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UNIVERSITÉ CATHOLIQUE DE LOUVAIN

D I S C U S S I O N
P A P E R

2012/30

Solvency analysis of defined benefit pension schemes

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Solvency Analysis of Defined Benefit Pension Schemes

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Abstract. Defined Benefit Pension Schemes (DB) are affected by a lot of different risks able to put in danger the viability of the system. A solvency analysis seems therefore to be essential as in insurance but it must take into account the specificities of pension liabilities. In particular, pension funds are characterized by a long term aspect and a limited need of liquidity. In this perspective, the purpose of this paper is to combine the three major risks affecting a DB plan (market, inflation and longevity risks) and to look at their effect on the solvency of the pension fund.

Keywords. Inflation risk, IAS norms, longevity risk, market risk, solvency requirement, value at risk.

M.S.C. classification. 60G15,91B30,91G30.
J.E.L. classification. C10,G23,J32.

1 Introduction

In Solvency II, solvency requirements for insurance companies are based on the idea that risk can be handled if a sort of buffer capital is available to deaden the impact of financial and demographic instability ([6],[13]). The purpose of this paper is to propose a similar but adapted risk based approach for a defined benefit pension scheme(DB plan). Following the IAS norms, we use as funding technique the so called projected unit credit cost method (see for instance [2]) in order to compute contributions and actuarial liabilities. Two main risks are then considered in a stochastic environment: investment risk and inflation risk. In a first step, mortality before retirement is not considered; afterwards, mortality is introduced in a deterministic way and a complete risk model is constructed. The long term aspect of pension liability is taken systematically into account by

analyzing the risks not only on a one year horizon,as in Solvency 2, but until maturity (retirement age). An ALM approach for the assets and liabilities of the scheme is proposed and various classical risk measures are applied to the surplus of the pension fund (probability of default, value at risk). The effect of the duration of the liability is clearly illustrated and is surely one of the key factors if we want to consider solvency measures for pension plans. This result is line with classical considerations about the time effect on risky investments (see for instance [3],[4],[5],[8],[11],[12]).
The paper is organized as follows. In section 2 we describe the general framework of the model in terms of asset and liability structure of the DB scheme. Section 3 is based on a static risk measurement and analyzes the influence of the time horizon on the probability of default at maturity. In section 4, we compute a solvency level using a classical value at risk approach. In section 5, we introduce the longevity risk using a deterministic approach based on the difference between the real mortality and the a priori mortality. Numerical examples are offered in Section 6. Finally, some conclusions are traced.

2 Model for a DB pension scheme

We consider a Defined Benefit pension scheme based on final salary. At time $t = 0$ an affiliate aged x is entering the scheme with an initial salary $S(0)$.
At retirement age, at time $t = T$, a lump sum will be paid, expressed as a multiple of its last wage, for instance:

$$B = NbS \tag{1}$$

where:
N is the years of service credited by the scheme
S is the final salary
B is benefit to pay (lump sum)
b is the coefficient (for instance 1%)
In order to compute the contributions to the pension fund, we will use the IAS norms and in particular the projected unit credit cost method as funding technique.
We need then the following assumptions:

1. a fixed discount rate: the risk free rate r;

2. a salary scale: the salary at time t denoted by $S(t)$ will follow a stochastic evolution given by :

$$dS(t) = \mu S(t)dt + \eta S(t)dz(t) \tag{2}$$

where
μ is the avarage salary increase;
η is the volatility on salary evolution;
z is standard Brownian motion.

Using a best estimate approach, the contribution for the first year of service (or the normal cost) is then given by :

$$NC_0 = bS(0)e^{(\mu-r)T} \tag{3}$$

(present value at the risk free rate of the average projected benefit at retirement age)
At time t ($t = 1, 2, .., T-1$), the normal cost will have the same form:

$$NC_t = bS(t)e^{(\mu-r)(T-t)} \tag{4}$$

We can also introduce a loading factor β on the contribution; the normal cost becomes then:

$$NC_t = bS(t)(1+\beta)e^{(\mu-r)(T-t)} \tag{5}$$

The actuarial liability AL at time t is then given by ($t = 0, 1, .., T-1$):

$$AL_t = (t+1)bS(t)(1+\beta)e^{(\mu-r)(T-t)} \tag{6}$$

On the asset side we assume each contribution is invested in a Geometric Brownian motion (see for instance [9]) whose evolution is solution of :

$$dA(s) = \delta A(s)ds + \sigma A(s)dw(s) \tag{7}$$

where
δ is mean return of the investment fund;
σ is volatility of the return;
w is standard Brownian motion.
The two sources of risk (inflation and market risks) are off course correlated:

$$corr(w(t), z(t)) = \rho t$$

Considering first the risk attached to the first year of contribution , we could take as initial condition:

$$A(0) = NC_0$$

Then the corresponding final asset is given by (projection between $t = 0$ and $t = T$):

$$A_0(T) = NC_0 e^{\delta - \frac{\sigma^2}{2}T + \sigma w(t)} = bS(0)(1+\beta)e^{\mu+\delta-r-\frac{\sigma^2}{2}T+\sigma w(t)} \tag{8}$$

More generally we could consider the risk between time t and time T by computing the future evolution till maturity of the investment of the actuarial liability AL existing at time t in the reference asset A (investment risk between time t and time T):

$$A_t(T) = AL_t e^{\delta - \frac{\sigma^2}{2}(T-t)+\sigma w(T)-w(t)} = bS(t)(1+\beta)e^{\mu+\delta-r-\frac{\sigma^2}{2}(T-t)+\sigma w(T)-w(t)} \tag{9}$$

with initial condition $A_t(t) = AL_t$.
In an ALM approach, these asset values must be compared to their respective liability counterparts.
We obtain successively:

– for the final liability corresponding to the first year contribution:

$$L_0(T) = bS(0)e^{\mu - \frac{\eta^2}{2}T + \eta z(T)} \tag{10}$$

– for the final liability corresponding to the actuarial liability until time t:

$$L_t(T) = bS(t)e^{\mu - \frac{\eta^2}{2}(T-t) + \eta(z(T) - z(t))} \tag{11}$$

In the next sections, we will compare the final assets given by eq. (9) with the final liabilities given by eq.(11).

3 Probability of Default

A first interesting question is to look at the probability of default at maturity without any extra resource (i.e. the risk to have not enough assets at maturity to pay the required pension benefit). In particular we can consider this probability as a function of the residual time $T - t$. This probability computed at time t $(t = 0, 1, .., T - 1)$ is given by:

$$\varphi(t,T) = P(A_t(T) < L_t(T)) = P(Y(t,T) < M) \tag{12}$$

where

$$\begin{aligned} &Y(t,T) = \sigma(w(T) - w(t)) - \eta(z(T) - z(t)) = N(0, \bar{\sigma}^2(T-t)) \\ &\bar{\sigma}^2 = \sigma^2 + \eta^2 - 2\rho\sigma\eta \\ &M = (r - \delta + \frac{\sigma^2}{2} - \frac{\eta^2}{2})(T - t) - ln(1 + \beta) \end{aligned} \tag{13}$$

So finally the probability of default at maturity depends on the residual time and is given by:

$$\begin{aligned} &\varphi(t,T) = \Phi(a(T-t)) \\ &a(s) = \frac{(r - \delta + \frac{\sigma^2}{2} - \frac{\eta^2}{2})\sqrt{s} - \frac{ln(1-\beta)}{\sqrt{s}}}{\bar{\sigma}} \end{aligned} \tag{14}$$

with $\Phi = distributionfunctionN(0,1)$
.

4 Value at Risk approach

In order to control this probability of default, we could as in Solvency 2 introduce a solvency level based on a value at risk approach ([1],[7],[10]).
We will use the following notations:
SC is the solvency capital using a value at risk methodology;

VaR is the Value a Risk;
$\alpha(N)$ is a chosen safety level for a horizon of N years (for instance 99.5% on one year in Solvency 2)
For this safety level we can choice the following value based on yearly independent default probabilities (probabilities of default of $(1-\alpha)$ independently each year):

$$\alpha_N = \alpha^N \tag{15}$$

We will assume that the solvency capital is invested in the reference investment fund; so we can define this solvency capital $SC(t,T)$ at time t ($t = 0, 1, .., T-1$) for the investment and inflation risks between time t and time T as solution of:

$$P\{A_t(T) + SC(t,T)\frac{A_t(T)}{A_t(t)} < L_t(T)\} = 1 - \alpha_{T-t} \tag{16}$$

Using (9) and (11), this condition becomes:

$$P\{tbS(t)(1+\beta) + SC(t,T)e^{\delta - \frac{\sigma^2}{2}(T-t)+\sigma w(T)-w(t)} < tbS(t)e^{\mu - \frac{\eta^2}{2}(T-t)+\eta(z(T)-z(t))}\}$$
$$= 1 - \alpha_{T-t}$$

After direct computation, we obtain the following value for the solvency capital:

$$SC(t,T) = tbS(t)\{e^{(\mu-\delta)(T-t)+z_{\alpha(T-t)}\sigma\sqrt{T-t}+\frac{(\sigma^2-\eta^2)(T-t)}{2}} - e^{(\mu-r)(T-t)}(1+\beta)\} \tag{17}$$

where
$z_\beta = \beta$ is the quantile of the normal distribution on such that $\Phi(z_\beta) = \beta$ We can express the solvency capital as a percentage of the actuarial liability AL given by eq.(6) (solvency level in percent):

$$SC^{\%}(t,T) = \frac{SC(t,T)}{AL_t} = \frac{1}{1+\beta}e^{-(\delta-r)(T-t)+z_{\alpha(T-t)}\bar{\sigma}\sqrt{T-t}+\frac{(\sigma^2-\eta^2)(T-t)}{2}} - 1 \tag{18}$$

We can observe that this relative level does not depend on the average salary increase.
In particular if we look at a one year risk (as in Solvency 2), we get:

$$SC^{\%}(0,1) = \frac{SC(0,1)}{AL_0} = \frac{1}{1+\beta}e^{-(\delta-r)+z_\alpha\bar{\sigma}+\frac{(\sigma^2-\eta^2)}{2}} - 1$$

5 Introduction of the Longevity Risk

Until now, two risk factors have been considered: investment and inflation. However, another risk source has to be introduced to outline a more complete risk model for pension plans: longevity. In this paper, since we are dealing with the case of payment of a lump sum at retirement and not a pension annuity, the pension provider has to evaluate just the probability that the affiliate dies before retirement, ignoring his remaining lifetime afterwards. In order to take into account this eventuality, the formulae developed in the previous sections have to be modified as follows. First of all, normal cost and actuarial liabilities becomes:

$$NC_t = bS(t)(1+\beta)e^{(\mu-r)(T-t)} \;{}_{T-t}p_{x+t} \tag{19}$$

$$AL_t = (t+1)bS(t)(1+\beta)e^{(\mu-r)(T-t)} \;{}_{T-t}p_{x+t} \tag{20}$$

where ${}_{T-t}p_{x+t}$ is the probability at age x to survive until T, calculated according to an a priori valuation mortality table, used by pension provider to estimate the actuarial liabilities. Longevity risk arises when the real ex-post survival probabilities differs from the a priori ones.
Let ${}_{T-t}\tilde{p}_{x+t}$ be the real survival probability; the liability at maturity seen from time t becomes:

$$L_t(T) = tbS(t)e^{\mu-\frac{\eta^2}{2}(T-t)+\eta(z(T)-z(t))} \;{}_{T-t}\tilde{p}_{x+t} \tag{21}$$

Consequently, the probability of default computed at time t $(t = 0, 1, .., T-1)$ is given by:

$$\begin{array}{l} \varphi(t,T) = P(A_t(T) < L_t(T)) = P(Y(t,T) < M) \\ M = (r-\delta+\frac{\sigma^2}{2}-\frac{\eta^2}{2})(T-t) - ln(1+\beta) + ln(\frac{{}_{T-t}\tilde{p}_{x+t}}{{}_{T-t}p_{x+t}}) \end{array} \tag{22}$$

and $Y/t, T)$ is computed according to eq.(13).
So finally as in the previous section the probability of default at maturity depends mainly on the residual time and is given by:

$$\varphi(t,T) = \Phi(\frac{(r-\delta+\frac{\sigma^2}{2}-\frac{\eta^2}{2})(T-t) - ln(1+\beta) + ln(\frac{{}_{T-t}\tilde{p}_{x+t}}{{}_{T-t}p_{x+t}})}{\bar{\sigma}\sqrt{T-t}}) \tag{23}$$

After direct computation, we obtain the following value for the solvency capital:

$$SC^{\%}(0,1) = \frac{SC(0,1)}{AL_0} = \frac{1}{1+\beta}e^{-(\delta-r)+z_\alpha\bar{\sigma}+\frac{(\sigma^2-\eta^2)}{2}} \frac{{}_{T-t}\tilde{p}_{x+t}}{{}_{T-t}p_{x+t}} - 1 \tag{24}$$

The general framework traced can be specified through the introduction of a given mortality model, as in the case of the well known Gompertz life table, which is based on the assumption of exponential mortality intensity. For sake of example, valuation and real life table can be derived as two different Gompertz tables:

$${}_{T-t}p_{x+t} = exp(-\int_t^T \mu_{x+s}ds) = exp(-\mu_{x+s}\frac{e^{\gamma(T-t)}-1}{\gamma})$$

$${}_{T-t}\tilde{p}_{x+t} = exp(-\int_t^T \tilde{\mu}_{x+s}ds) = exp(-\tilde{\mu}_{x+s}\frac{e^{\kappa(T-t)}-1}{\kappa})$$

Under this assumption, equations (23) and (24) are transformed into (25) and (26):

$$\varphi(t,T) = \Phi\left(\frac{(r-\delta+\frac{\sigma^2}{2}-\frac{\eta^2}{2})(T-t) - ln(1+\beta) + \mu_{x+s}\frac{e^{\gamma(T-t)}-1}{\gamma} - \tilde{\mu}_{x+s}\frac{e^{\kappa(T-t)}-1}{\kappa}}{\bar{\sigma}\sqrt{T-t}}\right) \quad (25)$$

$$SC^{\%}(0,1) = \frac{SC(0,1)}{AL_0} = \frac{1}{1+\beta}e^{-(\delta-r)(T-t)+z_\alpha\bar{\sigma}\sqrt{T-t}+\frac{(\sigma^2-\eta^2)(T-t)}{2}}e^{\mu_{x+s}\frac{e^{\gamma(T-t)}-1}{\gamma}-\tilde{\mu}_{x+s}\frac{e^{\kappa(T-t)}-1}{\kappa}} - 1 \quad (26)$$

6 Numerical Example

In this section, we carry out an applicative analysis, in which we compute the described quantities under given scenarios, in order to highlight how the solvency position of a DB changes during the time. To this aim, we have set the following financial parameters:

risk free rate	$r = 2\%$
mean return of the fund	$\delta = 6\%$
volatility of the fund	$\sigma = 10\%$
average increase of salary	$\mu = 5\%$
volatility of the salary	$\eta = 5\%$
correlation	$\rho = 50\%$
no safety loading is considered	$\beta = 0$

With respect to the demographic assumptions, we have assumed that the affiliate subscribes the pension plan at 35 years and the retirement age is 65. The valuation table used is the Italian male mortality table of the year 2006 downloaded from the Human Mortality Database, from which the values are derived. We have implemented the model without considering mortality and then with the introduction of mortality risk through two different ex post mortality tables. In the first case (a), we have considered that the force of mortality in the real table is smaller than the a priori mortality to take into account the effects of survival improvements. In the second case (b) we have considered the opposite situation. Under the scenario a) and b) the mortality tables are derived modifying the a priori mortality intensity according to the following assumptions:

a) $\tilde{\mu}'_{x+t} = \mu_{x+t}(1 - \Delta t)$
b) $\tilde{\mu}''_{x+t} = \mu_{x+t}(1 + \Delta t)$

with $\Delta = 2\%$

Both the valuation table and the real tables are fitted to the Gompertz law, producing the following parameters:

$\gamma = 0.004768$ for the valuation table
$\kappa' = 0.003038$ for the real table under the assumption a)
$\kappa'' = 0.005512$ for the real table under the assumption b)

Figure 1 shows the probability of default as a function of the residual time $T-t$ under three different hypothesis: no mortality (eq. (12)-(14)), survival improvements (a) and survival decreasing (b) (eq. (23)):

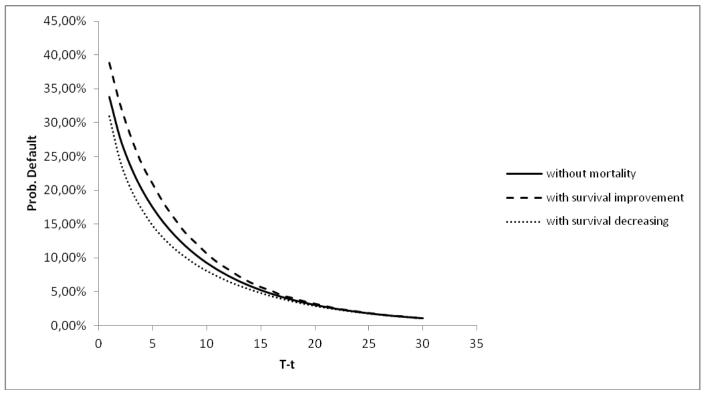

Fig. 1. Probability of default

We can see clearly a time effect: for short residual time to maturity this probability is quite high but it decreases rapidly for long residual time. Moreover, the longevity risk overdraws the time effect: in the case of survival improvements the probability of default is higher than that calculated ignoring the mortality and decreases more rapidly during the time. On the contrary, in the case of survival decreasing the probability of default is lower than that calculated ignoring the mortality and decreases less rapidly during the time.

In a second step of our application, we have computed the solvency capital under the same assumptions drawn so far. In addition, we have set:

safety level on one year: $\alpha = 99.5\%$
safety level on N years: $\alpha_N = \alpha^N$

Figure 2 shows then the evolution of the solvency level in percent as a function of the residual time $T-t$. Negative values for the SCR correspond to cases where no additional solvency is needed.
We can also observe as in Figure 1 a time effect. As expected, in face of an increase of the probability of default in the case of survival improvements the solvency capital increases too; on the contrary, in the case of survival decreasing the actuarial liabilities decreases and a lower solvency capital is needed.

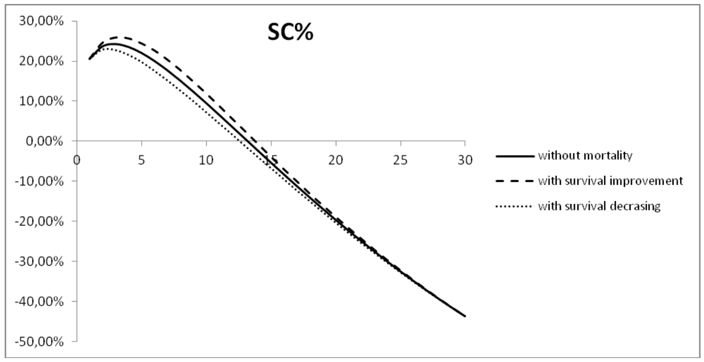

Fig. 2. Solvency Capital

7 Final Remarks

The solvency analysis is an important issue in the risk valuation of a pension plan.. A similar approach as in Solvency 2 turns out to be necessary also for pension funds, but must take into account a long term aspect and a limited need for liquidity.

In this paper we have highlighted the importance to extend solvency evaluation to pension funds through an integrated analysis of the risks that have influence on pension assets and liabilities. In this context, the determination of the solvency capital has been influenced by the way to measure the underlying risks and to integrate time in the process; this time aspect is particularly important for long term liabilities.

Further works are planned to extend the framework outlined in this paper to more general cases, as the generalization to stochastic longevity models, the introduction of pension annuity rather than a lump sum paid at retirement and the extension of longevity risk to the decumulation phase.

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