Statistics for spectral tail processes of heavy–tailed Markov chains

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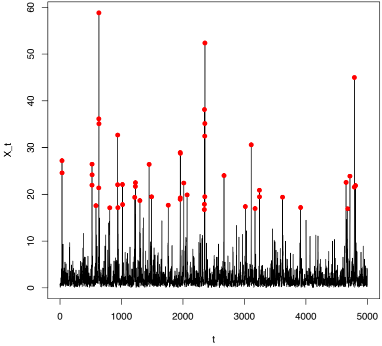
Modelling temporal tail dependence

- Time series $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$
- Strictly stationary
- Heavy–tailed
- What happens after an extreme event?



Tools for measuring extremal dependence

- Temporal dependence:
  - the extremal index: measure of clustering in the extremes,
  - the tail dependence coefficient: $\lim_{x\to\infty} \Pr(\|\boldsymbol{X}_t\| > x \mid \|\boldsymbol{X}_0\| > x)$.
- Cross–sectional and temporal dependence:
  - the extremogram: a correlogram for extreme events (Davis and Mikosch, 2009),
  - the tail process: limit in law of the process $\left(x^{-1}\boldsymbol{X}_t \mid \|\boldsymbol{X}_0\| > x\right)$ (Basrak and Segers, 2009).

Statistics for spectral processes of heavy–tailed M. chains

1. When does the tail process exists?
   - Regular variation
   - Examples
2. The tail process
   - What is it?
   - Why is it useful?
   - Examples
3. Tail processes of heavy–tailed Markov chains
   - Probabilistic overview
   - Statistics
   - Numerical simulations

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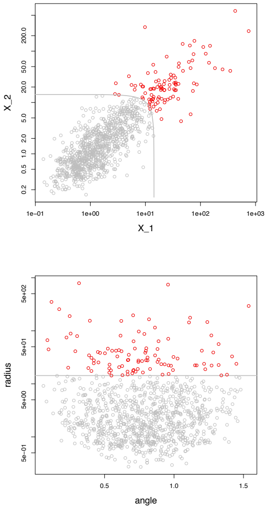
Regularly varying random vectors

A random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ is regularly varying with index $\alpha > 0$ if

$$\left(x^{-1}\boldsymbol{X} \mid \|\boldsymbol{X}\| > x\right) \xrightarrow{d} \boldsymbol{Y} = \boldsymbol{\Theta}\|\boldsymbol{Y}\|,$$

as $x \to \infty$, with $\boldsymbol{\Theta}$ and $\|\boldsymbol{Y}\|$ being independent and

$$\Pr(\|\boldsymbol{Y}\| > y) = y^{-\alpha} \text{ for } y \geq 1.$$



Regularly varying sequences

The time series $(\boldsymbol{X}_t)_{t \in Z}$ is said to be jointly regularly varying with index $\alpha > 0$ if for all integers $k \leq l$ the random vector $(\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l)$ is regularly varying with index α.

Wide range of stationary regularly varying sequences

- X_t i.i.d. RV(α) $\Leftrightarrow$ X is RV with the same index α
- Linear processes $X_t = \sum_{j=0}^{\infty} \phi_j Z_{t-j}$ with RV i.i.d real–valued noise Z_t under conditions on the deterministic sequence ϕ_j
- Models for returns $X_t = \sigma_t Z_t$, where σ_t is stationary non–negative sequence and Z_t is i.i.d.
  - Stochastic volatility where σ_t and Z_t and are independent under conditions: $\mathrm{E}\sigma^{\alpha+\delta} < \infty$ for some $\delta > 0$ and Z_t is i.i.d RV(α)
  - GARCH model
- Stochastic recurrence equations $X_t = A_t X_{t-1} + B_t$ where (A_t, B_t) is an i.i.d. $\mathbf{R}^2$–valued sequence

Statistics for spectral processes of heavy–tailed M. chains

The tail process

The tail process $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$ of $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is defined as the limit in law of

$$\left(x^{-1}\boldsymbol{X}_r,\ldots,x^{-1}\boldsymbol{X}_s \mid \|\boldsymbol{X}_0\| > x\right) \xrightarrow{d} (\boldsymbol{Y}_r,\ldots,\boldsymbol{Y}_s)\,,$$

as $x\to\infty$, for $r,s\in\mathbb{Z}$ with $r\le s$.

Assume that $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is jointly regularly varying with index $\alpha>0$.
Then, there exists $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$ with $\Pr(\|\boldsymbol{Y}_0\|>y)=y^{-\alpha}$ for $y\ge 1$.

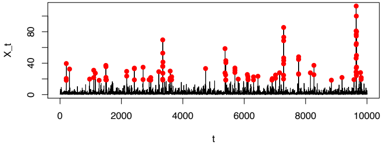
The spectral tail process

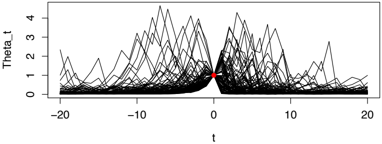
Writing $\Theta_t = \boldsymbol{Y}_t / \|\boldsymbol{Y}_0\|$ for $t \in \mathbb{Z}$ gives

$$\left(\frac{\boldsymbol{X}_r}{\|\boldsymbol{X}_0\|}, \ldots, \frac{\boldsymbol{X}_s}{\|\boldsymbol{X}_0\|} \mid \|\boldsymbol{X}_0\| > x \right) \xrightarrow{d} (\Theta_r, \ldots, \Theta_s).$$

The process $(\Theta_t)_{t\in\mathbb{Z}}$ is the spectral tail process of $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ and is independent of $\|\boldsymbol{Y}_0\|$.

Reversing time





Restricting $\mathbf{\Theta}_t$ i.e. the limit in law

$$(\boldsymbol{X}_t/\|\boldsymbol{X}_0\| \mid \|\boldsymbol{X}_0\| > x)$$

- $t \geq 0$ gives forward and
- $t \leq 0$ gives backward

  spectral tail process.

The relation between those two can be characterized by the following ($d = 1$, $X_t > 0$, $s > 0$):

$$\Pr(\Theta_s > c) = \mathrm{E}\left\{\Theta_{-s}^{\alpha}\mathbf{1}(\Theta_{-s} < 1/c)\right\}, \ c > 0.$$

The tail process embeds known tail asymptotic quantities

- The tail dependence coefficient:

$$\lambda_t := \lim_{u\to\infty} \Pr\left(\|\boldsymbol{X}_t\| > u \mid \|\boldsymbol{X}_0\| > u\right) = \mathrm{E}\left\{\min\left(\|\boldsymbol{\Theta}_t\|^\alpha, 1\right)\right\}.$$

- The extremogram:

$$\begin{aligned}\gamma_{AB}(h) :=& \lim_{n\to\infty} n \operatorname{cov}\left(\mathbf{1}_{\left\{a_n^{-1}\boldsymbol{X}_0\in A\right\}}, \mathbf{1}_{\left\{a_n^{-1}\boldsymbol{X}_h\in B\right\}}\right)\\ =& \Pr\left(\boldsymbol{Y}_h \in B,\ \boldsymbol{Y}_0 \in A\right).\end{aligned}$$

- The extremal index:

$$\begin{aligned}\theta :=& \lim_{r\to\infty}\lim_{u\to\infty} \Pr\left(\max_{t=1,\dots,r}\|\boldsymbol{X}_t\| \le u \mid \|\boldsymbol{X}_0\| > u\right)\\ =& \mathrm{E}\left(\sup_{t\ge 0}\|\boldsymbol{\Theta}_t\|^\alpha - \sup_{t\ge 1}\|\boldsymbol{\Theta}_t\|^\alpha\right).\end{aligned}$$

- Cross–sectional and temporal dependence...

The spectral tail process for particular time series

- For i.i.d. time series:

$$\boldsymbol{Y}_t = \boldsymbol{\Theta}_t = 0 \quad \text{when} \quad t \neq 0.$$

- For the stochastic recurrence equation

$$\boldsymbol{X}_t = \boldsymbol{A}_t \boldsymbol{X}_{t-1} + \boldsymbol{B}_t, \quad t \in \mathbb{Z},$$

for some i.i.d. sequence $(\boldsymbol{A}_t, \boldsymbol{B}_t)$, $t \in \mathbb{Z}$, of random $d \times d$ matrices $\boldsymbol{A}_t$ and random d–dimensional vectors $\boldsymbol{B}_t$:

$$\boldsymbol{\Theta}_t = \boldsymbol{A}_t^* \cdots \boldsymbol{A}_2^* \boldsymbol{A}_1^* \boldsymbol{\Theta}_0, \quad t > 0.$$

Statistics for spectral processes of heavy–tailed M. chains

When X_t is a positive–valued Markov chain

The spectral process can be fully characterized only by Θ_1 since

$$\Theta_0 = 1, \ \Theta_{\pm t} = \prod_{i=1}^{t} A_{\pm i}, \quad t \geq 1,$$

where $A_1, A_{-1}, A_2, A_{-2}, \ldots$ are independent. Moreover, for all integers $t \geq 1$, $\mathcal{L}(A_t) := \nu$ and $\mathcal{L}(A_{-t}) := \nu^*$ and

$$\mathcal{L}(X_1/X_0 \mid X_0 > x) \to \mathcal{L}(\Theta_1) = \nu,$$

$$\mathcal{L}(X_{-1}/X_0 \mid X_0 > x) \to \mathcal{L}(\Theta_{-1}) = \nu^*.$$

The measures ν and ν^* are related with each other through

$$\Pr\left(\Theta_1 > c\right) = \mathrm{E}\left\{\Theta_{-1}^{\alpha} \mathbf{1}\left(\Theta_{-1} < 1/c\right)\right\}, \ c > 0.$$

The forward estimator

The forward estimator is an empirical version of

$$(X_1/X_0 \mid X_0 > x) \xrightarrow{d} (\Theta_1),$$

and is defined by:

$$\widehat{\overline{F}}_n(c) = \frac{\sum_{t=1}^{n-1} \mathbf{1}(X_{t+1}/X_t > c,\ X_t > u_n)}{\sum_{t=1}^{n} \mathbf{1}(X_t > u_n)}.$$

The backward estimator

The backward estimator exploits the time–change formula

$$\Pr\left(\Theta_1 > c\right) = \mathrm{E}\left\{\Theta_{-1}^{\alpha} \mathbf{1}\left(\Theta_{-1} < 1/c\right)\right\}, \; c > 0,$$

and is defined by:

$$\widehat{\overline{F}}'_n(c) = \frac{\sum_{t=2}^{n} \left(X_{t-1}/X_t\right)^{\alpha} \mathbf{1}\left(X_{t-1}/X_t < 1/c, \; X_t > u_n\right)}{\sum_{t=1}^{n} \mathbf{1}\left(X_t > u_n\right)}.$$

The mixture estimator

The mixture estimator is a convex combination of the forward estimator and the backward estimator, i.e.:

$$\widehat{\overline{F}}''_n(c) = \alpha(c)\,\widehat{\overline{F}}'_n(c) + \beta(c)\,\widehat{\overline{F}}_n(c)\,, \quad \alpha(c) + \beta(c) = 1.$$

This may bring further reduction in asymptotic variance.

Asymptotic normality of the estimators

Denote $v_n = \Pr(X_t > u_n)$,

- **The forward estimator**

$$(nv_n)^{1/2}\left\{\widehat{\overline{F}}_n(c) - \Pr(X_1/X_0 > c|\, X_0 > u_n)\right\} \overset{n\to\infty}{\longrightarrow} N\{0, \omega(c)\}$$

$$\omega(c) = \Pr(\Theta_1 > c)\Pr(\Theta_1 \leq c)$$

- **The backward estimator**

$$(nv_n)^{1/2}\left\{\widehat{\overline{F}}'_n(c) - \Pr(X_1/X_0 > c|\, X_0 > u_n)\right\} \overset{n\to\infty}{\longrightarrow} N\{0, \omega'(c)\}$$

$$\omega'(c) = \mathrm{E}\left\{\Theta_1^{-\alpha}\mathbf{1}(\Theta_1 > c)\right\} - \{\Pr(\Theta_1 > c)\}^2$$

- **The mixture estimator**

$$(nv_n)^{1/2}\left\{\widehat{\overline{F}}''_n(c) - \Pr(X_1/X_0 > c|\, X_0 > u_n)\right\} \overset{n\to\infty}{\longrightarrow} N\{0, \omega''(c)\}$$

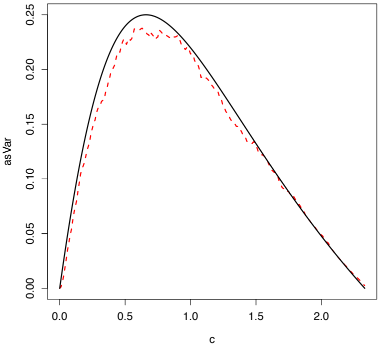
Setting – skip this slide?

At 12:29 PM, people will just be longing for sandwiches. Skip!

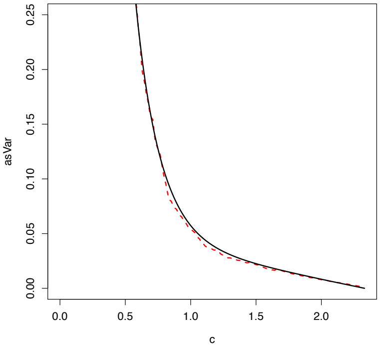
We consider the following stochastic recurrence equation:

- $X_t = A_t X_{t-1} + B_t, \; t \in \mathbb{Z}$,
- $A_t = \tan(c_0 U_t), \; B_t = V_t$,
- $(U_t)_{t\in\mathbb{Z}}, \; (V_t)_{t\in\mathbb{Z}}$ i.i.d. uniform $(0,1)$,
- $c_0 \approx 1.16556$,
- X_t is stationary and regularly varying with $\alpha = 2$,
- $\mathcal{L}(\Theta_1) = \mathcal{L}(A_1)$
- n=10000,
- $u_n = q(99\%)$
- $nv_n = 1000$ extremes,
- 1000 repetitions.

Results I – MC approximation of the asymptotic variance

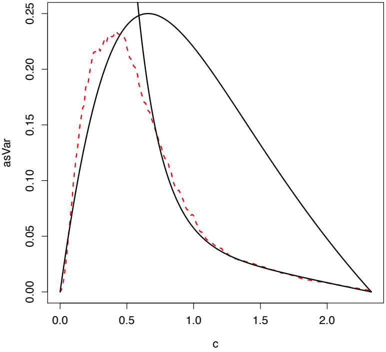


Forward: $\widehat{\overline{F}}_n(c)$



Backward: $\widehat{\overline{F}}'_n(c)$

Results II – MC approximation of the asymptotic variance



Mixture: $\widehat{\overline{F}}_n''(c)$

We set $\alpha(c) = \widehat{F}(c)$ and $\beta(c) = 1 - \widehat{F}(c)$.

Message

- The tail process models cross–sectional and temporal dependence of extremes of stationary time series.
- The tail process exists if the underlying time series is regularly varying.
- The radial component ($\|\boldsymbol{Y}_0\|$) and the angular component ($\boldsymbol{\Theta}_t$) are independent.
- The forward and the backward tail processes are related with each other.
- When the underlying time series is Markovian then the spectral tail process can be characterize by Θ_1.
- We propose asymptotically normal estimators for the law of Θ_1.

Outlook

- How to choose weights of the mixture estimator?
- What to do with the unknown index of regular variation α?
- How to estimate Θ_t and describe asymptotic behaviour of the estimator?
- How to model the tail process when we no longer assume that $X_t > 0$?
- How to tackle $d > 1$ case and cross–sectional analysis after extreme shocks?

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Thank you

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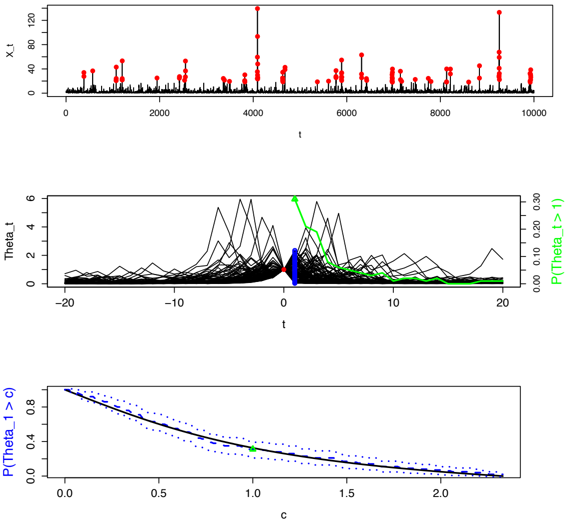
X_t
0
40
80
120
0
2000
4000
6000
8000
10000
t
Theta_t
0
2
4
6
−20
−10
0
10
20
t
P(Theta_t > 1)
0.00
0.10
0.20
0.30
P(Theta_1 > c)
0.0
0.4
0.8
0.0
0.5
1.0
1.5
2.0
c