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Abstract. We briefly recall some essential notions on interest rates and zero-coupon bonds. We then define a sound mathematical framework to study a model of the short rate in which the parameters are allowed to vary according to an underlying semi-Markov process. We give some properties of the short rate in our model. We follow by studying the notion of risk-neutral martingale measures in this context. Finally, we discuss the pricing of interest-rate derivatives. In particular, we show that the price of a zero-coupon bond has to satisfy a system of integro-differential equations that is influenced both by the market price of risk and by the market price of regime switch risk.
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1 Introduction

Modelling the uncertainty about the future behavior of interest rates has become a very active topic of research. Some classical continuous-time models include the Vasicek model (see Vasicek [15]), the Hull and White model (see Hull and White [8]) or the CIR model (See Cox *et al.* [2]).

Regime switching models of interest rates have gained some interest in the literature. The idea is to model the fact that the economic environment is not constant through time and that this should be reflected in the model via a change of the value of the parameters. Some papers that deal with this are Landén [11] and Wu and Zeng [16].

Most of the existing literature focuses on homogeneous Markov switching models. However, many authors have shown that markets exhibit some characteristics that are not well captured by homogeneous Markov switching models (let us cite Hong and Li [7], Easly and O'Hara [5] and [6], Diebold and Rudebusch [3] and Durland and McCurdy [4]). An interesting extension that better fits the data is the class of semi-Markov regime switching models. These are flexible and more general than homogeneous Markov models. Our paper deals with such a model, specifically a semi-Markov switching extension of the Vasicek model of the short rate of interest. The aim is to provide a sound mathematical framework for this model and to derive equations that allow to price interest-rate derivatives in this framewok.

2 Basic notation

We consider a financial market defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a filtration $\mathcal{F}_t$ and a brownian motion W. We suppose that it is defined for all times $t \in [0, T]$.

We recall some notions about interest rate theory (see Björk [1] for more on this subject).

Definition 1. A zero coupon bond with maturity T (also called a T-bond) is a contract which guarantees the holder a payment of one unit of currency at time T. We denote by $p(t, T)$ the price of a T-bond a time t. We suppose that $p(t, T)$ is a strictly positive adapted process for all $t \in [0, T]$.

Definition 2. The instantaneous forward rate with maturity T contracted at t is defined by

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T}$$

Definition 3. The instantaneous short rate at time t is defined by

$$r(t) = f(t, t)$$

Our paper will provide a model for the evolution of the short rate r_t. For the moment, we simply assume that r_t is adapted to the filtration $\mathcal{F}_t$. Given the short rate, the money account process or risk free asset (that will serve as numeraire) is defined by

$$B_t = \exp\left\{ \int_0^t r(s) ds \right\}$$

This allows us to introduce risk neutral martingale measures that will be useful in the pricing of interest rate derivatives.

Definition 4. A risk neutral martingale measure will be a measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ and such that for every T, the quantity

$$\frac{p(t, T)}{B_t}$$

is a $\mathbb{P}^*$-martingale.

3 Semi-Markov regime switching model

We define the set $E \subset \mathbb{R}$ by $E = \{1, ..., m\}$ for a fixed $m \in \mathbb{N}$ and we define $\mathcal{E}$ as the sigma-algebra of all the parts of E.

For each $n \in \mathbb{N}$, let (X_n, T_n) be a pair of random variables taking values in $E \times \mathbb{R}^+$. We suppose that the process $(X, T) = \{X_n, T_n; n \geq 0\}$ is a homogeneous Markov renewal process with state space E. The associated semi-Markov kernel is denoted by $Q_{ij}(t)$. We denote by P the transition matrix of the embedded Markov chain.

Remark 1. Given the number of states is finite, the number of jumps in a finite time interval is almost surely finite (for a proof see Pyke [13]).

We impose some regularity conditions on the Markov renewal process:

- No fictitious transitions are allowed i.e. $P_{ii} = 0$.
- No instantaneous transitions are allowed i.e. $Q_{ij}(0) = 0$.
- All states in E "communicate" at all times i.e. $Q_{ij}(t) > 0$, $\forall t > 0$.

Definition 5. Let us define s_t by

$$s_t := \sup(n \geq 0 : T_n \leq t)$$

with $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$ and Y_t as

$$Y_t := X_{s_t}$$

Process Y is called a semi-Markov process with kernel Q.

Definition 6. We define $\mathcal{F}_t$ as the completed filtration generated by process Y_t and W_t i.e., $\mathcal{F}_t = \sigma(Y_s, W_s, N, N \in \mathcal{N}, s \leq t)$ where $\mathcal{N}$ is the collection of all null sets.

The aim is to use Y with the usual tools of stochastic calculus. A step in that direction is made with the following result.

Lemma 1. *Y_t is a semimartingale.*

Proof. It is easy to show that Y is an adapted càdlàg finite variation process.

We denote the set of all possible jumps of Y by Z i.e. $Z = \{z_{ij} = i - j; i, j \in E, i \neq j\}$. Given there are m states, the set Z comprises $m(m-1)$

elements. Let Z_n denote the size of the n^{th} jump of Y. The jump measure of Y is the random integer-valued measure μ on $(0,\infty)\times Z$ defined by

$$\mu = \sum_{n=1}^{\infty} 1\!\!1_{(T_n,Z_n)}$$

We can write

$$Y_t = Y_0 + \int_0^t \int_Z z\mu(ds,dz)$$

This can be written as

$$Y_t = Y_0 + \sum_{z_{ij}\in Z} z_{ij} N_t(z_{ij})$$

where

$$N_t(z_{ij}) = \sum_{n\geq 1} 1\!\!1_{\{T_n\leq t\}} 1\!\!1_{\{Z_n=z_{ij}\}}$$

Proposition 1. *The $\mathbb{P}$-compensator associated with the jump measure μ is given by*

$$\nu(ds,\{z_{ij}\}) = \lambda_s(z_{ij})ds \tag{1}$$

where the intensity $\lambda_s(z_{ij})$ is defined as

$$\lambda_s(z_{ij}) = \sum_{n\geq 0} 1\!\!1_{\{T_n<s\leq T_{n+1}\}} \frac{P_{j,i}\, g(j,i,t-T_n)}{1-\sum_{i\neq j} Q_{j,i}(t-T_n)} 1\!\!1_{\{X_n=j\}}$$

where $g(j,i,t-T_n)$ is the density of the waiting time distribution between state j and i calculated at time $t-T_n$.

Proof. This follows from the general theory of marked point processes and properties of semi-Markov processes (for more details see [12]).

Remark 2. It follows that the intensity associated with process $N_t(z_{ij})$ is simply $\lambda_t(z_{ij})$.

We introduce the backward recurrence time K_t. This process represents the time continuously spent in a state since the last regime switch. We can write

$$K_t = t - T_{N_t(Z)} = t - \sum_{z_{ij}\in Z} \int_0^t K_{s^-} dN_s(z_{ij})$$

We now turn to our model of the short rate. Under $\mathbb{P}$, the short rate is supposed to have the following dynamics:

$$dr_t = (a(Y_t) - b(Y_t)r_t)dt + \sigma(Y_t)dW_t \tag{2}$$

where we define $a(Y_t)$, $b(Y_t)$ and $\sigma(Y_t)$ by

$$a(Y_t) = \sum_{i=1}^{m} a_i \mathbb{1}_{\{Y_t=i\}}$$
$$b(Y_t) = \sum_{i=1}^{m} b_i \mathbb{1}_{\{Y_t=i\}}$$
$$\sigma(Y_t) = \sum_{i=1}^{m} \sigma_i \mathbb{1}_{\{Y_t=i\}}$$

for some given constants $(a_1, ..., a_m)$, $(b_1, ..., b_m)$ and $(\sigma_1, ..., \sigma_m)$ (all the σ_i's are strictly positive).

Proposition 2. *Equation 2 is well defined.*

Proof. The proof is exactly similar to that in Hunt [9], proposition 3.

Remark 3. Equation 2 is an extension of the well-known Vasicek model (see Vasicek [15]) where we allow for the parameters of the model to switch between different states and the switching is controlled by a semi-Markov process Y.

In the classical Vasicek model, r_t is a mean-reverting process. In our setting, we have

Proposition 3. *For $T_n \leq t < T_{n+1}$, $r_t|\mathcal{F}_{T_n} \sim N(\mu, \sigma^2)$ where*

$$\mu = r_{T_n} e^{-b(Y_{T_n})(t-T_n)} + \frac{a(Y_{T_n})}{b(Y_{T_n})}(1 - e^{-b(Y_{T_N})(t-T_n)})$$
$$\sigma^2 = \frac{\sigma^2(Y_{T_n})}{2b(Y_{T_n})}(1 - e^{-2b(Y_{T_n})(t-T_n)})$$

Proof. This is a direct application of Itô's formula and of the properties of semi-Markov processes.

Remark 4. Proposition 3 tells us that for $T_n \leq t < T_{n+1}$, r_t starts in r_{T_n} but moves away from this value as time goes by and that the process tends to locally mean revert around value $\frac{a(Y_{T_n})}{b(Y_{T_n})}$ until the next jump.

4 Martingale measures and derivative pricing

Let N_t represent the multivariate point process ($m(m-1)$-dimensional) whose components are given by $(N_t(z_{ij}))_{z_{ij}\in Z}$. Let λ_t be the multivariate intensity associated to process N_t whose components are given by $(\lambda_t(z_{ij}))_{z_{ij}\in Z}$.

We discuss the existence of risk neutral measures by following this version of the Girsanov theorem.

Theorem 1. *Let θ be a progressively measurable process such that*

$$\int_0^t \theta_s^2 ds < \infty$$

Consider the multivariate point process N_t previously defined with $(\mathbb{P}, \mathcal{F}_t)$-intensity λ_t. Consider a predictable multivariate process $(\psi_t(z_{ij}))_{z_{ij} \in Z}$ such that $\mathbb{P}$-a.s. and for $t \in [0, T]$

$$\sum_{z_{ij} \in Z} \int_0^t \psi_s(z_{ij}) \lambda_s(z_{ij}) ds < \infty$$

Define the process L by:

$$L_t = \exp\left\{-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s dW_s\right\} \prod_{z_{ij} \in Z} \left[\exp\left\{\int_0^t (1 - \psi_s(z_{ij}))\lambda_s(z_{ij}) ds\right\} \prod_{n=1}^{N_t(z_{ij})} \psi_{T_n}(z_{ij})\right]$$

And suppose that for all finite t:

$$\mathbb{E}^{\mathbb{P}}[L_t] = 1$$

Define a probability measure $\mathbb{Q}$ on $\mathcal{F}$ by

$$d\mathbb{Q} = L_t d\mathbb{P}$$

Then, every measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ has the structure above. Furthermore, let $W_t^{\mathbb{Q}}$ be defined as

$$dW_t^{\mathbb{Q}} = dW_t - \theta_t dt$$

then $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-brownian motion. We denote by $N_t^{\mathbb{Q}}$ the multivariate point process N_t whose $\mathbb{Q}$-intensity given by $\lambda_t^{\mathbb{Q}}(z_{ij}) := (\psi_t(z_{ij})\lambda_t(z_{ij}))_{z_{ij} \in Z}$.

Proof. For a proof see [10].

It follows from theorem 1 that under any risk neutral measure $\mathbb{Q}$, we have the following dynamics for processes r_t, Y_t and K_t:

$$\begin{aligned} dr_t &= (a(Y_t) + \theta_t \sigma(Y_t) - b(Y_t) r_t) dt + \sigma(Y_t) dW_t^{\mathbb{Q}} \\ Y_t &= Y_0 + \sum_{z_{ij} \in Z} z_{ij} N_t^{\mathbb{Q}}(z_{ij}) \\ K_t &= t - \sum_{z_{ij} \in Z} \int_0^t K_{s^-} dN_s^{\mathbb{Q}}(z_{ij}) \end{aligned} \tag{3}$$

Specifying a risk neutral measure requires the knowledge of θ, the market price of risk but also the $m(m-1)$ other parameters i.e. the $\psi_t(z_{ij})$'s. These represent the market price of regime switch risks for jumps from state j to state i.

Remark 5. It is clear from equation 3 and proposition 3 that the distribution of $r_t|\mathcal{F}_{T_n}$ under $\mathbb{Q}$ is the same as that in proposition 3 with $a(Y_{T_n})$ replaced by $a(Y_{T_n}) + \theta_{T_n}\sigma(Y_{T_n})$.

Suppose a measure $\mathbb{Q}$ has been chosen. As far as pricing is concerned, it is well known that the price P_t at time t of a contingent claim whose payoff is given by an $\mathcal{F}_T$ measurable square integrable random variable H is given by

$$P_t = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds} H|\mathcal{F}_t] \tag{4}$$

Remark 6. Equation 4 implies that we treat Y_t as an observable variable as argued in Silvestrov and Stenberg [14].

The process (r_t, Y_t) does not -in general- satisfy the Markov property but process (r_t, Y_t, K_t) does and so we can write:

$$P_t = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds} H|r_t, Y_t, K_t]$$

In particular the price of a zero-coupon T-bond is given by

$$P_t = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds}|r_t, Y_t, K_t] = f(t, r, y, k)$$

This leads to the following result

Theorem 2. *The price $f(t, r, y, k)$ of a zero-coupon bond is given by the solution to the following system of integro-differential equations (one for each possible state i)*

$$rf = \mathcal{L}f + \mathcal{S}f \qquad \forall (t, r, k) \in [0, T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$$

where (with the subscript on f indicating the partial derivatives)

$$\mathcal{L}f = f_t(t, r, i, k) + f_k(t, r, i, k) + (a(i) + \theta\sigma(i) - b(i)r) f_r(t, r, i, k) + \frac{1}{2} f_{rr}(t, r, i, k)\sigma^2(i)$$

$$\mathcal{S}f = \sum_{j \neq i} (f(t, r, j, 0) - f(t, r, i, k))\lambda_t^{\mathbb{Q}}(z_{ji})$$

with boundary condition:

$$f(T, r, i, k) = 1 \qquad \forall i \in E\, \forall r,\, k \in \mathbb{R}_0^+$$

Proof. This follows from Feynman-Kac theory and the fact that (r_t, Y_t, K_t) is a Markov process.

Remark 7. In theorem 2, we clearly see the impact of the market price of risk and the market prices of regime switch risk on the price of a zero-coupon bond.

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