-(\beta_f+\beta_d)(t-2)}}{\beta_f+\beta_d} \end{array} \right]
$$

$$Var[M] = E\left\{ \begin{array}{l} \rho_f^2 \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right]^2 + \rho_d^2 \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] + \\ +2\rho_f \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] \left[\int_0^t \left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] \end{array} \right\}$$

$$M = B + L$$

$$Var[M] = Var[B] + Var[L] + 2\,\mathrm{cov}[B, L]$$

$$\begin{aligned} Var[M] = &\frac{\rho_f^2}{\beta_f^3}\left[-2 - e^{-2\beta_f t} + 4\beta_f - e^{-2\beta_f(t-2)} + 4e^{-2\beta_f} - 2e^{-\beta_f t} + 2e^{-2\beta_f(t-1)} \right] + \\ &+ \frac{\rho_d^2}{2\beta_d^3}\left[-2 - e^{-2\beta_d t} + 4\beta_d - e^{-2\beta_d(t-2)} + 4e^{-2\beta_d} - 2e^{-\beta_d t} + 2e^{-2\beta_d(t-1)} \right] + \\ &+ \frac{2\rho_f \rho_d \rho_{f,d}}{\beta_f \cdot \beta_d} \left[\begin{array}{l} \frac{2}{\beta_f + \beta_d} - \frac{e^{-(\beta_f+\beta_d)\cdot t}}{\beta_f + \beta_d} + \frac{e^{-2\beta_f}}{\beta_f} + \frac{e^{-2\beta_f}}{\beta_f + \beta_d} + \frac{e^{-\beta_f \cdot t - \beta_d(t-2)}}{\beta_f + \beta_d} + 2 + \frac{e^{-2\beta_d}}{\beta_d} \frac{e^{-2\beta_d}}{\beta_f + \beta_d} + \\ + \frac{e^{-\beta_f(t-2) - \beta_d \cdot t}}{\beta_f + \beta_d} - \frac{1}{\beta_f} - \frac{1}{\beta_d} - \frac{e^{-(\beta_f+\beta_d)(t-2)}}{\beta_f + \beta_d} \end{array} \right] \end{aligned}$$

$$F_t \cdot F_{t-1} \cdot D_{t-1} = e^{2\gamma_f + \gamma_d + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] + \rho_d \left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right]}$$

$$F_t \cdot F_{t-1} \cdot D_{t-1} = e^N$$

$$N = 2\gamma_f + \gamma_d + \rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] + \rho_d \left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right]$$

$$E[N] = 2\gamma_f + \gamma_d$$

$$Var[N] = E\left\{ \left[\rho_f \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] + \rho_d \left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] \right]^2 \right\}$$

$$
= E\left\{ \begin{array}{l} \rho_f^2 \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right]^2 + \rho_d^2 \left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] + \\ +2\rho_f \rho_d \left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] \left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] \end{array} \right\}
$$

$$
E\left[\left[\int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \right] \left[\int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \right] \right] =
$$

$$
= E\left[\begin{array}{l} \int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) - \int_0^t \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) + \\ - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^{t-1} \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) dW_d(u) + \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) dW_f(u) \cdot \int_0^{t-2} \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) dW_d(u) \end{array} \right]
$$

$$
= \int_0^{t-1} \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) \rho_{f,d} du - \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) \rho_{f,d} du +
$$

$$
- \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d} \right) \rho_{f,d} du + \int_0^{t-2} \left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f} \right) \cdot \left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d} \right) \rho_{f,d} du
$$

$$
= \int_0^{t-1} \frac{1-e^{-\beta_f(t-u)} - e^{-\beta_d(t-1-u)} + e^{-\beta_f(t-u)-\beta_d(t-1-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du - \int_0^{t-2} \frac{1-e^{-\beta_f(t-u)} - e^{-\beta_d(t-2-u)} + e^{-\beta_f(t-u)-\beta_d(t-2-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du +
$$

$$
- \int_0^{t-2} \frac{1-e^{-\beta_f(t-2-u)} - e^{-\beta_d(t-1-u)} + e^{-\beta_f(t-2-u)-\beta_d(t-1-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du + \int_0^{t-2} \frac{1-e^{-\beta_f(t-2-u)} - e^{-\beta_d(t-2-u)} + e^{-(\beta_f+\beta_d)(t-2-u)}}{\beta_f \cdot \beta_d} \rho_{f,d} du
$$

$$
= \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-u)}}{\beta_f} - \frac{e^{-\beta_d(t-1-u)}}{\beta_d} + \frac{e^{-\beta_f(t-u)-\beta_d(t-1-u)}}{\beta_f+\beta_d} \right]_0^{t-1} - \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-u)}}{\beta_f} - \frac{e^{-\beta_d(t-2-u)}}{\beta_d} + \frac{e^{-\beta_f(t-u)-\beta_d(t-2-u)}}{\beta_f+\beta_d} \right]_0^{t-2} +
$$

$$
- \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-2-u)}}{\beta_f} - \frac{e^{-\beta_d(t-1-u)}}{\beta_d} + \frac{e^{-\beta_f(t-2-u)-\beta_d(t-1-u)}}{\beta_f+\beta_d} \right]_0^{t-2} + \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[u - \frac{e^{-\beta_f(t-2-u)}}{\beta_f} - \frac{e^{-\beta_d(t-2-u)}}{\beta_d} + \frac{e^{-(\beta_f+\beta_d)(t-2-u)}}{\beta_f+\beta_d} \right]_0^{t-2}
$$

$$
= \frac{\rho_{f,d}}{\beta_f \cdot \beta_d} \left[\begin{array}{l} 1 - \frac{e^{-\beta_f}}{\beta_f} \frac{e^{-\beta_f}}{\beta_f+\beta_d} - \frac{e^{-\beta_f \cdot t-\beta_d(t-1)}}{\beta_f+\beta_d} + \frac{e^{-2\beta_f}}{\beta_f} + \frac{e^{-2\beta_f}}{\beta_f+\beta_d} + \frac{e^{-\beta_f \cdot t-\beta_d(t-2)}}{\beta_f+\beta_d} + \frac{e^{-\beta_d}}{\beta_d} \frac{e^{-\beta_d}}{\beta_f+\beta_d} + \\ + \frac{e^{-\beta_f(t-2)-\beta_d(t-1)}}{\beta_f+\beta_d} - \frac{1}{\beta_d} + \frac{1}{\beta_f+\beta_d} - \frac{e^{-(\beta_f+\beta_d)(t-2)}}{\beta_f+\beta_d} \end{array} \right]
$$

$$N = B + C$$

$$Var[N] = Var[B] + Var[C] + 2\operatorname{cov}[B, C]$$

$$Var[N] == \frac{\rho_f^2}{\beta_f^3}\left[-2 - e^{-2\beta_f t} + 4\beta_f - e^{-2\beta_f(t-2)} + 4e^{-2\beta_f} - 2e^{-\beta_f t} + 2e^{-2\beta_f(t-1)}\right] +$$

$$+\frac{\rho_d^2}{2\beta_d^3}\left[2\beta_d - 2 - e^{-2\beta_d(t-1)} + 4e^{-\beta_d t(t-1)} - e^{-2\beta_d(t-2)} + 4e^{-\beta_d(t-2)} + 2e^{-\beta_d} - 4e^{-\beta_d t} - 4e^{-\beta_d(t-1)} + 2e^{-\beta_d(2t-1)}\right] +$$

$$+\frac{2\rho_f\rho_d\rho_{f,d}}{\beta_f \cdot \beta_d}\left[\begin{array}{l} 1 - \frac{e^{-\beta_f}}{\beta_f}\frac{e^{-\beta_f}}{\beta_f + \beta_d} - \frac{e^{-\beta_f \cdot t - \beta_d(t-1)}}{\beta_f + \beta_d} + \frac{e^{-2\beta_f}}{\beta_f} + \frac{e^{-2\beta_f}}{\beta_f + \beta_d} + \frac{e^{-\beta_f \cdot t - \beta_d(t-2)}}{\beta_f + \beta_d} + \frac{e^{-\beta_d}}{\beta_d}\frac{e^{-\beta_d}}{\beta_f + \beta_d} + \\ + \frac{e^{-\beta_f(t-2) - \beta_d(t-1)}}{\beta_f + \beta_d} - \frac{1}{\beta_d} + \frac{1}{\beta_f + \beta_d} - \frac{e^{-(\beta_f + \beta_d)(t-2)}}{\beta_f + \beta_d} \end{array}\right]$$

$$F_t \cdot D_t \cdot D_{t-1} = e^{\gamma_f + 2\gamma_d + \rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right] + \rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u) - \int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]}$$

$$F_t \cdot D_t \cdot D_{t-1} = e^O$$

$$O = \gamma_f + 2\gamma_d + \rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right] + \rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u) - \int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]$$

$$E[O] = \gamma_f + 2\gamma_d$$

$$Var[O] = E\left\{\left[\rho_f\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right] + \rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u) - \int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\right]^2\right\}$$

$$= E\left\{\begin{array}{l} \rho_f^2\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]^2 + \rho_d^2\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u) - \int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right] + \\ +2\rho_f\rho_d\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u) - \int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u) - \int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right] \end{array}\right\}$$

$$E\left[\left[\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\right]\left[\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\right]=$$

$$=E\left[\begin{array}{l}\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)-\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)+\\-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^t\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)dW_d(u)+\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\end{array}\right]$$

$$=\int_0^t\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)\rho_{f,d}du-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)\rho_{f,d}du+$$
$$-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-u)}}{\beta_d}\right)\rho_{f,d}du+\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)\rho_{f,d}du$$

$$=\int_0^t\frac{1-e^{-\beta_f(t-u)}-e^{-\beta_d(t-u)}+e^{-(\beta_f+\beta_d)(t-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du-\int_0^{t-2}\frac{1-e^{-\beta_f(t-u)}-e^{-\beta_d(t-2-u)}+e^{-\beta_f(t-u)-\beta_d(t-2-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du+$$
$$-\int_0^{t-1}\frac{1-e^{-\beta_f(t-1-u)}-e^{-\beta_d(t-u)}+e^{-\beta_f(t-1-u)-\beta_d(t-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du+\int_0^{t-2}\frac{1-e^{-\beta_f(t-1-u)}-e^{-\beta_d(t-2-u)}+e^{-\beta_f(t-1-u)-\beta_d(t-2-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du$$

$$=\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-u)}}{\beta_f}-\frac{e^{-\beta_d(t-u)}}{\beta_d}+\frac{e^{-(\beta_f+\beta_d)(t-u)}}{\beta_f+\beta_d}\right]_0^t-\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-u)}}{\beta_f}-\frac{e^{-\beta_d(t-2-u)}}{\beta_d}+\frac{e^{-\beta_f(t-u)-\beta_d(t-2-u)}}{\beta_f+\beta_d}\right]_0^{t-2}+$$
$$-\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-1-u)}}{\beta_f}-\frac{e^{-\beta_d(t-u)}}{\beta_d}+\frac{e^{-\beta_f(t-1-u)-\beta_d(t-u)}}{\beta_f+\beta_d}\right]_0^{t-1}+\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-1-u)}}{\beta_f}-\frac{e^{-\beta_d(t-2-u)}}{\beta_d}+\frac{e^{-\beta_f(t-1-u)-\beta_d(t-2-u)}}{\beta_f+\beta_d}\right]_0^{t-2}$$

$$=\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[\begin{array}{l}\frac{1}{\beta_f+\beta_d}-\frac{e^{-(\beta_f+\beta_d)\cdot t}}{\beta_f+\beta_d}+\frac{e^{-2\beta_f}}{\beta_f}-\frac{e^{-2\beta_f}}{\beta_f+\beta_d}+\frac{e^{-\beta_f\cdot t-\beta_d(t-2)}}{\beta_f+\beta_d}+1+\frac{e^{-\beta_d}}{\beta_d}-+\frac{e^{-\beta_d}}{\beta_f+\beta_d}+\frac{e^{-\beta_f(t-1)-\beta_d\cdot t}}{\beta_f+\beta_d}+\\-\frac{e^{-\beta_f}}{\beta_f}-\frac{1}{\beta_d}+\frac{e^{-\beta_f}}{\beta_f+\beta_d}-\frac{e^{-\beta_f(t-1)-\beta_d(t-2)}}{\beta_f+\beta_d}\end{array}\right]$$

$$O=A+L$$

$$Var[O]=Var[A]+Var[L]+2\operatorname{cov}[A,L]$$

$$Var[O]=\frac{\rho_f^2}{2\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f(t-1)}+2e^{-\beta_f}+2e^{-\beta_f(2t-1)}\right]+$$

$$+\frac{\rho_d^2}{2\beta_d^3}\left[-2-e^{-2\beta_d t}+4\beta_d-e^{-2\beta_d(t-2)}+4e^{-2\beta_d}-2e^{-\beta_d t}+2e^{-2\beta_d(t-1)}\right]+$$

$$+\frac{2\rho_f\rho_d\rho_{f,d}}{\beta_f\cdot\beta_d}=\left[\begin{array}{l}\frac{1}{\beta_f+\beta_d}-\frac{e^{-(\beta_f+\beta_d)\cdot t}}{\beta_f+\beta_d}+\frac{e^{-2\beta_f}}{\beta_f}-\frac{e^{-2\beta_f}}{\beta_f+\beta_d}+\frac{e^{-\beta_f\cdot t-\beta_d(t-2)}}{\beta_f+\beta_d}+1+\frac{e^{-\beta_d}}{\beta_d}-+\frac{e^{-\beta_d}}{\beta_f+\beta_d}+\frac{e^{-\beta_f(t-1)-\beta_d\cdot t}}{\beta_f+\beta_d}+\\ -\frac{e^{-\beta_f}}{\beta_f}-\frac{1}{\beta_d}+\frac{e^{-\beta_f}}{\beta_f+\beta_d}-\frac{e^{-\beta_f(t-1)-\beta_d(t-2)}}{\beta_f+\beta_d}\end{array}\right]$$

$$F_{t-1}\cdot D_{t-1}=e^{\gamma_f+\gamma_d+\rho_f\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]+\rho_d\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]}$$

$$F_{t-1}\cdot D_{t-1}=e^P$$

$$P=\gamma_f+\gamma_d+\rho_f\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]+\rho_d\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]$$

$$E[P]=\gamma_f+\gamma_d$$

$$Var[P]=E\left\{\left[\rho_f\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]+\rho_d\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\right]^2\right\}$$

$$=E\left\{\begin{array}{l}\rho_f^2\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]^2+\rho_d^2\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]+\\ +2\rho_f\rho_d\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\end{array}\right\}$$

$$E\left[\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\right]\left[\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\right]\right]=$$

$$=E\left[\begin{array}{l}\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)-\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)+\\-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^{t-1}\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)dW_d(u)+\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)dW_f(u)\cdot\int_0^{t-2}\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)dW_d(u)\end{array}\right]$$

$$=\int_0^{t-1}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)\rho_{f,d}du-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-1-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)\rho_{f,d}du+$$
$$-\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-1-u)}}{\beta_d}\right)\rho_{f,d}du+\int_0^{t-2}\left(\frac{1-e^{-\beta_f(t-2-u)}}{\beta_f}\right)\cdot\left(\frac{1-e^{-\beta_d(t-2-u)}}{\beta_d}\right)\rho_{f,d}du$$

$$=\int_0^{t-1}\frac{1-e^{-\beta_f(t-1-u)}-e^{-\beta_d(t-1-u)}+e^{-(\beta_f+\beta_d)(t-1-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du-\int_0^{t-2}\frac{1-e^{-\beta_f(t-1-u)}-e^{-\beta_d(t-2-u)}+e^{-\beta_f(t-1-u)-\beta_d(t-2-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du+$$
$$-\int_0^{t-2}\frac{1-e^{-\beta_f(t-2-u)}-e^{-\beta_d(t-1-u)}+e^{-\beta_f(t-2-u)-\beta_d(t-1-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du+\int_0^{t-2}\frac{1-e^{-\beta_f(t-2-u)}-e^{-\beta_d(t-2-u)}+e^{-(\beta_f+\beta_d)(t-2-u)}}{\beta_f\cdot\beta_d}\rho_{f,d}du$$

$$=\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-1-u)}}{\beta_f}-\frac{e^{-\beta_d(t-1-u)}}{\beta_d}+\frac{e^{-(\beta_f+\beta_d)(t-1-u)}}{\beta_f+\beta_d}\right]_0^{t-1}-\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-1-u)}}{\beta_f}-\frac{e^{-\beta_d(t-2-u)}}{\beta_d}+\frac{e^{-\beta_f(t-1-u)-\beta_d(t-2-u)}}{\beta_f+\beta_d}\right]_0^{t-2}+$$
$$-\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-2-u)}}{\beta_f}-\frac{e^{-\beta_d(t-1-u)}}{\beta_d}+\frac{e^{-\beta_f(t-2-u)-\beta_d(t-1-u)}}{\beta_f+\beta_d}\right]_0^{t-2}+\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[u-\frac{e^{-\beta_f(t-2-u)}}{\beta_f}-\frac{e^{-\beta_d(t-2-u)}}{\beta_d}+\frac{e^{-(\beta_f+\beta_d)(t-2-u)}}{\beta_f+\beta_d}\right]_0^{t-2}$$

$$=\frac{\rho_{f,d}}{\beta_f\cdot\beta_d}\left[\begin{array}{l}1+\frac{2}{\beta_f+\beta_d}-\frac{e^{-(\beta_f+\beta_d)\cdot(t-1)}}{\beta_f+\beta_d}+\frac{e^{-\beta_f}}{\beta_f}-\frac{e^{-\beta_f}}{\beta_f+\beta_d}+\frac{e^{-\beta_f\cdot(t-1)-\beta_d(t-2)}}{\beta_f+\beta_d}+\frac{e^{-\beta_d}}{\beta_d}+\frac{e^{-\beta_d}}{\beta_f+\beta_d}+\frac{e^{-\beta_f(t-2)-\beta_d\cdot(t-1)}}{\beta_f+\beta_d}+\\-\frac{1}{\beta_f}-\frac{1}{\beta_d}-\frac{e^{-(\beta_f+\beta_d)(t-2)}}{\beta_f+\beta_d}\end{array}\right]$$

$P=$

$$Var[P]=\frac{\rho_f^2}{2\beta_f^3}\left[2\beta_f-2-e^{-2\beta_f(t-1)}+4e^{-\beta_f t(t-1)}-e^{-2\beta_f(t-2)}+4e^{-\beta_f(t-2)}+2e^{-\beta_f}-4e^{-\beta_f t}-4e^{-\beta_f(t-1)}+2e^{-\beta_f(2t-1)}\right]+$$

$$+\frac{\rho_d^2}{2\beta_d^3}\left[2\beta_d-2-e^{-2\beta_d(t-1)}+4e^{-\beta_d t(t-1)}-e^{-2\beta_d(t-2)}+4e^{-\beta_d(t-2)}+2e^{-\beta_d}-4e^{-\beta_d t}-4e^{-\beta_d(t-1)}+2e^{-\beta_d(2t-1)}\right]+$$

$$+\frac{2\rho_f\rho_d\rho_{f,d}}{\beta_f\cdot\beta_d}\left[\begin{array}{l}1+\frac{2}{\beta_f+\beta_d}-\frac{e^{-(\beta_f+\beta_d)\cdot(t-1)}}{\beta_f+\beta_d}+\frac{e^{-\beta_f}}{\beta_f}-\frac{e^{-\beta_f}}{\beta_f+\beta_d}+\frac{e^{-\beta_f\cdot(t-1)-\beta_d(t-2)}}{\beta_f+\beta_d}+\frac{e^{-\beta_d}}{\beta_d}+\frac{e^{-\beta_d}}{\beta_f+\beta_d}+\frac{e^{-\beta_f(t-2)-\beta_d\cdot(t-1)}}{\beta_f+\beta_d}+\\-\frac{1}{\beta_f}-\frac{1}{\beta_d}-\frac{e^{-(\beta_f+\beta_d)(t-2)}}{\beta_f+\beta_d}\end{array}\right]$$

$$E[X\cdot Y]=\frac{e^{E[M]+\frac{Var[M]}{2}}}{p^2(x_0,1,t-2)}+\frac{e^{E[N]+\frac{Var[N]}{2}}}{p(x_0,1,t-2)}(1+h_{t-1})+\frac{e^{E[O]+\frac{Var[P]}{2}}}{p(x_0,1,t-2)}+e^{E[P]+\frac{Var[P]}{2}}\cdot(1+h_{t-1})$$

$$Cov[X,Y]=\frac{e^{E[M]+\frac{Var[M]}{2}}}{p^2(x_0,1,t-2)}+\frac{e^{E[N]+\frac{Var[N]}{2}}}{p(x_0,1,t-2)}(1+h_{t-1})+\frac{e^{E[O]+\frac{Var[P]}{2}}}{p(x_0,1,t-2)}+e^{E[P]+\frac{Var[P]}{2}}\cdot(1+h_{t-1})+$$

$$-\frac{e^{E[B]+\frac{Var[B]}{2}+E[L]+\frac{Var[L]}{2}}}{p^2(x_0,1,t-2)}-\frac{e^{E[B]+\frac{Var[B]}{2}+E[C]+\frac{Var[C]}{2}}}{p(x_0,1,t-2)}(1+h_{t-1})-\frac{e^{E[A]+\frac{Var[A]}{2}+E[L]+\frac{Var[L]}{2}}}{p(x_0,1,t-2)}+$$

$$-e^{E[A]+\frac{Var[A]}{2}+E[C]+\frac{Var[C]}{2}}\cdot(1+h_{t-1})$$

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