

I N S T I T U T D E S T A T I S T I Q U E
B I O S T A T I S T I Q U E E T
S C I E N C E S A C T U A R I E L L E S
(I S B A)

UNIVERSITÉ CATHOLIQUE DE LOUVAIN



D I S C U S S I O N
P A P E R

2012/34

Approximations for quantiles of life expectancy and annuity values using the parametric improvement rate approach to modelling and projecting mortality

DENUIT, M., HABERMAN, S. and A. RENSHAW

Approximations for quantiles of life expectancy and annuity values using the parametric improvement rate approach to modelling and projecting mortality

Abstract

In this paper, we develop accurate approximations for medians of life expectancy and life annuity pure premiums viewed as functions of future mortality trends as predicted by parametric models of the improvement rates in mortality. Numerical illustrations show that the comonotonic approximations perform well in this case, which suggests that they can be used in practice to evaluate the consequences of the uncertainty in future death rates. Prediction intervals based on 5% and 95% quantiles are also considered but appear to be wider compared to simulated ones. This provides the practitioner with a conservative shortcut, thereby avoiding the problem of simulations within simulations in, for instance, Solvency 2 calculations.

Key words and phrases: Life annuity, life expectancy, mortality projection, comonotonicity, simulation.

1 Introduction

Forecasting mortality in actuarial studies is generally based on extrapolation methods that capture the pattern in historical mortality rates by means of appropriate parametric predictor structures. Foremost among such structures is the Poisson log-bilinear specification proposed by Brouhns, Denuit and Vermunt (2002) and Renshaw and Haberman (2003) in line with the seminal paper by Lee and Carter (1992). Recently, Haberman and Renshaw (2012) have introduced and investigated parametric mortality projection methods based on mortality improvement rates (as opposed to mortality rates). This approach provides an efficient alternative to the direct parametric modelling and projecting of mortality rates.

In this paper, we consider present values of life annuity benefits as functions of the unknown life table applying in the future to the policyholders (as well as life expectancies, corresponding to zero interest rate). Deriving the exact distribution for this random variable requires extensive simulations of numerical evaluations. Therefore, we take the comonotonic approximations proposed by Denuit and Dhaene (2007) and Denuit (2007) in the random walk with drift case and extended to general *ARIMA* models by Denuit, Haberman and Renshaw (2010). Specifically, we adapt this approach to the parametric projection models targeting mortality improvement rates (rather than mortality rates) proposed by Haberman and Renshaw (2012). The approach developed in the present paper helps avoid the requirement to conduct simulations with simulations in, for instance, Solvency 2 reserving calculations. Numerical illustrations show that the comonotonic approximations perform well for medians (and other central quantiles), which suggests that they can be used in practice to evaluate the consequences of the uncertainty in future death rates. Prediction intervals based on 5% and 95% quantiles are also considered but appear to be wider compared to simulated ones. This provides the practitioner with a conservative shortcut avoiding simulations within simulations.

This paper is organised as follows. Section 2 describes the mortality projection method based on parametric improvement rates. The comonotonic approximations are derived in Section 3. Section 4 is devoted to numerical illustrations. The final Section 5 briefly discusses the results.

2 Mortality improvement rates

We consider a rectangular data array, partitioned into unit squares of size one year corresponding to ages $x = x_1, x_2, \dots, x_k$ and periods $t = t_1, t_2, \dots, t_n$. Denote $m_x(t) \equiv m_{x,t}$ the central rate of mortality (or death rate) at age x in period t .

Referring to Haberman and Renshaw (2012), under their *Route II* approach we consider the period-based mortality improvement rates (*MIR*) given by

$$Z_{x,t} = 2 \frac{1 - m_{x,t}/m_{x,t-1}}{1 + m_{x,t}/m_{x,t-1}}.$$

Following modelling and extrapolation, the *MIR* are converted to mortality rates (*MR*) using the reverse relationship

$$m_{x,t_n+j} = m_{x,t_n+j-1} \frac{(2 - z_{x,t_n+j})}{(2 + z_{x,t_n+j})} = m_{x,t_n+j-1} g(z_{x,t_n+j})$$

where the function g is defined as

$$g(z) = \frac{2 - z}{2 + z}.$$

Given the nature of z , which typically take values well within the range $(-0.5, 0.5)$ as can be seen for Figure 3 in Renshaw and Haberman (2012), we are interested in g for $z \in (-1, 2)$, say, where g is positive and decreasing.

In this paper, we consider

$$Z_{x+j,t_n+i} \equiv Z_{ji} = \beta_{x+j} \kappa_{t_n+i},$$

where Z_{ji} and κ_{t_n+i} are random variables and the β_{x+j} are considered as known constants. Then conditional on t_0 we have

$$Z_{ji} \sim N(\mu_{ji}, \sigma_{ji}^2)$$

with

$$\mu_{ji} = \beta_{x+j} E(\kappa_{t_n+i}), \quad \sigma_{ji}^2 = (\beta_{x+j})^2 \text{Var}(\kappa_{t_n+i}).$$

3 Comonotonic approximations

In this section, we show that the theoretical arguments which formed the basis of Denuit, Haberman and Renshaw (2010) can be extended to provide approximations for quantiles of life expectancy and annuity predictions under parametric improvement rate modelling as defined in Section 2. There are, however some fundamental differences, as stressed below.

As in Section 2, we decompose the incremental mortality rate changes into

$$Z_{x+j, t_n+i} = \beta_{x+j} \kappa_{t_n+i} \sim N(\mu_{ji}, \sigma_{ji}^2).$$

Here, the κ_{t_n+i} obey some stochastic process. The next result shows that assuming that the incremental mortality rate changes are perfectly correlated provides a conservative upper bound on future death rates. In this paper, we concentrate on u-type approximations for quantiles as the numerical study performed in Denuit, Haberman and Renshaw (2010) showed that they were more accurate than their l-type counterparts.

Before proceeding with this result, let us recall the definition of some useful stochastic order relations. For more details, we refer the interested reader to Denuit, Dhaene, Goovaerts and Kass (2005). The increasing convex order, or stop-loss order (denoted as \leq_{ICX}) is defined for random variables X and Y as follows: $X \leq_{ICX} Y$ if $E[h(X)] \leq E[h(Y)]$ for all the non-decreasing convex functions h for which the expectations exist. In words, $X \leq_{ICX} Y$ means that X tends to be “smaller” and “less variable” than Y . The supermodular order (denoted as \leq_{SM}) is defined for random vectors (X_1, \dots, X_n) and (Y_1, \dots, Y_n) as $(X_1, \dots, X_n) \leq_{SM} (Y_1, \dots, Y_n)$ if $E[h(X_1, \dots, X_n)] \leq E[h(Y_1, \dots, Y_n)]$ for all the supermodular functions h for which the expectations exist. Recall that a (regular) supermodular function has a non-negative mixed partial derivative with respect to each pair of distinct components. In words, $(X_1, \dots, X_n) \leq_{SM} (Y_1, \dots, Y_n)$ means that the components of (X_1, \dots, X_n) are less positively dependent than the components of (Y_1, \dots, Y_n) .

Property. Let $Z \sim N(0,1)$. We then have the following upper bound on the death rate at age $x+j$ in calendar year t_n+j :

$$m_{x+j,t_n+j} \leq_{ICX} m_{x+j,t_n} \prod_{i=1}^j g(\mu_{ji} + \sigma_{ji}Z).$$

Proof. Whatever the dependent structure between the κ_{t_n+i} , $i = 1, 2, 3, \dots$, we have from Proposition 6.3.7 of Denuit, Dhaene, Goovaerts and Kaas (2005) that

$$(Z_{x+j,t_n+1}, Z_{x+j,t_n+2}, \dots, Z_{x+j,t_n+j}) \leq_{SM} (\mu_{j1} + \sigma_{j1}Z, \mu_{j2} + \sigma_{j2}Z, \dots, \mu_{jj} + \sigma_{jj}Z).$$

The more the κ_{t_n+i} are positively related, the closer is the incremental mortality rate random vector to the upper bound in the \leq_{SM} sense. Now, we get from Property 3.4.61(ii) of Denuit *et al.* (2005) that

$$(g(Z_{x+j,t_n+1}), \dots, g(Z_{x+j,t_n+j})) \leq_{SM} (g(\mu_{j1} + \sigma_{j1}Z), \dots, g(\mu_{jj} + \sigma_{jj}Z))$$

also holds. From Proposition 6.3.9 of Denuit *et al.* (2005), we finally see that

$$\prod_{i=1}^j g(Z_{x+j,t_n+i}) \leq_{ICX} \prod_{i=1}^j g(\mu_{ji} + \sigma_{ji}Z)$$

from which the announced result follows since a ranking in the \leq_{ICX} sense is not affected by location shifts. This completes the proof.

Now, let us denote as ${}_dP_x(t_n | \boldsymbol{\kappa})$ the random d -year survival probability for an individual aged x in calendar year t_n , that is, the conditional probability that this individual reaches age $x+d$ in year t_n+d , given the vector $\boldsymbol{\kappa}$ of the κ_i . It is formally defined as

$${}_dP_x(t_n | \boldsymbol{\kappa}) = \exp(-S_d)$$

where

$$S_d = m_{x,t_n} + \sum_{j=1}^{d-1} m_{x+j,t_n} \prod_{i=1}^j g(Z_{x+j,t_n+i}).$$

We know from Proposition 3.4.29 of Denuit *et al.* (2005) that

$$S_d \leq_{ICX} S_d^u = m_{x,t_n} + \sum_{j=1}^{d-1} m_{x+j,t_n} \prod_{i=1}^j g(\mu_{ji} + \sigma_{ji}Z).$$

Unlike previous applications, the d -year survival probability ${}_dP_x(t_n | \boldsymbol{\kappa}) = \exp(-S_d)$ and its approximation $\exp(-S_d^u)$ do not share the same expected value in the present setting, that is, the equality $E[\exp(-S_d^u)] = E[{}_dP_x(t_n | \boldsymbol{\kappa})]$ does not hold in general and we do not know which expectation is larger. Hence, we cannot compare these two random variables by means of \leq_{ICX} . Therefore, we take $\exp(-S_d^u)$ as a convenient approximation to ${}_dP_x(t_n | \boldsymbol{\kappa})$ and we investigate its accuracy in the next section, based on numerical illustrations.

As a final comment, let us mention that the approach developed in the present section also applies to alternative specifications for $Z_{x,t}$. For instance, the comonotonic approximations also hold for models with a cohort effect as long as the individual under interest belongs to a cohort whose effect can be estimated from the available historical data. Specifically, we can also consider

$$Z_{x+j,t_n+i} = \beta_{x+j} \kappa_{t_n+i} + t_{t_n-x+i-j} \sim N(\mu_{ji}, \sigma_{ji}^2).$$

with

$$\mu_{ji} = \beta_{x+j} E(\kappa_{t_n+i}) + t_{t_n-x+i-j}, \quad \sigma_{ji}^2 = (\beta_{x+j})^2 \text{Var}(\kappa_{t_n+i})$$

as long as the cohort effect ι can be considered as constant (*i.e.* estimated from past data).

4. Numerical illustrations

Let us consider a basic life annuity contract paying 1 unit of currency at the end of each year, as long as the annuitant survives. The random life annuity single premium, that is, the conditional expectation of the payments made to an annuitant aged x in the year t_n given $\kappa_{t_n}, \kappa_{t_n+1}, \kappa_{t_n+2}, \dots$ is

$$a_x(t_n | \boldsymbol{\kappa}) = \sum_{d \geq 1} {}_dP_x(t_n | \boldsymbol{\kappa}) v(0, d),$$

where $v(\cdot, \cdot)$ is the discount factor (precisely, $v(s, t)$ is the present value at time s of a unit payment made at time t). Note that $a_x(t_n | \boldsymbol{\kappa})$ corresponds to the generation aged x in calendar year t_n , and accounts for future mortality improvements experienced by this particular cohort. Clearly, $a_x(t_n | \boldsymbol{\kappa})$ is a random variable that depends on the future

trajectory $\kappa_{t_n}, \kappa_{t_n+1}, \kappa_{t_n+2}, \dots$. An analytical computation of the distribution function of $a_x(t_n | \boldsymbol{\kappa})$ is out of reach.

From the approximation S_d^u derived for S_d , we get the following approximation for the random survival probabilities

$${}_d P_x(t_n | \boldsymbol{\kappa}) \approx \exp\left(-F_{S_d^u}^{-1}(1-U)\right)$$

where U is uniformly distributed on the interval $(0,1)$. Note that the same random variable U is used for all of the values of d , making the approximations to the conditional survival probabilities comonotonic. Hence, we obtain the following approximation for $a_x(t_n | \boldsymbol{\kappa})$

$$a_x(t_n | \boldsymbol{\kappa}) \approx \sum_{d \geq 1} \exp\left(-F_{S_d^u}^{-1}(1-U)\right) v(0, d).$$

Since this approximation is a sum of comonotonic random variables, its quantile functions is additive. So, we obtain the following approximations for the quantile function $F_{a_x(t_n | \boldsymbol{\kappa})}^{-1}(z)$ of $a_x(t_n | \boldsymbol{\kappa})$

$$F_{a_x(t_n | \boldsymbol{\kappa})}^{-1}(z) \approx \sum_{d \geq 1} \exp\left(-F_{S_d^u}^{-1}(1-z)\right) v(0, d)$$

where $F_{S_d^u}^{-1}$ is given by

$$F_{S_d^u}^{-1}(z) \approx m_{x,t_n} + \sum_{j=1}^{d-1} m_{x+j,t_n} \prod_{i=1}^j g\left(\mu_{ji} + \sigma_{ji} \Phi^{-1}(z)\right).$$

The random cohort life expectancy $e_x^\square(t_n | \boldsymbol{\kappa})$ is the conditional expected remaining lifetime of an individual aged x in year t_n , given $\kappa_{t_n}, \kappa_{t_n+1}, \kappa_{t_n+2}, \dots$. Keeping the assumption that deaths are uniformly distributed over each calendar year, this demographic indicator is given by

$$e_x^\square(t_n | \boldsymbol{\kappa}) = \frac{1}{2} + \sum_{d \geq 1} {}_d P_x(t_n | \boldsymbol{\kappa}).$$

We use the \square superscript to indicate that we work along a diagonal band in the Lexis diagram. Except for the additive constant $1/2$, $e_x^\square(t_n | \boldsymbol{\kappa})$ coincides with $a_x(t_n | \boldsymbol{\kappa})$ if we

let the interest rate tend to zero. As was the case for $a_x(t_n | \boldsymbol{\kappa})$, an analytic computation of the distribution function of $e_x^\square(t_n | \boldsymbol{\kappa})$ is out of reach.

From the approximation S_d^u derived for S_d , we get the approximation for $e_x^\square(t_n | \boldsymbol{\kappa})$

$$e_x^\square(t_0 | \boldsymbol{\kappa}) \approx \frac{1}{2} + \sum_{d \geq 1} \exp(-S_d^u).$$

Since the S_d^u 's are sums of comonotonic random variables, their quantile functions are additive. Moreover, the z th quantile of $\exp(-S_d^u)$ is $\exp(-F_{S_d^u}^{-1}(1-z))$. This provides the following approximation for the quantile function $F_{e_x^\square(t_n | \boldsymbol{\kappa})}^{-1}(z)$ of $e_x^\square(t_n | \boldsymbol{\kappa})$

$$F_{e_x^\square(t_n | \boldsymbol{\kappa})}^{-1}(z) \approx \frac{1}{2} + \sum_{d \geq 1} \exp(-F_{S_d^u}^{-1}(1-z)).$$

For the numerical results which follow, we use the 1961-2006 USA male and female mortality experiences with deaths and matching exposures by individual calendar year for individual ages 20-104 (the full age range being 0-109). Preliminary analysis including an analysis of residuals (not reproduced) is supportive of the inclusion of the cohort effects terms ι_{t-x} for males but not for females. Hence, we report the results for the respective H_1 formulation for males and the LC formulation for females by depicting the fitted parameter values in Figure 1. Also included are the period component time series forecasts using the selected $AR(1)$ process for males and the simple $AR(0)$ process for females, fitted as Gaussian regression models. Thus, for males, we have chosen to model the MIR Gaussian structure using the so-called H_1 formulation:

$$Z_{x+j,t_n+i} = \beta_{x+j} \kappa_{t_n+i} + \iota_{t_n-x+i-j} \sim N(\mu_{ji}, \sigma_{ji}^2)$$

whereas for females, we have modelled the structure based on the LC formulation:

$$Z_{x+j,t_n+i} = \beta_{x+j} \kappa_{t_n+i} \sim N(\mu_{ji}, \sigma_{ji}^2)$$

where

$$\mu_{ji} = \beta_{x+j} E(\kappa_{t_n+i}) + \iota_{t_n-x+i-j} \text{ or } \mu_{ji} = \beta_{x+j} E(\kappa_{t_n+i}) \text{ and } \sigma_{ji}^2 = (\beta_{x+j})^2 \text{Var}(\kappa_{t_n+i}).$$

Using these parameters estimates and forecasts we tabulate (Table 1: 1st and 3rd panels) details for life expectancy and 4% annuity predictions, computed by cohort trajectory for

ages 40, 45, 50, ...75 focused on the year 2006. For convenience, we have not used the topping-out procedure advocated by Haberman and Renshaw (2012) for dealing with extrapolating the life table to the oldest ages. For comparison, we also tabulate (Table 1: 2nd and 4th panels) the respective equivalent life expectancies and 4% annuity predictions generated by the simulation method described in Haberman and Renshaw (2012), using a total of 2,000 simulations for each age. Referring to Table 1 and Figure 1 we note the following points:

- On comparing like for like, there is an exceptionally close agreement between the matching theoretical and simulated median predictions. However, the interval prediction widths in the theoretical cases are much wider when compared with the matching simulated cases.
- We note the narrowness of the simulated males prediction intervals, which are appreciably narrower than equivalent simulated intervals for the England & Wales male mortality experience depicted in Figure 8 of Haberman & Renshaw (2012), where topping-out by age has been applied but this seems to have little effect on increasing the interval widths.
- With the exception of a few isolated ages in the male experience, the beta parameters are positive over the full age range for males and females and therefore for both modelling structures. It would be possible to adapt the algorithms so that the beta parameters are constrained to be positive. We note further that the period index forecasts for mortality improvement rates are negative for males using H_1 but positive for females using LC .

5. Discussion

Combining a conservative shift with non necessary conservative ones, the approximations derived in the present paper appear to be very accurate in the centre of the distribution (around the median) but tend to over estimate the tails (left and right). The dominant effect seems thus to be conservative in the tails (on the basis of the numerical illustrations). Using the proposed easy-to-compute approximation may thus be a good strategy as it would considerably reduce the computational burden and save time.

Acknowledgements

The financial support of PARC “Stochastic Modelling of Dependence” 2012-17 awarded by the *Communauté française de Belgique* is gratefully acknowledged by Michel Denuit.

References

Brouhns, N., Denuit, M., Vermunt, J.K. 2002. A Poisson log-bilinear approach to the construction of projected life tables. *Insurance: Mathematics and Economics* 31, 373-393.

Denuit, M. 2007. Distribution of the random future life expectancies in log-bilinear mortality projection models. *Lifetime Data Analysis*, 13, 381-397.

Denuit, M., Dhaene, J. 2007. Comonotonic bounds on the survival probabilities in the Lee-Carter model for mortality projection. *Computational and Applied Mathematics* 203, 169-176.

Denuit, M., Dhaene, J., Goovaerts, M.J., Kaas, R. 2005. *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. Wiley, New York.

Denuit, M., Haberman, S., Renshaw A.E. 2010. Comonotonic approximations to quantiles of life annuity conditional expected present values: extensions to general ARIMA models and comparison with the bootstrap. *ASTIN Bulletin* 40, 331-349.

Haberman, S., Renshaw A.E. 2009. On age-period-cohort parametric mortality rate projections. *Insurance: Mathematics and Economics* 45, 255-270.

Haberman, S., Renshaw A.E. 2012. Parametric mortality improvement rate modelling and projecting. *Insurance: Mathematics and Economics* 50, 309-333.

Lee, R.D., Carter, L., 1992. Modelling and forecasting the time series of US mortality. *Journal of the American Statistical Association* (with discussion) 87, 659-272.

Renshaw, A.E., Haberman, S. 2003. Lee-Carter mortality forecasting with age-specific enhancement. *Insurance: Mathematics and Economics* 33, 255-272.

Renshaw, A.E., Haberman, S. 2006. A cohort based extension to the Lee-Carter model for mortality reduction factors. *Insurance: Mathematics and Economics* 38, 556-570.

<i>age\quantile</i>	life expectancy			4% annuity value		
	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>
40	36.55	45.12	54.92	18.33	19.86	21.25
45	32.60	40.04	48.89	17.28	18.84	20.33
50	28.74	35.09	42.92	16.10	17.66	19.23
55	24.97	30.29	37.02	14.78	16.30	17.91
60	21.33	25.67	31.27	13.33	14.77	16.36
65	17.88	21.32	25.81	11.77	13.09	14.60
70	14.63	17.25	20.67	10.11	11.27	12.63
75	11.64	13.52	15.95	8.41	9.36	10.49

USA females- LC: theoretical predictions

<i>age\quantile</i>	life expectancy			4% annuity value		
	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>
40	43.60	45.13	46.59	19.62	19.86	20.09
45	38.60	40.02	41.51	18.56	18.83	19.11
50	33.82	35.10	36.44	17.36	17.66	17.95
55	29.15	30.28	31.42	15.99	16.30	16.60
60	24.73	25.67	26.70	14.46	14.77	15.09
65	20.47	21.31	22.12	12.77	13.09	13.39
70	16.58	17.25	17.98	10.98	11.27	11.59
75	12.98	13.51	14.06	9.09	9.36	9.63

USA females- LC: simulated predictions

<i>age\quantile</i>	life expectancy			4% annuity value		
	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>
40	40.24	43.96	47.50	18.73	19.43	20.06
45	34.70	37.67	40.62	17.41	18.07	18.68
50	29.57	31.87	34.23	15.93	16.54	17.12
55	25.65	27.44	29.28	14.62	15.16	15.68
60	22.09	23.46	24.88	13.27	13.74	14.20
65	18.46	19.48	20.54	11.73	12.12	12.52
70	15.00	15.71	16.45	10.06	10.37	10.69
75	11.72	12.17	12.64	8.27	8.50	8.73

USA males- H1: theoretical predictions

<i>age\quantile</i>	life expectancy			4% annuity value		
	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>	<i>0.05</i>	<i>0.5</i>	<i>0.95</i>
40	43.38	43.93	44.45	19.31	19.43	19.54
45	37.15	37.65	38.14	17.94	18.07	18.18
50	31.46	31.86	32.27	16.42	16.53	16.65
55	27.09	27.43	27.76	15.05	15.15	15.26
60	23.19	23.45	23.72	13.64	13.73	13.83
65	19.26	19.47	19.67	12.03	12.12	12.20
70	15.55	15.70	15.87	10.30	10.37	10.45
75	12.06	12.17	12.27	8.44	8.49	8.55

USA males- H1: simulated predictions

Table 1. USA male & female 2006 life expectancy and 4% annuity quantile predictions, ages 40(05)75. Comparison of theoretical & simulated predictions