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Almost Marginal Conditional Stochastic Dominance

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Abstract

Marginal Conditional Stochastic Dominance (MCSD) developed by Shalit and Yitzhaki (1994) gives the conditions under which all risk-averse individuals prefer to increase the share of one risky asset over another in a given portfolio. In this paper, we extend this concept to provide conditions under which most (and not all) risk-averse investors behave in this way. Instead of stochastic dominance rules, almost stochastic dominance is used to assess the superiority of one asset over another in a given portfolio. Almost stochastic dominance means that the expected utility of most risk-averse investors can be improved by increasing the share of the dominant asset at the expense of the dominated one, excluding investors with extreme forms of preferences. Switching from MCSD to Almost MCSD (AMCSD) helps to reduce the inconsistency between common practice in asset allocation and the elegant decision rules in modern portfolio theory inspired from stochastic dominance relations.

Keywords: marginal conditional stochastic dominance, almost stochastic dominance, asset allocation, optimal investment.

JEL classification: D81

1 Introduction

The most common investment rule is certainly the mean-variance (MV) rule which states that one portfolio dominates another whenever it has at least the same mean and at most the same variance. The MV-efficient set is easy to compute, and in some cases even to express analytically which explains why the MV rule has become most widely accepted throughout the financial profession. On the other hand, Expected utility (EU) maximization lies at the heart of modern investment theory and practice. To be analytically consistent with EU maximization, the MV rule requires strong assumptions (such as quadratic utility functions or normally distributed returns) which seldom hold in practice. However, EU requires the specification of the investor's utility function which appears extremely difficult.

Stochastic dominance (SD) is an alternative approach which avoids all these shortcomings by considering the preferences shared by all the rational decision-makers. Therefore, it does not require a specific utility function nor a specific return distribution. Furthermore, it uses the whole probability distribution rather than the usual MV parameters of standard deviation and mean return. The second order stochastic dominance (SSD) rule is appropriate for the class of all risk-averse EU maximizers. It has the advantage that it requires no restrictions on probability distributions nor on investors' utility functions outside of the requirement that investors be risk-averse, EU maximizers.

Given a portfolio of assets, marginal conditional stochastic dominance (MCSD) has been introduced by Yitzhaki and Olkin (1991) and Shalit and Yitzhaki (1994) as a condition under which all risk-averse EU utility maximizer individuals prefer to increase the share of one risky asset over that of another. Specifically, these authors consider risk-averse investors holding a given portfolio of risky assets and derive criteria expressed in terms of the joint probability distribution of the assets and of the underlying portfolio to ensure that the share of an asset is increased at the expense of another in the portfolio. This helps to detect inefficiency and to improve inefficient portfolios. MCSD has been successfully applied to solve asset allocation problems by several authors, including Clark et al. (2011), Clark and Kassimatis (2012a,b), Shalit and Yitzhaki (2010). MCSD expresses the conditions under which all risk averse investors holding a specific portfolio prefer one asset to another. It is a less demanding concept and more adapted to empirical analysis than SSD because it considers only marginal changes of holding risky assets in a given portfolio. However, even if the existence of an MCSD improvement ensures that the current porfolio is not efficient, its absence does not prove portfolio efficiency.

Despite their theoretical attractiveness, MV and SSD rules may create paradoxes in the sense that they fail to distinguish between some risky prospects whereas it is obvious that the vast majority of investors would prefer one over the other. This is why Bali et al. (2009) considered almost stochastic dominance (ASD) as a viable alternative. ASD corresponds to all utility functions after eliminating pathological preferences, keeping only the economically relevant utility functions. Bali et al. (2009) demonstrated that the ASD rule unambiguously supports some common practice, like advising an higher stock to bond ratio for long investment horizons. Switching from SD to ASD thus allows to provide a theoretical support for the practitioners' view within the EU paradigm. The study conducted by these authors suggest that modifying MCSD into almost MCSD (AMCSD) may also help to analyze economic behavior under risk. This is the subject of the present work.

In this paper, MCSD is weakened to ensure that most (but not all) risk-averse decision-makers increase the share of one risky asset over another. This extension of MCSD to AMCSD is inspired from almost stochastic dominance rules introduced by Leshno and Levy (2002), suitably corrected by Tzeng et al. (2012). Specifically, restrictions are imposed on the marginal utility function and on its derivative to exclude extreme forms of preferences, not shared by real-world investors. Then, the condition leading to MCSD is adapted to correspond to utilities defining almost second-degree stochastic dominance. As pointed out by Levy et al. (2010), investment rules based on stochastic dominance may cover theoretical preferences that are not encountered in practice: there are situations where stochastic dominance is unable to rank two portfolios whereas experimentally 100% of the subjects reveal a clear cut ranking. The switch from MCSD to AMCSD can be expected to avoid such paradoxical results.

The remainder of the paper is organized as follows. The next section extends MCSD to AMCSD. Section 3 discusses a numerical example comparing the two concepts. Section 4 concludes.

2 Almost Marginal Conditional Stochastic Dominance

2.1 Marginal Conditional Stochastic Dominance

Assume that a risk-averse investor with a utility function u holds a portfolio with n risky assets. Let w_0 be the initial wealth, X_i denote the rate of return on risky asset i and α_i be the investment proportion on asset i, $i = 1, 2, \ldots, n$. A portfolio α is defined by the shares α_i such that $\sum_{i=1}^{n} \alpha_i = 1$. The final wealth of the investor is given by $W = w_0 \left(1 + \sum_{i=1}^{n} \alpha_i X_i\right)$. Henceforth, we normalize the initial wealth w_0 to unity so that $W = 1 + \sum_{i=1}^{n} \alpha_i X_i$.

The goal of the investor is to select the weights to maximize $E[u(W)]$. Given a portfolio α, Shalit and Yitzhaki (1994) have established that it is optimal to increase the weight α_k of asset k at the expense of asset j if, and only if,

$$E\left[u'(W)\left(X_k - X_j\right)\right] \geq 0. \tag{1}$$

Asset k dominates asset j according to MCSD if the condition (1) is fulfilled for all risk-averse investors, that is, for all concave utility u.

Let R denote the portfolio return, i.e.,

$$R = \sum_{i=1}^{n} \alpha_i X_i.$$

Shalit and Yitzhaki (1994) proved that for a given portfolio α, asset k dominates asset j according to MCSD if, and only if, the inequality

$$E[X_k | R \leq r] \geq E[X_j | R \leq r]$$

holds for all the return levels r. This is easily deduced from (1) by taking the kinked utilities $u(x) = \min\{x, r\}$. In words, MCSD favors assets performing better in adverse situations (i.e. when the portfolio underperforms $\Leftrightarrow R \leq r$).

The next section shows how to define AMCSD from MCSD, avoiding extreme forms of preferences.

2.2 From MCSD to AMCSD

MCSD is based on all the non-decreasing and concave utility functions, that is, on the utility functions in

$$U_2 = \left\{\text{utility functions } u|u' \geq 0 \text{ and } u'' \leq 0\right\}.$$

As explained in Leshno and Levy (2002), U_2 contains some extreme utility functions which presumably rarely represent real-world investors' preferences. The prototype is $u(x) = \min\{x, r\}$ for some constant r. Note that such utilities form the representative set of non-decreasing and concave utility functions used by Hadar and Seo (1988).

To reveal a preference for most investors, but not for all of them, we restrict U_2 to a subset of it. Specifically, following Leshno and Levy (2002), let us further impose restrictions on the utility function and define

$$U_2^*(\varepsilon) = \left\{u \in U_2 \middle| -u''(x) \leq \inf\left\{-u''(x)\right\}\left(\frac{1}{\varepsilon} - 1\right) \text{ for all } x\right\}, \tag{2}$$

where $\varepsilon \in \left(0, \frac{1}{2}\right)$. The range of the parameter ε which controls the area of violation has been discussed empirically by Levy et al. (2010)[1]. The following result characterizes the situations where asset j is dominated by asset k for all investors with $u \in U_2^*(\varepsilon)$. Before stating it formally, we need to introduce some additional notation. Let $\mu_i(r)$ denote the conditional expected return of asset i when the portfolio return is r, i.e.,

$$\mu_i(r) = E[X_i|R = r].$$

Henceforth, we assume without real loss of generality that the return is bounded and valued is some interval $[a, b]$ of the real line. Furthermore, define

$$\begin{aligned} B(t) &= \int_a^t \big(\mu_k(r) - \mu_j(r)\big) dF_R(r) \\ &= \Big(E[X_k|R \leq t] - E[X_j|R \leq t]\Big) F_R(t) \\ \Omega &= \{t \in [a, b]\, | B(t) < 0\} \end{aligned}$$

[1]However, in that paper, the definition of ASSD is not correct as pointed out by Tzeng et al. (2012). Thus, there are some doubts about the estimated ε derived by these authors.

and Ω^c denote the complement of Ω in $[a, b]$. MCSD requires $B(t) \geq 0$ for all t, that is, $\Omega = \emptyset$. If this is not the case, Ω represents the set of violation for MCSD.

Proposition 1 *Given portfolio α, asset k dominates asset j for all individuals with preferences represented by the utility function $u \in U_2^*(\varepsilon)$ if, and only if,*

$$\int_{\Omega} (-B(t)dt) \leq \varepsilon \int_a^b |B(t)|\, dt \tag{3}$$

and $E[X_k] \geq E[X_j]$.

The proof of this result can be found in appendix. Together with the comparison of expected returns, condition (3) provides the operational way to check for AMCSD in a given portfolio.

3 Numerical illustrations

An example is introduced to demonstrate the application of Proposition 1. Assume that there exist three independently distributed risky assets. The distributions of the rates of return for these three assets are respectively

$$X_1 = \begin{cases} -10\% \text{ with probability } \frac{1}{2} \\ +15\% \text{ with probability } \frac{1}{2} \end{cases}$$

$$X_2 = \begin{cases} -11\% \text{ with probability } \frac{1}{2} \\ +50\% \text{ with probability } \frac{1}{2} \end{cases}$$

$$X_3 = \begin{cases} -15\% \text{ with probability } \frac{1}{2} \\ +25\% \text{ with probability } \frac{1}{2} \end{cases}$$

We further assume that the weights in the current portfolio are $\alpha_1 = 25\%$, $\alpha_2 = 50\%$ and $\alpha_3 = 25\%$. Table 1 shows the distribution of the portfolio returns and the assets conditional expected returns. Shalit and Yitzhaki (1994) related MCSD to Absolute Concentration Curves (ACCs) defined as follows. The ACC for asset i with respect to the portfolio α is

$$ACC_i(p) = \int_{-\infty}^{F_R^{-1}(p)} \mu_i(r) dF_R(r) = E[X_i | R \leq F_R^{-1}(p)]$$

where $F_R^{-1}(p)$ is the pth quantile of the distribution function F_R formally defined as

$$F_R^{-1}(p) = \inf\{\xi \in \mathbb{R} | F_R(\xi) \geq p\}.$$

Table 2 shows the ACCs and the function $B(t)$ involved in Proposition 1.

Table 1 The portfolio returns and assets conditional expected returns

r	$\Pr[R = r]$	$F_R(r)$	$\mu_1(r)$	$\mu_2(r)$	$\mu_3(r)$
-11.75	0.125	0.125	-10	-11	-15
-5.5	0.125	0.25	15	-11	-15
-1.75	0.125	0.375	-10	-11	25
4.5	0.125	0.5	15	-11	25
18.75	0.125	0.625	-10	50	-15
25	0.125	0.75	15	50	-15
28.75	0.125	0.875	-10	50	25
35	0.125	1	15	50	25

Table 2 ACCs and $B(\cdot)$

F_R	ACC_1	ACC_2	ACC_3	$B(t)\|_{2vs1}$	$B(t)\|_{2vs3}$	$B(t)\|_{3vs1}$
0	0	0	0	0	0	0
0.125	-1.25	-1.375	-1.875	-0.125	0.5	-0.625
0.25	0.625	-2.75	-3.75	-3.375	1	-4.375
0.375	-0.625	-4.125	-0.625	-3.5	-3.5	0
0.5	1.25	-5.5	2.5	-6.75	-8	1.25
0.625	0	0.75	0.625	0.75	0.125	0.625
0.75	1.875	7	-1.25	5.125	8.25	-3.125
0.875	0.625	13.25	1.875	12.625	11.375	1.25
1	2.5	19.7	5	17	14.5	2.5

From Table 2, the ACCs for the three assets reveal that they do not MCSD dominate each other since the violation set $\Omega \neq \emptyset$. On basis of the criteria of MCSD, no suggestion can be made to improve the efficiency of the portfolio.

Now, let us assume that ε is equal to 0.3. Table 3 shows the two criteria of AMCSD suggested in Proposition 1. Note that the criteria that $E[X_k] \geq E[X_j]$ can help us to reduce the comparison among assets. Thus, in both Tables 2 and 3, only the results of 2 vs 1, 2 vs 3 and 3 vs 1 are demonstrated.

Table 3 The criteria of AMCSD

	2 vs 1	2 vs 3	3 vs 1
Differences in expectations	17	14.5	2.5
$\varepsilon \int_a^b \lvert B(t) \rvert \, dt - \int_\Omega \big(- B(t) \big) dt$	1.025	2.675	-4

From Table 3, asset 2 AMCSD dominates both assets 1 and 3 for all investors in $U_2^*(\varepsilon = 0.3)$. Thus, the expected utility of all investors in $U_2^*(\varepsilon = 0.3)$ can be further improved by increasing the weight on asset 2. On the basis of AMCSD, the current portfolio is not efficient. Asset 2 does not MCSD dominate either assets 1 or 3 because MCSD seeks for the condition for all risk-averse investors. In this example, it is obvious that asset 2 could be an attractive alternative for most investors. Therefore, with respect to the current portfolio, most investors, e.g., those with utility function $u \in U_2^*(\varepsilon = 0.3)$ are inclined to invest more on asset 2.

4 Discussion

This paper develops a new methodology for improving existing portfolios, based on MCSD but replacing stochastic dominance with almost stochastic dominance. This helps to exclude extreme forms of preferences, not shared by real-world investors. Given the increasing importance of MCSD in constructing efficient portfolios, we believe that the extension proposed in the present paper will be useful for active asset management. In this respect, switching from MCSD to AMCSD helps to reduce the inconsistency between common practice in asset allocation and the elegant decision rules in modern portfolio theory inspired from stochastic dominance relations. This is done by considering only economically relevant utility functions, i.e. by excluding pathological preferences. In our simple numerical example, we discovered that a portfolio without possible MCSD improvement may appear as dominated based on AMCSD. In this case, AMCSD tends to favor investing less on assets with lower means which is in line with common practice in asset management.

To end with, let us mention that AMCSD can also be extended to higher orders, accounting for attitudes towards risk beyond risk aversion. Recall that the Nth order stochastic dominance is based on the common preferences shared by all the decision-makers with utility function in

$$U_N = \left\{ \text{utility function u} | (-1)^{n+1} u^{(n)} \geq 0,\, n = 1, 2, ..., N \right\},$$

where $u^{(n)}$ denotes the nth derivative of the utility function u, and $N > 2$. Besides risk aversion, U_N entails behavioral traits such as prudence ($N = 3$), temperance ($N = 4$), and more generally risk apportionment of order N in the terminology of Eeckhoudt and Schlesinger (2006).

Almost Nth order stochastic dominance introduced by Tzeng et al. (2012) excludes extreme forms of preferences by restricting to

$$U_N^*(\varepsilon_N) = \left\{ u \in U_N \middle| (-1)^{N+1} u^{(N)}(x) \leq \inf \left\{ (-1)^{N+1} u^{(N)}(x) \right\} \left(\frac{1}{\varepsilon_N} - 1 \right) \text{ for all } x \right\}.$$

Now, starting from $B^{(1)}(t) = B(t)$, let us define iteratively for $n = 2, 3, ...N$

$$B^{(n)}(t) = \int_a^t B^{(n-1)}(s) ds,$$

$\Omega_n = \left\{ t \in [a, b] : B^{(n)}(t) < 0 \right\}$, and Ω_n^c as the complement of Ω_n in $[a, b]$. Following the reasoning that lead to Proposition 1, we can show that given portfolio α, asset k dominates asset j for all individuals with preferences $u \in U_N^*(\varepsilon_N)$, $N > 2$, if and only if

$$\int_{\Omega_N} \big(-B^{(N)}(t) \big) dt \leq \varepsilon_N \int_a^b \left| B^{(N)}(t) \right| dt$$

and $B^{(n)}(b) \geq 0$, $n = 2, 3, ...N$. This provides the necessary and sufficient condition that asset j is dominated by asset k for all investors with $u \in U_N^*(\varepsilon_N)$.

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Proof of Proposition 1

"If" part:

Let us first prove that (3) together with $E[X_k] \geq E[X_j]$ ensure that inequality (1) holds true. To this end,

$$\begin{aligned} E\left[u'(W)\left(X_k - X_j\right)\right] &= E\left[u'(W)\left(E[X_k|W] - E[X_j|W]\right)\right] \\ &= \int_a^b u'(t)\big(\mu_k(t) - \mu_j(t)\big)dF_R(t) \\ &= \int_a^b u'(t)dB(t). \end{aligned}$$

Integrating the above equation by parts yields

$$E\left[u'(W)\left(X_k - X_j\right)\right] = u'(b)B(b) + \int_a^b \big(-u''(t)\big)B(t)dt.$$

Since

$$B(b) = \int_a^b \big(\mu_k(t) - \mu_j(t)\big)dF_R(t) = E[X_k] - E[X_j],$$

the conditions $u' > 0$ and $E[X_k] \geq E[X_j]$ ensure that $u'(b)B(b) \geq 0$. Furthermore,

$$\begin{aligned} & \int_a^b \big(-u''(t)\big)B(t)dt \\ = \ & \int_\Omega \big(-u''(t)\big)B(t)dt + \int_{\Omega^c} \big(-u''(t)\big)B(t)dt \\ \geq \ & \int_\Omega \sup\left\{-u''(t)\right\} B(t)dt + \int_{\Omega^c} \inf\left\{-u''(t)\right\} B(t)dt \\ = \ & \sup\left\{-u''(t)\right\} \int_\Omega B(t)dt + \inf\left\{-u''(t)\right\} \int_{\Omega^c} B(t)dt \\ = \ & \big(\sup\left\{-u''(t)\right\} + \inf\left\{-u''(t)\right\}\big) \int_\Omega B(t)dt + \inf\left\{-u''(t)\right\} \left(\int_{\Omega^c} B(t)dt - \int_\Omega B(t)dt\right) \\ = \ & -\big(\sup\left\{-u''(t)\right\} + \inf\left\{-u''(t)\right\}\big) \int_\Omega \big(-B(t)\big)dt + \inf\left\{-u''(t)\right\} \int_a^b |B(t)|\, dt. \end{aligned}$$

Since $u \in U_2^*(\varepsilon)$, based on the definition of $U_2^*(\varepsilon)$, we have

$$\sup\left\{-u''(t)\right\} \leq \inf\left\{-u''(t)\right\} \left(\frac{1}{\varepsilon} - 1\right)$$

$$\Leftrightarrow \varepsilon \leq \frac{\inf\{-u''(t)\}}{\sup\{-u''(t)\} + \inf\{-u''(t)\}}.$$

Thus, condition (3) ensures that

$$\int_{\Omega} \big(- B(t) \big) dt \leq \frac{\inf\{-u''(t)\}}{\sup\{-u''(t)\} + \inf\{-u''(t)\}} \int_a^b |B(t)|\, dt.$$

Rearranging the above equation yields

$$-\big(\sup\big\{-u''(t)\big\} + \inf\big\{-u''(t)\big\} \big) \int_{\Omega} \big(- B(t) \big) dt + \inf\big\{-u''(t)\big\} \int_a^b |B(t)|\, dt \geq 0.$$

Thus, we have $\int_a^b \big(- u''(t) \big) B(t) dt \geq 0$ and hence $E\left[u'(W)\left(X_k - X_j\right)\right] \geq 0$, which ends the proof of the "if" part.

"Only if" part:

We would like to prove that if

$$\int_{\Omega} \big(- B(t) \big) dt > \varepsilon \int_a^b |B(t)|\, dt \text{ or } E[X_k] < E[X_j],$$

then there exists one individual with preferences $u \in U_2^*(\varepsilon)$ such that inequality (1) is violated. Let us first show the following statement:

$$\int_{\Omega} \big(- B(t) \big) dt > \varepsilon \int_a^b |B(t)|\, dt \Rightarrow \exists u \in U_2^*(\varepsilon) \text{ such that (1) is violated.}$$

Let $\underline{\theta}$ and $\bar{\theta}$ be two positive real numbers such that $\varepsilon = \frac{\underline{\theta}}{\bar{\theta}+\underline{\theta}}$. Assume that $\Omega = [c, d]$, where $a \leq c \leq d \leq b$. Define a marginal utility function as follows:

$$u'(x) = \begin{cases} \underline{\theta}\,(b-d) + \bar{\theta}\,(d-c) + \underline{\theta}\,(c-x) & \text{if } a \leq x \leq c \\ \underline{\theta}\,(b-d) + \bar{\theta}\,(d-x) & \text{if } c \leq x \leq d \\ \underline{\theta}\,(b-x) & \text{if } d \leq x \leq b \end{cases},$$

which belongs to $U_2^*(\varepsilon)$. Since $u'(b)=0$, we have

$$\begin{aligned} E\left[u'(W)\left(X_k - X_j\right)\right] &= \int_a^b \big(-u''(t)\big)B(t)dt \\ &= \bar{\theta}\int_c^d B(t)dt + \underline{\theta}\int_{\Omega^C} B(t)dt \\ &= -\left(\bar{\theta}+\underline{\theta}\right)\int_a^b \big(-B(t)\big)dt + \underline{\theta}\int_a^b |B(t)|\,dt. \end{aligned}$$

Since $\varepsilon = \frac{\underline{\theta}}{\bar{\theta}+\underline{\theta}}$, and

$$\int_\Omega \big(-B(t)\big)dt > \frac{\underline{\theta}}{\bar{\theta}+\underline{\theta}}\int_a^b |B(t)|\,dt,$$

we have $E\left[u'(W)\left(X_k - X_j\right)\right] < 0$, which is the desired result.

Now, let us prove that

$$E[X_k] < E[X_j] \Rightarrow \exists u \in U_2^*(\varepsilon) \text{ such that (1) is violated.}$$

Define

$$u'(x) = c - \delta x,$$

where δ is a positive constant small enough to ensure $c > \delta b$. Thus, $u \in U_2^*(\varepsilon)$. Furthermore,

$$\begin{aligned} E\left[u'(W)\left(X_k - X_j\right)\right] &= u'(b)B(b) + \int_a^b \big(-u''(t)\big)B(t)dt \\ &= \left(c-\delta b\right)B(b) + \delta\int_a^b B(t)dt. \end{aligned}$$

Now, if c is such that

$$\left(c-\delta b\right)\big(E[X_k]-E[X_j]\big) + \delta\int_a^b B(t)dt < 0 \Leftrightarrow c > \delta b - \frac{\delta\int_a^b B(t)dt}{E[X_k]-E[X_j]},$$

we have $E\left[u'(W)\left(X_k - X_j\right)\right] < 0$ which contradicts (1) and completes the proof.