INSTITUT DE STATISTIQUE

BIOSTATISTIQUE ET

SCIENCES ACTUARIELLES

(ISBA)

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

DISCUSSION
PAPER

2012/14

UNI- AND MULTIDIMENSIONAL RISK ATTITUDES: SOME UNIFYING THEOREMS

DENUIT, M. and B. REY

UNI- AND MULTIDIMENSIONAL RISK ATTITUDES: SOME UNIFYING THEOREMS

MICHEL DENUIT
Institut de Statistique, Biostatistique et Sciences Actuarielles
Université Catholique de Louvain, Louvain-la-Neuve, Belgium

BÉATRICE REY
Laboratoire SAF - ISFA
Université de Lyon, Université Lyon 1, Lyon, France
beatrice.rey-fournier@univ-lyon1.fr

April 23, 2012

Abstract

The notion of (additive) risk apportionment introduced by Eeckhoudt and Schlesinger (2006) is a preference for a particular class of lotteries combining sure reductions and zero-mean risks. It is equivalent to determining the sign of higher-order derivatives of the utility function. The notion of multiplicative risk apportionment has been defined by Wang and Li (2010) by means of similar lottery preferences. The notion of cross risk apportionment introduced by Eeckhoudt, Rey and Schlesinger (2007) and further studied in Jokung (2011) extends these concepts to the case of two attributes. The present paper aims to provide a unified approach to these closely related notions based on a bivariate model introduced in Denuit, Eeckhoudt and Rey (2010), allowing for a better understanding of changes in risk in the additive, multiplicative and 2-attribute cases. In this setting, it is shown that aversion to increasing the correlation parameter and the impact of initial wealth levels both lead to the different forms or risk apportionment.

1   Introduction and motivation

Individual risk attitudes are described by means of behavioral traits including risk aversion, prudence, temperance and their higher-order extensions. Some economists define these concepts by signing the derivatives of the utility function. Others resort to preferences over pairs of simple lotteries. Specifically, Eeckhoudt and Schlesinger (2006) introduced risk apportionment of order s, $s = 1, 2, \ldots$, by imposing preferences over (seemingly) simple lotteries[1]. Their central result links risk apportionment to the sign of the sth derivative of the utility function, showing equivalence to Ekern (1980)'s definition of sth degree increase in risk. These higher-order risk attitudes entail a preference for combining relatively good outcomes with bad ones and can be interpreted as a desire to disaggregate the harms of unavoidable risks and losses.

Rather than subtracting a fixed level of wealth and/or adding zero-mean risks, as in Eeckhoudt and Schlesinger (2006), Eeckhoudt, Etner and Schroyen (2009) considered proportional changes, reducing wealth by a fixed proportion and/or multiplying wealth by a mean-one random variable.

Wang and Li (2010) further examined changes in multiplicative risk and how these changes affect preferences. Specifically, Wang and Li (2010) defined multiplicative risk apportionment via a lottery preference similar to Eeckhoudt and Schlesinger (2006). They established that multiplicative risk apportionment corresponds to sth order relative risk aversion coefficient above the threshold s, extending the results of Eeckhoudt, Etner and Schroyen (2009) dealing only with $s \in \{1, 2\}$. Extensions from Ekern (1980)'s sth degree increase in risk to sth order stochastic dominance can be found in Eeckhoudt, Schlesinger and Tsetlin (2009) for additive risks and in Chiu, Eeckhoudt and Rey (2010) for multiplicative risks.

Additive and multiplicative risk apportionment are univariate notions, defined from a one-argument utility function. When the preferences of the decision-maker depend on several attributes. Eeckhoudt, Rey and Schlesinger (2007) introduced the notion of cross risk apportionment extending the concept of correlation aversion due to Epstein and Tanny (1980) to cross prudence and cross temperance. Jokung (2011) generalized cross risk apportionment to any order s.

In this paper, we derive in a unified way all the results summarized above, using the bivariate model introduced in Denuit, Eeckhoudt and Rey (2010), extending the one initially proposed by Epstein and Tanny (1980). First, we exploit the aversion to increasing the correlation parameter, which represents an aversion to probability spreads in 4-state lotteries. For 50-50 marginal lotteries and specific values of the correlation parameter, we recover the results obtained by Eeckhoudt and Schlesinger (2006), Eeckhoudt, Rey and Schlesinger (2007), Eeckhoudt, Etner and Schroyen (2009), Wang and Li (2010), Chiu, Eeckhoudt and Rey (2010), and Jokung (2011). Second, we show that these results also follow from the natural idea that the sensitivity to such detrimental changes should decrease with initial wealth.

The rest of the paper is organized as follows. In Section 2, we review the risk appor-

---

[1]These lotteries were characterized by Roger (2011) who established that they only differ by their moments of order greater than or equal to s. See also Ebert (2010).

tionment concepts used in the paper and we relate these notions to stochastic dominance rules. By means of specific 4-state lotteries, we show in Section 3 that decision-makers who dislike probability spreads, i.e. transfers of probability mass from the inner wealth levels to the outer wealth levels, are precisely those who exhibit risk apportionment. Section 4 is devoted to the multiplicative case. In Section 5 we measure the dislike for correlation by utility premiums and we show that risk apportionment is a consequence of the intuitive idea that a decision-maker should be less sensitive to an increase in the correlation parameter when he gets richer.

All the random variables considered in this paper are valued in some interval $[a, b]$ of the real line, the bivariate random vectors being valued in a rectangle $[a_1, b_1] \times [a_2, b_2]$ of the real plane. Whatever the function g, a property like $g \geq 0$ is meant to hold over its domain. Henceforth, we denote as u', u'', and u''' the first derivative, the second derivative, and the third derivative of the utility function u. More generally, we write $u^{(s)}$ for the sth derivative of u, $s = 1, 2, 3, 4, \ldots$; the notations u', u'', and u''' and $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$, respectively, can be used interchangeably. As decision-makers are usually assumed to be non-satiated and risk-averse, u is non-decreasing and concave. If u is differentiable, this means that $u' \geq 0$ and $u'' \leq 0$. Given a 2-attribute utility function v defined on (a subset of) the real plane, we denote as $v^{(i,j)}$ the (i,j)th cross derivative of v with respect to x_1 and x_2, that is, $v^{(i,j)}(x_1, x_2) = \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} v(x_1, x_2)$. Henceforth, the notation u is used for a single attribute utility function whereas v denotes a 2-attribute utility function.

2 Risk apportionment

2.1 Additive risk apportionment

Recall the definitions of the lotteries introduced by Eeckhoudt and Schlesinger (2006). Let J_s, $s = 1, 2, \ldots$, be a sequence of independent, Bernoulli distributed random variables with parameter $\frac{1}{2}$, that is,

$$\Pr[J_s = 0] = \Pr[J_s = 1] = \frac{1}{2}.$$

Consider a positive constant $k > 0$ and a sequence $\mathcal{E}_1, \mathcal{E}_2, \ldots$ of independent zero-mean risks (i.e. $E[\mathcal{E}_i] = 0$ for all i) with finite moments, independent of the J_s. Then, starting from $A_1 = -k$, $B_1 = 0$, $A_2 = \mathcal{E}_1$ and $B_2 = 0$, define iteratively

$$
\begin{aligned}
A_s &= J_{s-2}B_{s-2} + (1 - J_{s-2})(A_{s-2} + \mathcal{E}_{[s/2]}) \\
B_s &= J_{s-2}A_{s-2} + (1 - J_{s-2})(B_{s-2} + \mathcal{E}_{[s/2]})
\end{aligned}
$$

where $[s/2]$ is the largest integer smaller than or equal to $s/2$. A decision-maker exhibits risk apportionment of order s, if he prefers $w + B_s$ over $w + A_s$ for all initial wealth levels w, for all sure losses k and zero-mean risks $\mathcal{E}_i$.

For instance, the lotteries for prudence $(s = 3)$ are

$$
\begin{aligned}
A_3 &= J_1 B_1 + (1 - J_1)(A_1 + \mathcal{E}_1) = (1 - J_1)(\mathcal{E}_1 - k) \\
B_3 &= J_1 A_1 + (1 - J_1)(B_1 + \mathcal{E}_1) = J_1(-k) + (1 - J_1)\mathcal{E}_1.
\end{aligned}
$$

A prudent decision-maker, thus, prefers to disaggregate the sure loss $-k$ and the zero-mean risk $\mathcal{E}_1$. That is, he prefers to have the two items in different, rather than in the same of two equally likely states of nature. In other words, he disaggregates the two harms of a sure loss and a zero-mean risk.

Eeckhoudt, Schlesinger and Tsetlin (2009) further established that a decision-maker exhibiting risk apportionment prefers not to group the two relatively bad lotteries in the same state, where bad is defined via higher-order stochastic dominance.

Eeckhoudt and Schlesinger (2006) proved that a differentiable utility function u satisfies risk apportionment of order s if, and only if, it fulfils the condition $(-1)^{s+1}u^{(s)} \geq 0$. The common preferences of all the decision-makers satisfying risk apportionment of degree s, thus, correspond to the concept of increase in sth degree risk introduced by Ekern (1980). Formally, given two random variables X and Y, we write $X \preceq_{s,=} Y$ if every decision-maker whose preferences exhibit risk apportionment of order s prefers Y over X, that is, if the inequality $E[u(X)] \leq E[u(Y)]$ holds for every utility function u such that $(-1)^{s+1}u^{(s)} \geq 0$. Note that such random variables have then to share the same first $s-1$ moments, that is, $E[X^k] = E[Y^k]$ has to hold for $k = 1, 2, \ldots, s-1$ (which explains the meaning of the equality sign involved in the subscript to $\preceq_{s,=}$).

The common preferences of all the decision-makers satisfying risk apportionment of orders 1 to s correspond to sth order stochastic dominance. Formally, given two random variables X and Y, we write $X \preceq_s Y$ if every decision-maker whose preferences exhibit risk apportionment of orders 1 to s prefers Y over X, that is, if the inequality $E[u(X)] \leq E[u(Y)]$ holds for every utility function u such that $(-1)^{k+1}u^{(k)} \geq 0$ for $k = 1$ to s.

When the first $s-1$ moments of X and Y are equal, sth-order stochastic dominance coincides with Ekern's (1980) concept of increase in sth-degree risk. More precisely,

$$X \preceq_{s,=} Y \Leftrightarrow X \preceq_s Y \text{ and } E[X^k] = E[Y^k] \text{ for } k = 1, 2, \ldots, s-1.$$

As an example, X is an increase in second-degree risk over Y if Y dominates X via second-order stochastic dominance and both random variables have equal mean. This is what Rothshild and Stiglitz (1970) define as a "mean-preserving increase in risk". Similarly, Menezes, Geiss and Tressler (1980) describe an increase in third-degree risk, which is also called an "increase in downside risk" corresponding to $X \preceq_3 Y$ with $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$.

2.2 Cross risk apportionment

Decisions under risk are often multidimensional, where the decision-maker's preferences depend on several attributes. For example, an individual might be concerned about both the level of wealth and the condition of health. As another example, consider an intertemporal model in which preferences depend on the lifetime path of consumption. In this paper, we consider the case of two attributes. To ease the exposition, we speak about two wealth levels even if attributes may be more general, related to health, environment or represent various goods, for instance.

Eeckhoudt, Rey and Schlesinger (2007) defined a preference ordering over a set of simple lotteries, extending the concept of correlation aversion due to Epstein and Tanny (1980). Following the approach suggested by Eeckhoudt and Schlesinger (2006) for risk apportionment

in one dimension, this lead to multivariate analogues of prudence and temperance, labelled as cross prudence and cross temperance. These particular lottery preferences turn out to be equivalent to signing the cross derivatives of the 2-attribute utility function and are therefore referred to as cross risk apportionment. Cross risk apportionment has been further studied by Jokung (2011) who extended to 2-argument utility functions Eeckhoudt, Schlesinger and Tsetlin (2009)'s result relating risk apportionment to the preference for "combining good with bad". Jokung (2011) also extended Eeckhoudt, Rey and Schlesinger (2007) to higher orders, as explained next.

Let $J_{(s_1,s_2)}$ be a sequence of independent, Bernoulli distributed random variables with parameter $\frac{1}{2}$, that is,

$$\Pr[J_{(s_1,s_2)} = 0] = \Pr[J_{(s_1,s_2)} = 1] = \frac{1}{2}, \quad s_1, s_2 = 1, 2, \ldots$$

According to Jokung (2011), preferences are said to satisfy cross risk apportionment of order (s_1, s_2) if the decision-maker always prefers $J_{(s_1,s_2)}(w_1 + B_{s_1}, w_2 + A_{s_2}) + (1 - J_{(s_1,s_2)})(w_1 + A_{s_1}, w_2 + B_{s_2})$ over $J_{(s_1,s_2)}(w_1 + B_{s_1}, w_2 + B_{s_2}) + (1 - J_{(s_1,s_2)})(w_1 + A_{s_1}, w_2 + A_{s_2})$ whatever the initial wealth levels w_1 and w_2. According to Jokung (2011, Theorem 5), this is equivalent to the condition $(-1)^{s_1+s_2+1} v^{(s_1,s_2)} \geq 0$ imposed on the 2-attribute utility function v. For $s_1, s_2 \in \{1, 2\}$, cross risk apportionment reduces to correlation aversion, cross prudence and cross temperance. If the 2-attribute utility function v satisfies cross risk apportionment of order (k_1, k_2) for all $k_1 = 0, 1, \ldots, s_1$, $k_2 = 0, 1, \ldots, s_2$, such that $k_1 + k_2 \geq 1$, that is, if $(-1)^{k_1+k_2+1} v^{(k_1,k_2)} \geq 0$ for all $k_1 = 0, 1, \ldots, s_1$, $k_2 = 0, 1, \ldots, s_2$, such that $k_1 + k_2 \geq 1$ then it is said to exhibit correlation aversion of order (s_1, s_2) in the terminology of Denuit and Eeckhoudt (2010).

The common preferences of all the decision-makers exhibiting correlation aversion of order (s_1, s_2) generate the bivariate (s_1, s_2)th stochastic dominance studied in Denuit, Lefevre and Mesfioui (1999). More specifically, given two pairs of random variables (X_1, X_2) and (Y_1, Y_2), (X_1, X_2) is said to be smaller than (Y_1, Y_2) in the bivariate (s_1, s_2)th stochastic dominance, denoted by $(X_1, X_2) \preceq_{(s_1,s_2)} (Y_1, Y_2)$, when $E[v(X_1, X_2)] \leq E[v(Y_1, Y_2)]$ for all the utility functions v exhibiting correlation aversion of order (s_1, s_2), that is, for all the utility functions v such that $(-1)^{k_1+k_2+1} v^{(k_1,k_2)} \geq 0$ for all $k_1 = 0, 1, \ldots, s_1$, $k_2 = 0, 1, \ldots, s_2$, with $k_1 + k_2 \geq 1$.

3   Correlation increasing transformation and expected utility

Let us consider the bivariate random vector (I_1, I_2) with distribution

$$\begin{aligned}
\Pr[I_1 = 0, I_2 = 0] &= p_1 p_2 + \rho \\
\Pr[I_1 = 1, I_2 = 0] &= (1 - p_1) p_2 - \rho \\
\Pr[I_1 = 0, I_2 = 1] &= p_1 (1 - p_2) - \rho \\
\Pr[I_1 = 1, I_2 = 1] &= (1 - p_1)(1 - p_2) + \rho
\end{aligned}$$

where, without loss of generality, we assume that $p_1 \leq p_2$, and that ρ is such that $-p_1 p_2 \leq \rho \leq p_1(1 - p_2)$.

Compared to the case when I_1 and I_2 are mutually independent, we see that ρ is added to the probability mass at (0,0) and (1,1), whereas the same quantity is subtracted from the probability mass at (0,1) and (1,0). Clearly, the covariance between the binary random variables I_1 and I_2 is given by

$$\mathbb{Cov}[I_1, I_2] = \Pr[I_1 = 1, I_2 = 1] - \Pr[I_1 = 1]\Pr[I_2 = 1] = \rho$$

so that ρ can be considered as a correlation parameter between the random variables I_1 and I_2.

Consider a decision-maker with utility function u and initial wealth w facing the risky outcome $a_1I_1 + a_2I_2$ for some non-negative constants a_1 and a_2. An increase in the correlation parameter ρ is welfare deteriorating for a risk-averse decision-maker, as pointed out by Epstein and Tanny (1980).

Note that increasing ρ increases the correlation between the random variables a_1I_1 and a_2I_2 faced by the decision-maker. Considering the 4-state lottery $a_1I_1 + a_2I_2$, we also see that increasing ρ transfers some probability mass from the inner outcomes a_1 and a_2 to the outer outcomes 0 and $a_1 + a_2$. Any risk-averse decision-maker dislikes such a probability spread, i.e. an increase in the probability of getting the outer outcomes and a corresponding decrease in the probability of getting the inner ones.

Given independent random variables X_1, X_2, Y_1, and Y_2, independent from (I_1, I_2), the bivariate random vector $((1 - I_1)X_1 + I_1Y_1, (1 - I_2)X_2 + I_2Y_2)$ can be seen as a lottery with the following four outcomes:

$$((1 - I_1)X_1 + I_1Y_1, (1 - I_2)X_2 + I_2Y_2) = \begin{cases} (X_1, X_2) \text{ with probability } p_1p_2 + \rho, \\ (X_1, Y_2) \text{ with probability } p_1(1 - p_2) - \rho, \\ (Y_1, X_2) \text{ with probability } (1 - p_1)p_2 - \rho, \\ (Y_1, Y_2) \text{ with probability } (1 - p_1)(1 - p_2) + \rho. \end{cases}$$

Assume that $X_1 \preceq_{s_1} Y_1$ and $X_2 \preceq_{s_2} Y_2$ both hold for some s_1 and s_2. Then, $(X_1, X_2) \preceq_{(s_1,s_2)} (Y_1, Y_2)$ holds true, as well as $(X_1, X_2) \preceq_{(s_1,s_2)} (Y_1, X_2) \preceq_{(s_1,s_2)} (Y_1, Y_2)$ and $(X_1, X_2) \preceq_{(s_1,s_2)} (X_1, Y_2) \preceq_{(s_1,s_2)} (Y_1, Y_2)$. The interpretation of $((1-I_1)X_1 + I_1Y_1, (1-I_2)X_2 + I_2Y_2)$ in terms of a 4-state lottery shows that increasing ρ makes the extreme cases (X_1, X_2) and (Y_1, Y_2) with respect to $\preceq_{(s_1,s_2)}$ more likely to happen, compared to the intermediate cases (X_1, Y_2) and (Y_1, X_2).

The expected utility for a decision-maker with initial wealth levels (w_1, w_2), facing $((1 - I_1)X_1 + I_1Y_1, (1 - I_2)X_2 + I_2Y_2)$ is denoted as

$$\mathbb{U}(\rho, w_1, w_2) = E[v(w_1 + (1 - I_1)X_1 + I_1Y_1, w_2 + (1 - I_2)X_2 + I_2Y_2)]. \qquad (3.1)$$

The next result is central in our analysis. It shows that the expected utility (3.1) is a decreasing function of the correlation parameter ρ when the decision-maker exhibits cross risk apportionment (resp. correlation aversion) when the X_is precede the Y_is in Ekern (1980)'s increase in risk (resp. stochastic dominance).

Theorem 3.1. *In addition to $((1-I_1)X_1 + I_1Y_1, (1-I_2)X_2 + I_2Y_2)$ described above, consider another random vector $((1 - I_1')X_1 + I_1'Y_1, (1 - I_2')X_2 + I_2'Y_2)$ where (I_1', I_2') has the same*

distribution as (I_1, I_2), except that ρ is replaced with $\rho' > \rho$ (ρ' such that $-p_1 p_2 \leq \rho' \leq p_1(1 - p_2)$), that is,

$$
\begin{array}{rcl}
\Pr[I_1' = 0, I_2' = 0] & = & p_1 p_2 + \rho' \\
\Pr[I_1' = 1, I_2' = 0] & = & (1 - p_1) p_2 - \rho' \\
\Pr[I_1' = 0, I_2' = 1] & = & p_1 (1 - p_2) - \rho' \\
\Pr[I_1' = 1, I_2' = 1] & = & (1 - p_1)(1 - p_2) + \rho'.
\end{array}
$$

(i) Consider a 2-attribute utility function v exhibiting correlation aversion of order (s_1, s_2), X_1 and Y_1 such that $X_1 \preceq_{s_1} Y_1$, and X_2 and Y_2 such that $X_2 \preceq_{s_2} Y_2$, then

$$\mathbb{U}(\rho', w_1, w_2) \leq \mathbb{U}(\rho, w_1, w_2) \text{ whenever } \rho < \rho', \tag{3.2}$$

$$\Leftrightarrow \frac{\partial}{\partial \rho} \mathbb{U}(\rho, w_1, w_2) \leq 0.$$

(ii) Consider a 2-argument utility function v exhibiting cross risk apportionment of order (s_1, s_2), X_1 and Y_1 such that $X_1 \preceq_{s_1,=} Y_1$, and X_2 and Y_2 such that $X_2 \preceq_{s_2,=} Y_2$, then (3.2) holds true.

Proof. Without loss of generality, we can set $w_1 = w_2 = 0$ and simply denote $\mathbb{U}(\rho) = \mathbb{U}(\rho, 0, 0)$. Item (i) is a direct consequence of Denuit, Eeckhoudt and Rey (2010) who established that $X_1 \preceq_{s_1} Y_1$ and $X_2 \preceq_{s_2} Y_2$ imply

$$((1 - I_1')X_1 + I_1'Y_1, (1 - I_2')X_2 + I_2'Y_2) \preceq_{(s_1, s_2)} ((1 - I_1)X_1 + I_1 Y_1, (1 - I_2)X_2 + I_2 Y_2). \tag{3.3}$$

Let us now turn to (ii). It is easily seen that

$$
\begin{array}{rcl}
\mathbb{U}(\rho) & = & E[v((1 - I_1)X_1 + I_1 Y_1, (1 - I_2)X_2 + I_2 Y_2)] \\
& = & (p_1 p_2 + \rho)E[v(X_1, X_2)] + ((1 - p_1)p_2 - \rho)E[v(Y_1, X_2)] \\
& & + (p_1(1 - p_2) - \rho)E[v(X_1, Y_2)] + ((1 - p_1)(1 - p_2) + \rho)E[v(Y_1, Y_2)] \\
& = & \rho\Big(E[v(X_1, X_2)] - E[v(Y_1, X_2)] - E[v(X_1, Y_2)] + E[v(Y_1, Y_2)]\Big) \\
& & + \text{ constant with respect to } \rho.
\end{array}
$$

The coefficient multiplying ρ in the expression for $\mathbb{U}(\rho)$ is negative since the functions g defined as

$$g(x) = E[v(x, X_2)] - E[v(x, Y_2)]$$

is such that $(-1)^{s_1+1} g^{(s_1)} \geq 0$ provided $(-1)^{s_1+s_2+1} v^{(s_1, s_2)} \geq 0$. $\square$

The next corollary shows that the results about cross risk apportionment derived in Jokung (2011) can be viewed as a particular case of Theorem 3.1 for appropriate value of p_1, p_2, ρ and ρ'.

Corollary 3.2. (i) With $p_1 = p_2 = \frac{1}{2}$, $\rho = -\frac{1}{4}$ and $\rho' = \frac{1}{4}$, Theorem 3.1(i) gives

$$\mathbb{U}(\rho) = \frac{1}{2}E[v(X_1, Y_2)] + \frac{1}{2}E[v(Y_1, X_2)] \geq \mathbb{U}(\rho') = \frac{1}{2}E[v(X_1, X_2)] + \frac{1}{2}E[v(Y_1, Y_2)] \tag{3.4}$$

provided $X_1 \preceq_{s_1} Y_1$ and $X_2 \preceq_{s_2} Y_2$ which is Theorem 3 in Jokung (2011). Inequality (3.4) *expresses a preference for the lottery offering either* (X_1, Y_2) *or* (Y_1, X_2) *with equal probability over the one offering either* (X_1, X_2) *or* (Y_1, Y_2) *with equal probability by all the decision-makers exhibiting correlation aversion of order* (s_1, s_2).

(ii) *With* $p_1 = p_2 = \frac{1}{2}$, $\rho = -\frac{1}{4}$ *and* $\rho' = \frac{1}{4}$, *Theorem 3.1(ii) ensures that inequality (3.4) is valid provided* $X_1 \preceq_{s_1,=} Y_1$ *and* $X_2 \preceq_{s_2,=} Y_2$, *which is Theorem 5 in Jokung (2011). This inequality expresses a preference over lotteries described in (i) by all the decision-makers exhibiting cross risk apportionment of order* (s_1, s_2).

The notion of additive-risk apportionment arises when the 2-attribute utility function takes the additive form

$$v(x_1, x_2) = u(x_1 + x_2). \tag{3.5}$$

With this specification, the analysis is simplified because all notions of cross risk apportionment are captured by the expression of a unique derivative of u. Indeed, $v^{(k_1, k_2)} = u^{(k_1 + k_2)}$ when (3.5) holds. Theorem 3.1 then rewrites as follows.

Corollary 3.3. *(i) With* $p_1 = p_2 = \frac{1}{2}$, $\rho = -\frac{1}{4}$ *and* $\rho' = \frac{1}{4}$, *Theorem 3.1(i) gives*

$$\mathbb{U}(\rho) = \frac{1}{2}E[u(X_1 + Y_2)] + \frac{1}{2}E[u(Y_1 + X_2)]$$

$$\geq \mathbb{U}(\rho') = \frac{1}{2}E[u(X_1 + X_2)] + \frac{1}{2}E[u(Y_1 + Y_2)] \tag{3.6}$$

provided $X_1 \preceq_{s_1} Y_1$, $X_2 \preceq_{s_2} Y_2$ *and the single-attribute utility function u exhibits risk apportionment of degrees 1 to* $s_1 + s_2$, *which is Theorem 3 in Eeckhoudt, Schlesinger and Tsetlin (2009).*

(ii) *With* $p_1 = p_2 = \frac{1}{2}$, $\rho = -\frac{1}{4}$ *and* $\rho' = \frac{1}{4}$, *Theorem 3.1(ii) ensures that inequality (3.6) is valid provided* $X_1 \preceq_{s_1,=} Y_1$, $X_2 \preceq_{s_2,=} Y_2$ *and the single-attribute utility function u exhibits risk apportionment of degree* $s_1 + s_2$, *which is essentially the result obtained by Eeckhoudt and Schlesinger (2006) as shown in the corollary to Theorem 3 in Eeckhoudt, Schlesinger and Tsetlin (2009).*

4 Multiplicative risk-apportionment

In this section, we restrict our analysis to non-negative random variables. Rather than subtracting a fixed level of wealth k and/or adding zero-mean risks $\mathcal{E}_i$, as done in the preceding section, Eeckhoudt, Etner and Schroyen (2009) looked at proportional changes: reducing wealth by a fixed proportion and/or multiplying wealth by a mean-one random variable. In a paper dealing essentially with an interpretation of the signs of successive derivatives of u, Caballe and Pomansky (1996) introduced the sth degree coefficient of relative risk aversion $r_u^{(s)}$, $s = 1, 2, \ldots$, defined as $r_u^{(s)}(x) = -xu^{(s+1)}(x)/u^{(s)}(x)$. After Eeckhoudt, Etner and Schroyen (2009), Wang and Li (2010), and Chiu, Eeckhoudt and Rey (2010), multiplicative risk apportionment of order s is captured by the non-negative sign of the difference between $r_u^{(s)}$ and s, i.e. by the inequality $r_u^{(s)} \geq s$.

In this section, we show that this inequality naturally arises by considering the special case where the utility function v is multiplicative in x_1 and in x_2, that is,

$$v(x_1, x_2) = u(x_1 x_2). \tag{4.1}$$

The question is, thus, to derive conditions on u so that v exhibits cross risk apportionment or correlation aversion of order (s_1, s_2). The analysis is more difficult than in the additive case because risk apportionment is no longer captured by the unique derivative $u^{(s_1+s_2)}$.

The next result corresponds to Proposition 1 in Eeckhoudt, Etner and Schroyen (2009). It is derived here from a different perspective, allowing for a direct extension to higher orders.

Proposition 4.1. *Consider a single-attribute utility function u satisfying risk apportionment of order 1. Then, v defined from u by (4.1) exhibits correlation aversion of order (1,1) if, and only if, $r_u^{(1)} \geq 1$.*

Proof. A direct calculation gives

$$v^{(1,1)}(x_1, x_2) = u^{(1)}(x_1 x_2) + x_1 x_2 u^{(2)}(x_1 x_2).$$

If we require that v exhibits correlation aversion of order (1,1), that is, $v^{(1,1)} \leq 0$, we see that

$$v^{(1,1)}(x_1, x_2) \leq 0 \text{ for all } x_1, x_2 \geq 0$$
$$\Leftrightarrow \quad u^{(1)}(\xi) + \xi u^{(2)}(\xi) \leq 0 \text{ for all } \xi \geq 0$$
$$\Leftrightarrow \quad r_u^{(1)} \geq 1 \text{ provided } u^{(1)} \geq 0,$$

which ends the proof. $\square$

Note that $u^{(1)} \geq 0$ together with $r_u^{(1)} \geq 1$ imply $u^{(2)} \leq 0$.

The next result corresponds to Proposition 2 in Eeckhoudt, Etner and Schroyen (2009), transposed in our setting. It gives the conditions on u to ensure that v exhibits cross risk apportionment of order $(2,1)$, and conditions on u to ensure that v exhibits correlation aversion of orders $(2,1)$ and $(1,2)$.

Proposition 4.2. *Consider a single-attribute utility function u and define the 2-attribute utility v from (4.1). Then,*

(i) *if u exhibits risk apportionment of order 2 then v exhibits cross risk apportionment of order (2,1) if, and only if, $r_u^{(2)} \geq 2$.*

(ii) *if u exhibits risk apportionment of orders 1 and 2 then v exhibits correlation aversion of orders (2,1) and (1,2) if, and only if, $r_u^{(1)} \geq 1$ and $r_u^{(2)} \geq 2$.*

Proof. A direct calculation gives

$$v^{(2,1)}(x_1, x_2) = 2 x_2 u^{(2)}(x_1 x_2) + x_1 x_2^2 u^{(3)}(x_1 x_2).$$

8

If we require that v exhibits cross risk apportionment of order (2,1), we see that

$$v^{(2,1)}(x_1, x_2) \geq 0 \text{ for all } x_1, x_2 \geq 0$$
$$\Leftrightarrow \quad 2u^{(2)}(\xi) + \xi u^{(3)}(\xi) \geq 0 \text{ for all } \xi \geq 0$$
$$\Leftrightarrow \quad r_u^{(2)} \geq 2 \text{ provided } u^{(2)} \leq 0, \text{ which proves (i).}$$

Statement (ii) is a direct consequence of (i) together with Proposition 4.1. $\qquad\square$

However, Propositions 4.1-4.2 do not readily extend to correlation aversion of order (s_1, s_2) for higher values of s_1 and s_2 because controlling the sign of the higher degree partial derivatives of v involves coefficients of relative risk aversion of different degrees. Whereas increasing both s_1 and s_2 to values larger than 2 appears to be difficult, the analysis remains feasible if we fix the value of s_2 to 1 and then increase the value of s_1. For instance, with $s_1 = 3$, we have

$$v^{(3,1)}(x_1, x_2) = 3x_2^2 u^{(3)}(x_1 x_2) + x_1 x_2^3 u^{(4)}(x_1 x_2).$$

Hence, imposing $v^{(3,1)} \leq 0$ we see that

$$v^{(3,1)}(x_1, x_2) \leq 0 \text{ for all } x_1, x_2 \geq 0$$
$$\Leftrightarrow \quad 3u^{(3)}(\xi) + \xi u^{(4)}(\xi) \leq 0 \text{ for all } \xi \geq 0$$
$$\Leftrightarrow \quad r_u^{(3)}(\xi) \geq 3 \text{ provided } u \text{ satisfies risk apportionement of order 3.}$$

The next result shows that this extends to higher orders. Let us now fix $s_1 = 1$ and $s_2 = s \geq 1$.

Theorem 4.3. *Consider a single-attribute utility function u and define the 2-attribute utility v from (4.1). Then,*

(i) *if u exhibits risk apportionment of order s then v exhibits cross risk apportionment of order $(s, 1)$ if, and only if, $r_u^{(s)} \geq s$.*

(ii) *if u exhibits risk apportionment of orders 1 to s then v exhibits correlation aversion of orders $(s, 1)$ if, and only if, $r_u^{(k)} \geq k$ for $k = 1, 2, \ldots, s$.*

Proof. Starting from $v^{(1,1)}$, it is easily seen that

$$\frac{\partial^{k-1}}{\partial x_1^{k-1}} v^{(1,1)}(x_1, x_2) = k x_2^{k-1} u^{(k)}(x_1 x_2) + x_1 x_2^k u^{(k+1)}(x_1 x_2)$$

so that

$$(-1)^k v^{(k,1)}(x_1, x_2) \geq 0 \text{ for all } x_1, x_2 \geq 0$$
$$\Leftrightarrow \quad (-1)^k \left(k u^{(k)}(\xi) + \xi u^{(k+1)}(\xi) \right) \geq 0 \text{ for all } \xi \geq 0$$
$$\Leftrightarrow \quad r_u^{(k)}(\xi) \geq k \text{ for all } \xi \geq 0.$$

This ends the proof. $\qquad\square$

As v defined in (4.1) is symmetric in its arguments x_1 and x_2, v exhibiting correlation aversion of order $(s,1)$ is equivalent to v exhibiting correlation aversion of order $(1,s)$.

In view of Theorem 4.3, we are now ready to rewrite Theorem 3.1 in the multiplicative case if we fix $s_1 = 1$ and $s_2 = s \geq 1$. Specifically, we consider the degenerated lotteries $X_1 = a$ and $Y_1 = b$ where a and b are two positive constants such that $a < b$. Then, $X_1 \preceq_1 Y_1$ obviously holds. Considering $p_1 = p_2 = 1/2$, $\rho = -1/4$ and $\rho' = 1/4$, we obtain the following result.

Proposition 4.4. *(i) Consider a utility function u satisfying risk apportionment of order s such that $r_u^{(s)} \geq s$, two positive constants a and b such that $a < b$, and X_2 and Y_2 such that $X_2 \preceq_{s,=} Y_2$, then*

$$\frac{1}{2}\mathbb{E}[u(aX_2)] + \frac{1}{2}\mathbb{E}[u(bY_2)] \leq \frac{1}{2}\mathbb{E}[u(aY_2)] + \frac{1}{2}\mathbb{E}[u(bX_2)]. \tag{4.2}$$

Expressed in terms of preference over 50-50 lotteries, this is Proposition 3.1 in Wang and Li (2010) and Theorem 2(i) in Chiu, Eeckhoudt and Rey (2010).

(ii) Consider a utility function u satisfying risk apportionment of orders 1 to s such that $r_u^{(k)} \geq k$ for $k = 1, 2, \ldots, s$, two positive constants a and b such that $a < b$, and X_2 and Y_2 such that $X_2 \preceq_s Y_2$, then inequality (4.2) holds true. Expressed in terms of preference over 50-50 lotteries, this is Theorem 2(ii) in Chiu, Eeckhoudt and Rey (2010).

5 Risk apportionment as a lower sensitivity to ρ when richer

Whereas risk aversion means that the decision-maker dislikes an increase in the correlation parameter ρ when final wealth is given by $w + a_1 I_1 + a_2 I_2$, prudence means that the decision-maker is less sensitive to an increase in ρ when he gets richer. Denuit and Rey (2010) established that risk apportionment of any degree can be interpreted as a lower sensitivity to detrimental changes when the decision-maker gets richer.

Let us now establish that the natural idea that aversion to probability spreads in our specific 4-state lotteries should decrease as wealth increases leads to cross risk apportionment. From a mathematical point of view, this amounts to requiring that the expected utility is supermodular in the initial wealth level and correlation parameter ρ when the decision-maker is faced with these specific lotteries. Supermodularity here means a lower sensitivity to an increase in the correlation parameter due to increased initial wealth.

Considering cross risk apportionment, the analysis is more complex since there are now two initial wealth levels, one in each dimension. We are now ready to state the main result of this section.

Theorem 5.1. *Consider expected utility* (3.1).

(i) If $X_1 \preceq_{s_1} Y_1$ and $X_2 \preceq_{s_2} Y_2$ then

$$v \text{ exhibits correlation aversion of order } (s_1 + 1, s_2)$$

10

$$\Rightarrow (\rho, w_1) \mapsto \mathbb{U}(\rho, w_1, w_2) \text{ is supermodular}$$

$$v \text{ exhibits correlation aversion of order } (s_1, s_2 + 1)$$

$$\Rightarrow (\rho, w_2) \mapsto \mathbb{U}(\rho, w_1, w_2) \text{ is supermodular}$$

$$v \text{ exhibits correlation aversion of order } (s_1 + 1, s_2 + 1)$$

$$\Rightarrow \frac{\partial^3}{\partial \rho \partial w_1 \partial w_2} \mathbb{U}(\rho, w_1, w_2) \geq 0.$$

(ii) If $X_1 \preceq_{s_1,=} Y_1$ and $X_2 \preceq_{s_2,=} Y_2$ then

$$v \text{ exhibits cross risk apportionment of order } (s_1 + 1, s_2)$$

$$\Rightarrow (\rho, w_1) \mapsto \mathbb{U}(\rho, w_1, w_2) \text{ is supermodular}$$

$$v \text{ exhibits cross risk apportionment of order } (s_1, s_2 + 1)$$

$$\Rightarrow (\rho, w_2) \mapsto \mathbb{U}(\rho, w_1, w_2) \text{ is supermodular}$$

$$v \text{ exhibits cross risk apportionment of orders } (s_1 + 1, s_2) \text{ and } (s_1, s_2 + 1)$$

$$\Rightarrow \frac{\partial^3}{\partial \rho \partial w_1 \partial w_2} \mathbb{U}(\rho, w_1, w_2) \geq 0.$$

Proof. We only prove (i). The pain caused by an increase in the parameter ρ is given by

$$
\begin{aligned}
\frac{\partial \mathbb{U}(\rho, w_1, w_2)}{\partial \rho} = \; & E[v(w_1 + X_1, w_2 + X_2)] - E[v(w_1 + Y_1, w_2 + X_2)] \\
& - E[v(w_1 + X_1, w_2 + Y_2)] + E[v(w_1 + Y_1, w_2 + Y_2)].
\end{aligned}
$$

Define g and h respectively as

$$
\begin{aligned}
g(x, y) &= E[v(x, y + X_2)] - E[v(x, y + Y_2)] \\
h(x, y) &= E[v(x + X_1, y)] - E[v(x + Y_1, y)].
\end{aligned}
$$

This allows us to rewrite the previous pain as

$$
\begin{aligned}
\frac{\partial \mathbb{U}(\rho, w_1, w_2)}{\partial \rho} &= E[g(w_1 + X_1, w_2)] - E[g(w_1 + Y_1, w_2)] \\
&= E[h(w_1, w_2 + X_2)] - E[h(w_1, w_2 + Y_2)].
\end{aligned}
$$

To study the decision-maker's sensitivity to an increase in the correlation parameter ρ when he becomes richer in the first attribute, we must examine the sign of the following expression:

$$\frac{\partial^2 \mathbb{U}(\rho, w_1, w_2)}{\partial \rho \partial w_1} = E[g^{(1,0)}(w_1 + X_1, w_2)] - E[g^{(1,0)}(w_1 + Y_1, w_2)].$$

To study the decision-maker's sensitivity to an increase in the correlation parameter ρ when he becomes richer in the second attribute, we must examine the sign of the following expression:

$$\frac{\partial^2 \mathbb{U}(\rho, w_1, w_2)}{\partial \rho \partial w_2} = E[h^{(0,1)}(w_1, w_2 + X_2)] - E[h^{(0,1)}(w_1, w_2 + Y_2)].$$

11

It is then easy to verify that these expressions are non negative if the functions g and h are respectively such that $(-1)^{(s_1+1)} g^{(s_1+1,0)} \leq 0$ and $(-1)^{(s_2+1)} h^{(0,s_2+1)} \leq 0$, which is equivalent to require that v expresses correlation aversion of order (s_1+1, s_2) and correlation aversion of order (s_1, s_2+1), respectively. $\qquad\square$

Let us discuss the meaning of Theorem 5.1(i). Any decision-maker with a utility function v expressing correlation aversion of order (s_1, s_2) dislikes a simultaneous increase in the probability of getting the extreme outcomes $(w_1 + X_1, w_2 + X_2)$ (the worst one) and $(w_1 + Y_1, w_2 + Y_2)$ (the best one) and a corresponding decrease in the probability of getting the intermediate outcomes $(w_1 + X_1, w_2 + Y_2)$ and $(w_1 + Y_1, w_2 + X_2)$. Proposition 5.1 states that the pain caused by such a probability mass shift is decreasing in the initial wealth levels w_1 and w_2 provided v expresses correlation aversion of order (s_1+1, s_2) or of order (s_1, s_2+1). Hence, the extent to which the decision-maker dislikes a spread in the probabilities from the inner cases $(w_1 + X_1, w_2 + Y_2)$ and $(w_1 + Y_1, w_2 + X_2)$ to the outer cases $(w_1 + X_1, w_2 + X_2)$ and $(w_1 + Y_1, w_2 + Y_2)$ is decreasing with initial wealth levels w_1 and w_2. Moreover, this extent is supermodular in (w_1, w_2) provided v expresses correlation aversion of order (s_1+1, s_2+1). The interpretation for (ii) is similar.

Acknowledgements

References

- Caballe, J., Pomansky, A., 1996. Mixed risk aversion. Journal of Economic Theory 71, 485-513.

- Chiu, W.H., Eeckhoudt, L., Rey, B., 2010. On relative and partial risk attitudes: Theory and implications. Economic Theory, in press.

- Denuit, M., Eeckhoudt, L., 2010. Bivariate stochastic dominance and substitute risk (in)dependent utilities. Decision Analysis 7, 302-312.

- Denuit, M., Eeckhoudt, L., Rey, B., 2010. Some consequences of correlation aversion in decision science. Annals of Operations Research 176, 259-269.

- Denuit, M., Lefevre, Cl., Mesfioui, M., 1999. A class of bivariate stochastic orderings with applications in actuarial sciences. Insurance: Mathematics and Economics 24, 31-50.

- Denuit, M., Rey, B., 2010. Prudence, temperance, edginess, and risk apportionment as decreasing sensitivity to detrimental changes. Mathematical Social Sciences, 60, 137-143.

- Ebert, S., 2010. Moment characterization of higher-order risk preferences. Theory and Decision, in press.

- Eeckhoudt, L., Rey, B., Schlesinger, H., 2007. A good sign for multivariate risk taking. Management Science, 53, 117-124.

- Eeckhoudt, L., Etner, J., Schroyen, F., 2009. The values of relative risk aversion and prudence: a context-free interpretation. Mathematical Social Sciences 58, 1-7.

- Eeckhoudt, L., Schlesinger H., Tsetlin, I. 2009. Apportioning risks via stochastic dominance. Journal of Economic Theory 144, 994-1003.

- Eeckhoudt, L., Schlesinger, H., 2006. Putting risk in its proper place. American Economic Review 96, 280-289.

- Ekern, S. 1980. Increasing nth degree risk. Economics Letters 6, 329-333.

- Epstein, L.G., Tanny, S.M., 1980. Increasing generalized correlation: A definition and some economic consequences. Canadian Journal of Economics 13, 16-34.

- Jokung, O., 2011. Risk apportionment via bivariate stochastic dominance. Journal of Mathematical Economics, 47, 448-452.

- Menezes, C., Geiss, C., Tressler, J., 1980. Increasing downside risk. American Economic Review 70, 921-932.

- Roger, P., 2011. Mixed risk aversion and preference for risk disaggregation: A story of moments. Theory and Decision 70, 27-44.

- Rothshild, M., Stiglitz J.E., 1970. Increasing risk: 1. a definition. Journal of Economic Theory 2, 225-243.

- Wang, J., Li, J. 2010. Multiplicative risk apportionment. Mathematical Social Sciences 60, 79-81.